DIFFERENTIAL CHARACTERS OF DRINFELD MODULES AND DE RHAM COHOMOLOGY

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Abstract. We introduce δ-characters of Drinfeld modules, which are function-field analogues of Buium's δ-characters of p-adic elliptic curves and of Manin's differential characters of elliptic curves in differential algebra. We determine the structure of the group of δ-characters. This shows the existence of a family of interesting δ-modular functions on the moduli of Drinfeld modules. It also leads to a canonical subspace inside the de Rham cohomology of a Drinfeld module over a δ-base. This subspace has a canonical semi-linear Frobenius operator on it.

1. Introduction

The aim of this paper is to study the group of δ-characters of Drinfeld modules. The δ-characters are analogues of the Manin maps associated to elliptic curves in the sense of differential algebra and, even more closely, of Buium's δ-characters of p-adic elliptic curves. As a consequence of understanding these δ-characters, we produce an interesting short exact sequence of finite rank modules which maps to the Hodge sequence of the Drinfeld module. If we look at the image inside the de Rham cohomology group, then this image also has a natural semi-linear operator on it.

One of the reasons for studying Drinfeld modules, indeed the original reason, is that progress there comes easier than over number fields, yet remarkably often it can be translated back to the number field setting. In our case too, the theorems that we prove can also be reproved for elliptic curves over local fields of characteristic 0. However, we will not do so in this paper as it will require us to develop tools different from the ones required in the case of Drinfeld modules. But we do note that all the fundamental principles that go into studying Drinfeld modules also work for elliptic curves and we will look into it that aspect in a subsequent paper.

Fix $q = p^h$ where p is a prime and $h \geq 1$. Let X be a projective, geometrically connected, smooth curve over $\mathbb{F}_q$. Fix an $\mathbb{F}_q$-rational point ∞ on X. Consider the Dedekind domain $A = \mathcal{O}(X \setminus \{\infty\})$. Fix a maximal ideal $\mathfrak{p}$ of A, and let $\hat{A}$ denote the $\mathfrak{p}$-adic completion of A. Denote by $\hat{\mathfrak{p}}$ the maximal ideal of the complete, local ring $\hat{A}$ and $\iota : A \hookrightarrow \hat{A}$ the natural inclusion. Let $\pi \in A$ be such that $\iota(\pi)$ generates the maximal ideal $\hat{\mathfrak{p}}$ in $\hat{A}$. Since ι is an injection, by abuse of notation, we will consider π as an element of $\hat{A}$ as well. Let $k := A/\mathfrak{p} = \hat{A}/(\pi)$ and $q = |k|$.

Let R be an $\hat{A}$-algebra which is also π-adically complete and flat, or equivalently π-torsion free. Thus the structure map $\theta : A \to R$ is injective and hence one can

Date: March 26, 2017.

say that θ is of *generic characteristic.* Fix a lift of the q-power Frobenius ϕ on R which when restricted to $\hat{A}$ is identity. Do note here that the identity map on $\hat{A}$ indeed lifts the q-power Frobenius on $\hat{A}/\hat{\mathfrak{p}}$. Then one can consider the operator on R given by $\delta x = \frac{\phi(x)-x^q}{\pi}$. It is called the π-*derivation* associated to ϕ.

A $\hat{\mathfrak{p}}$-formal A-module scheme over $S = \operatorname{Spf} R$ is by definition a pair (E, φ), where E is a commutative group object in the category of formal S-schemes and $\varphi : A \to \operatorname{End}(E/S)$ is a ring map. Then the tangent space $T_0 E$ at the identity has two A-modules structures: one coming by restriction of the usual R-module structure to A, and the other coming from differentiating φ. We will say that (E, φ) is *strict* if these two A-module structures coincide, and *admissible* if it is both strict and isomorphic to $\mathbb{G}_{\mathrm{a}}$ as a group scheme. The group scheme $\hat{\mathbb{G}}_{\mathrm{a}}$ plays a second role here in that it admits an A-module structure $\varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}$ given by the usual scalar multiplication $\varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}(a)x = ax$. This role will be especially important for us, as our δ-characters have $(\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$ it as their target.

In analogy with Buium's arithmetic jet space [Bui2], we define the n-th jet space $J^n E$ of the Drinfeld module E to be the (π-adic) formal scheme over R with functor of points

$$(J^n E)(C) = E(W_n(C)),$$

where W_n is the function-field analogue of the usual Witt vector functor, which we recall in section 3. It has relative dimension $n+1$ over $\operatorname{Spf} R$. One might also call it the function-field Greenberg transform. Since E is an A-module formal scheme, $J^n E$ has a natural A-module structure $(J^n E, \varphi_{J^n E})$. However, we would like to remark here that for all $n \geq 1$, the $J^n E$ are not Anderson modules.

We then define the group $\mathbf{X}_n(E)$ of δ-characters to be the group of morphisms of A-module schemes over $\operatorname{Spf}(R)$ from $(J^n E, \varphi_{J^n E})$ to $(\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$. Since $\hat{\mathbb{G}}_{\mathrm{a}}$ has an R-linear structure as well, $\mathbf{X}_n(E)$ is naturally an R-module for all $n \geq 0$. Let $\mathbf{X}_\infty(E)$ be the limit of $\mathbf{X}_n(E)$ over n. Now for all $n \geq 0$, there is a canonical A-linear Frobenius morphism $\phi : J^{n+1}E \to J^n E$ lying over the endomorphism ϕ of $\operatorname{Spf}(R)$. Hence pulling back morphisms via ϕ as $\Theta \mapsto \phi^*\Theta$, endows $\mathbf{X}_\infty(E)$ with an action of ϕ^* and hence makes $\mathbf{X}_\infty(E)$ into a left module over the twisted polynomial ring $R\{\phi^*\}$ with commutation law $\phi^* r = \phi(r)\phi^*$.

We say E splits at m if $\mathbf{X}_m(E) \neq \{0\}$ but $\mathbf{X}_i(E) = \{0\}$ for all $0 \leq i \leq m-1$. Then we show that m satisfies $1 \leq m \leq r$, where r is the rank of E, and $\mathbf{X}_m(E)$ is a free R-module with basis element $\Theta_m \in \mathbf{X}_m(E)$ depending only on a chosen coordinate on E. In the case when the rank r is 2, the splitting condition coincides with the notion of canonical lifts on E, that is, $m = 1$ if and only if E admits a lift of Frobenius compatible with the A-module structure on E, otherwise, $m = 2$.

The structure of $\mathbf{X}_\infty(E)$ was first studied by Buium [Bui2] in the case of elliptic curves over p-adic rings R. He showed that $\mathbf{X}_\infty(E) \otimes_R K$ is generated by a single element as a $K\{\phi^*\}$-module, where $K = R[\frac{1}{p}]$. In this paper, by using different methods, one of our results is to show the stronger result that $\mathbf{X}_\infty(E)$ is generated by a single element as an $R\{\phi^*\}$-module:

Theorem 1.1. *Let E be a Drinfeld module that splits at m. Then the R-module $\mathbf{X}_m(E)$ is free of rank 1, and it freely generates $\mathbf{X}_\infty(E)$ as an $R\{\phi^*\}$-module in the sense that the canonical map $R\{\phi^*\} \otimes_R \mathbf{X}_m(E) \to \mathbf{X}_\infty(E)$ is an isomorphism.*

Let $u : J^nE \to E$ be the A-linear projection map and $N^n = \ker u$. Then N^n is a strict A-module of relative dimension n over Spf R, and we have the exact sequence

$$0 \to N^n \stackrel{i}{\to} J^nE \stackrel{u}{\to} E \to 0 \tag{1.1}$$

of A-module π-formal schemes. We show in theorem 9.1 that $\mathrm{Hom}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) = \{0\}$. Then if we apply the contravariant functor $\mathrm{Hom}_A(-, \hat{\mathbb{G}}_{\mathrm{a}})$ of A-module scheme morphisms to $(\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$, we obtain

$$0 \to \mathbf{X}_n(E) \stackrel{i^*}{\to} \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \stackrel{\partial}{\to} \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) \tag{1.2}$$

By [Ge1], we have $\mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) \simeq R^{r-1}$, where r is the rank of E.

For each $n \geq 1$, we show in proposition 8.2 that there is a lift of Frobenius $\mathfrak{f} : N^{n+1} \to N^n$ making the system $\{N^n\}$ into a prolongation sequence with respect the obvious projection map $u : N^{n+1} \to N^n$. We call $\mathfrak{f}$ the *lateral Frobenius*. However, $\mathfrak{f}$ is not compatible with i and $\phi : J^{n+1}E \to J^nE$ in the obvious way, that is, it is not true that $\phi \circ i = i \circ \mathfrak{f}$ holds. In fact, we can not expect it to be true because that would induce an A-linear lift of Frobenius on (E, φ_E) which is not the case to start with. Instead we have

$$\phi^2 \circ i = \phi \circ i \circ \mathfrak{f}$$

As a result, if $\mathfrak{f}^*$ denotes the pullback via $\mathfrak{f}$, we obtain the following commutative diagram for all $n \geq m$

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{X}_{n+1}(E) \arrow[r, "(i\circ\phi)^*"] & \mathrm{Hom}_A(N^{n+2}, \hat{\mathbb{G}}_{\mathrm{a}}) \\
0 \arrow[r] & \mathbf{X}_n(E) \arrow[r, "(i\circ\phi)^*"'] \arrow[u, "\phi^*"] & \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[u, "\mathfrak{f}^*"']
\end{tikzcd}$$

Then we define $\mathbf{H}_n(E) = \frac{\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})}{i^*\phi^*(\mathbf{X}_{n-1}(E))}$. The projection map $u : N^{n+1} \to N^n$ induces $u^* : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_{\mathrm{a}})$. It will then follow easily that u^* induces maps $u^* : \mathbf{H}_n(E) \to \mathbf{H}_{n+1}(E)$. Define $\mathbf{H}(E) = \lim_{\to} \mathbf{H}_n(E)$.

Similarly $\mathfrak{f} : N^{n+1} \to N^n$ will induce maps

$$\mathfrak{f}^* : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_{\mathrm{a}})$$

and $\mathfrak{f}^* : \mathbf{H}_n(E) \to \mathbf{H}_{n+1}(E)$. Hence we have a semi-linear endomorphism $\mathfrak{f}^* : \mathbf{H}(E) \to \mathbf{H}(E)$.

Let $\mathbf{I}_n(E) = \text{image } \partial \subseteq \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ as in 1.1 and let $\mathbf{I}(E) := \lim \mathbf{I}_n(E)$. Then we will show in section 10.2 that $\mathbf{I}(E)$ and $\mathbf{H}(E)$ are free R-modules of finite rank and satisfy the short exact sequence of free R-modules

$$0 \to \mathbf{X}_m(E) \to \mathbf{H}(E) \to \mathbf{I}(E) \to 0 \tag{1.3}$$

where $\mathbf{X}_m(E)$ is a free R-module of rank 1.

Recall from [Ge1, Ge2] that the elements in $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ are pairs (C, s) where

$$0 \to \hat{\mathbb{G}}_{\mathrm{a}} \to C \to E \to 0$$

is an extension of A-module schemes and s is a splitting of the extension of tangent spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & \mathrm{Lie}(C) \arrow[l, bend left, "s"] \arrow[r] & \mathrm{Lie}(E) \arrow[r] & 0 \end{tikzcd}$$

The groups $\mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ also fit in a Hodge sequence

$$0 \to \mathrm{Lie}(E)^* \to \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \to 0.$$

Of course, the de Rham cohomology for Drinfeld modules is, in fact, defined to be $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ in [Ge1]. Now given a $\Psi \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$, one can consider the push-out of exact sequence (1.1) by Ψ to obtain

$$0 \to \hat{\mathbb{G}}_{\mathrm{a}} \to E^*_\Psi \to E \to 0 \tag{1.4}$$

which is represented by the class as $\partial(\Psi) \in \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$. In section 8, we will show a way of attaching a canonical splitting of Lie algebras as follows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & \mathrm{Lie}(E^*_\Psi) \arrow[l, bend left, "s_{\mathrm{Witt}}"] \arrow[r] & \mathrm{Lie}(E) \arrow[r] & 0 \end{tikzcd}$$

In other words, for all $n \geq 1$, we can define $\Phi : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ given by $\Phi(\Psi) := (\partial(\Psi), s_{\mathrm{Witt}})$. We will show that the map Φ in fact descends to a map $\Phi : \mathbf{H}(E) \to \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and leads to the following map between short exact sequences which is our next result

Theorem 1.2. *We have the following map between exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{X}_m(E) \arrow[r] \arrow[d, "\Upsilon"'] & \mathbf{H}(E) \arrow[r] \arrow[d, "\Phi"] & \mathbf{I}(E) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathrm{Lie}(E)^* \arrow[r] & H_{dR}(E) \arrow[r] & \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & 0 \end{tikzcd}$$

Moreover, the operator $\mathfrak{f}^*$ on $\mathbf{H}(E)$ descends to its image under Φ.

Even though we show the above results for Drinfeld modules, our methods work for elliptic curves over p-adic fields as well. However, that will be discussed in a subsequent paper. Whether for elliptic curves or Drinfeld modules, the de Rham cohomology has a Frobenius operator obtained by identifying it with the crystalline cohomology. The comparison with our Frobenius operator on $\mathbf{H}(E)$ is a natural open question.

2. Notation

Let us fix some notation which will hold throughout the paper. Let $q = p^h$ where p is a prime and $h \geq 1$. Let X be a projective, geometrically connected, smooth curve over $\mathbb{F}_q$. Fix an $\mathbb{F}_q$-rational point ∞ on X. Let A denote the Dedekind domain $\mathcal{O}(X \setminus \{\infty\})$. Let $\mathfrak{p}$ be a maximal ideal of A, and let $\hat{A}$ denote the $\mathfrak{p}$-adic completion of A. Let t be an element of $\mathfrak{p} \setminus \mathfrak{p}^2$, and let π denote its image in $\hat{A}$. Then π generates the maximal ideal $\hat{\mathfrak{p}}$ of $\hat{A}$. Let k denote the residue field $A/\mathfrak{p}$, and let $\hat{q}$ denote its cardinality. Note that the quotient map $\hat{A} \to k$ has a unique section. Thus $\hat{A}$ is not just an $\mathbb{F}_q$-algebra but also canonically a k-algebra.

Now let R be an $\hat{A}$-algebra which is $\mathfrak{p}$-adically complete and flat, or equivalently π-torsion free. Thus the composition $\theta : A \to \hat{A} \to R$ is injective (assuming $R \neq \{0\}$) and hence one says that θ is of *generic characteristic*. Let us also fix an $\hat{A}$-algebra endomorphism $\phi : R \to R$ which lifts the $\hat{q}$-power Frobenius modulo $\mathfrak{p}R$:

$$\phi(x) \equiv x^{\hat{q}} \bmod \mathfrak{p}R.$$

Do note that the identity map on $\hat{A}$ does indeed lift the $\hat{q}$-power Frobenius on $\hat{A}/\hat{\mathfrak{p}}$.

Also note that not all rings R admit such a Frobenius lift; so the existence of ϕ does place a restriction on R. For our main results, R will in the end be a discrete valuation ring, most importantly the completion of the maximal unramified extension of $\hat{A}$. So the reader may assume this from the start. But some form of our results should hold in general, and with essentially the same proofs. This is of some interest, for instance when R is the coordinate ring of the moduli space of Drinfeld modules of a given rank. With an eye to the future, we have not assumed that R is a discrete valuation ring where it is easily avoided, in sections 3–8.

Let K denote $R[1/\pi]$, and for any R-module M write $M_K = K \otimes_R M$. Finally, let S denote $\operatorname{Spf} R$.

3. Function-field Witt Vectors

Witt vectors over Dedekind domains with finite residue fields were introduced in [Bo1]. We will only work over $\hat{A}$, which is the ring of integers of a local field of characteristic p, and here they were introduced earlier in [D76]. The basic results can be developed exactly as in any of the usual developments of the p-typical Witt vectors. The only difference is that in all formulas any p in a coefficient is replaced with a π and any p in an exponent is replaced with a $\hat{q}$.

3.1. **Frobenius lifts and π-derivations.** Let B be an R-algebra, and let C be a B-algebra with structure map $u : B \to C$. In this paper, a ring homomorphism $\psi : B \to C$ will be called a *lift of Frobenius* (relative to u) if it satisfies the following:

(1) The reduction mod π of ψ is the $\hat{q}$-power Frobenius relative to u, that is, $\psi(x) \equiv u(x)^{\hat{q}} \bmod \pi C$.
(2) The restriction of ψ to R coincides with the fixed ϕ on R, that is, the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "\psi"] & C \\ R \arrow[u] \arrow[r, "\phi"'] & R \arrow[u] \end{tikzcd}$$

A *π-derivation* δ from B to C means a set-theoretic map $\delta : B \to C$ satisfying the following for all $x, y \in B$

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) \\ \delta(xy) &= u(x)^{\hat{q}}\delta(y) + \delta(x)u(y)^{\hat{q}} + \pi\delta(x)\delta(y) \end{aligned}$$

such that for all $r \in R$, we have

$$\delta(r) = \frac{\phi(r) - r^{\hat{q}}}{\pi}.$$

When $C = B$ and u is the identity map, we will call this simply a π-derivation on B.

It follows that the map $\phi : B \to C$ defined as

$$\phi(x) := u(x)^{\hat{q}} + \pi\delta(x)$$

is a lift of Frobenius in the sense above. On the other hand, for any flat R-algebra B with a lift of Frobenius ϕ, one can define the π-derivation $\delta(x) = \frac{\phi(x) - x^{\hat{q}}}{\pi}$ for all $x \in B$.

Note that this definition depends on the choice of uniformizer π, but in a transparent way: if π' is another uniformizer, then $\delta(x)\pi/\pi'$ is a π'-derivation. This correspondence induces a bijection between π-derivations $B \to C$ and π'-derivations $B \to C$.

3.2. **Witt vectors.** We will present three different points of view on function-field Witt vectors, all parallel to the mixed characteristic case. But there is perhaps one unfamiliar element below, which is that we will work relative to our general base R, and it already has a lift of Frobenius. The consequence is that we need to pay attention to certain twists of the scalars by Frobenius, which are invisible over the absolute base $R = \hat{A}$. However this unfamiliar element has nothing to do with the difference between mixed and equal characteristic and only with the difference between the relative and the absolute setting.

Let B be an R-algebra with structure map $u : R \to B$.

(1) The ring $W(B)$ of *π-typical Witt vectors* can be defined as the unique (up to unique isomorphism) R-algebra $W(B)$ with a π-derivtion δ on $W(B)$ and an R-algebra homomorphism $W(B) \to B$ such that, given any R-algebra C with a π-derivation δ on it and an R-algebra map $f : C \to B$, there exists a unique R-algebra homomorphism $g : C \to W(B)$ such that the diagram

$$\begin{tikzcd} W(B) \arrow[d] & \\ B & C \arrow[l, "f"'] \arrow[ul, "g"'] \end{tikzcd}$$

commutes and $g \circ \delta = \delta \circ g$. Thus W is the right adjoint of the forgetful functor from R-algebras with π-derivation to R-algebras. For details, see section 1 of [Bo1]. This approach follows that of [Jo] to the usual p-typical Witt vectors.

(2) If we restrict to flat R-algebras B, then we can ignore the concept of π-derivation and define $W(B)$ simply by expressing the universal property above in terms of Frobenius lifts, as follows. Given a flat R-algebra B, the ring $W(B)$ is the unique (up to unique isomorphism) flat R-algebra $W(B)$ with a lift of Frobenius (in the sense above) $F : W(B) \to W(B)$ and an R-algbebra homomorphism $W(B) \to B$ such that for any flat R-algebra C with a lift of Frobenius ϕ on it and an R-algebra map $f : C \to B$, there exists a unique R-algebra homomorphism $g : C \to W(B)$ such that the diagram

$$\begin{tikzcd} W(B) \arrow[d] & \\ B & C \arrow[l, "f"'] \arrow[ul, "g"'] \end{tikzcd}$$

commutes and $g \circ \phi = F \circ g$.

(3) Finally, one can also define Witt vectors in terms of the Witt polynomials. For each $n \geq 0$ let us define B^{ϕ^n} to be the R-algebra with structure map $R \xrightarrow{\phi^n} R \xrightarrow{u} B$ and define the *ghost rings* to be the product R-algebras $\Pi_\phi^n B = B \times B^\phi \times \cdots \times B^{\phi^n}$ and $\Pi_\phi^\infty B = B \times B^\phi \times \cdots$. Then for all $n \geq 1$ there exists a *restriction*, or *truncation*, map $T_w : \Pi_\phi^n B \to \Pi_\phi^{n-1} B$ given by $T_w(w_0, \cdots, w_n) = (w_0, \cdots, w_{n-1})$. We also have the left shift *Frobenius* operators $F_w : \Pi_\phi^n B \to \Pi_\phi^{n-1} B$ given by $F_w(w_0, \ldots, w_n) = (w_1, \ldots, w_n)$. Note that T_w is an R-algebra morphism, but F_w lies over the Frobenius endomorphism ϕ of R.

Now as sets define

$$W_n(B) = B^{n+1}, \tag{3.1}$$

and define the set map $w : W_n(B) \to \Pi_\phi^n B$ by $w(x_0, \ldots, x_n) = (w_0, \ldots, w_n)$ where

$$w_i = x_0^{\hat{q}^i} + \pi x_1^{\hat{q}^{i-1}} + \cdots + \pi^i x_i \tag{3.2}$$

are the *Witt polynomials*. The map w is known as the *ghost* map. (Do note that under the traditional indexing our W_n would be denoted W_{n+1}.) We can then define the ring $W_n(B)$, the ring of truncated π-typical Witt vectors, by the following theorem as in the p-typical case [H05]:

Theorem 3.1. *For each $n \geq 0$, there exists a unique functorial R-algebra structure on $W_n(B)$ such that w becomes a natural transformation of functors of R-algebras.*

Note that, unlike with the usual Witt vectors in mixed characteristic, addition for function-field Witt vectors is performed componentwise. This is because the Witt polynomials (3.2) are additive. This might appear to defeat the whole point of Witt vectors and arithmetic jet spaces. But this is not so. The reason is that while the additive structure is the componentwise one, the A-module structure is not. So the difference is only that, unlike in mixed characteristic where $A = \mathbb{Z}$, a group structure is weaker than A-module structure. In fact, because the Witt polynomials are k-linear, the k-vector space structure on $W_n(B)$ is the componentwise one. This is just like with the p-typical Witt vectors, where multiplication by roots of $x^p - x$ can be performed componentwise.

3.3. **Operations on Witt vectors.** Now we recall some important operators on the Witt vectors. There are the *restriction*, or *truncation*, maps $T : W_n(B) \to W_{n-1}(B)$ given by $T(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1})$. Note that $W(B) = \lim_{\leftarrow} W_n(B)$. There is also the *Frobenius* ring homomorphism $F : W_n(B) \to W_{n-1}(B)$, which can be described in terms of the ghost map. It is the unique map which is functorial in B and makes the following diagram commutative

$$\begin{tikzcd} W_n(B) \arrow[r, "w"] \arrow[d, "F"'] & \Pi_\phi^n B \arrow[d, "F_w"] \\ W_{n-1}(B) \arrow[r, "w"'] & \Pi_\phi^{n-1} B^n \end{tikzcd} \tag{3.3}$$

As with the ghost components, T is an R-algebra map but F lies over the Frobenius endomorphism ϕ of R.

Next we have the *Verschiebung* $V : W_{n-1}(B) \to W_n(B)$ given by $V(x_0, \ldots, x_{n-1}) = (0, x_0, \ldots, x_{n-1})$. Let $V_w : \Pi_\phi^{n-1} B \to \Pi_\phi^n B$ be the additive map given by $V_w(w_0, .., w_{n-1}) = (0, \pi w_0, \ldots, \pi w_{n-1})$. Then the Verschiebung V makes the following diagram commute:

$$\begin{tikzcd} W_{n-1}(B) \arrow[r, "w"] \arrow[d, "V"'] & \Pi_\phi^{n-1} B \arrow[d, "V_w"] \\ W_n(B) \arrow[r, "w"'] & \Pi_\phi^n B \end{tikzcd} \tag{3.4}$$

For all $n \geq 0$ the Frobenius and the Verschiebung satisfy the identity

$$FV(x) = \pi x. \tag{3.5}$$

The Verschiebung is not a ring homomorphism, but it is k-linear.

Finally, we have the multiplicative Teichmüller map $[\ ] : B \to W_n(B)$ given by $x \mapsto [x] = (x, 0, 0, \ldots)$. Here in the function-field setting, $[\ ]$ is additive and even a homomorphism of k-algebras.

3.4. **Computing the universal map to Witt vectors.** Given an R-algebra C with a π-derivation δ and an R-algebra map $f : C \to B$, we will now describe the universal lift $g : C \to W(B)$. The explicit description of g leads us to proposition 3.2 which is used in section 11 in computations for Drinfeld modules of rank 2. The reader may skip this subsection without breaking continuity till then.

It is enough to work in the case where both B and C are flat over R. Then the ghost map $w : W(B) \to \Pi_\phi^\infty B$ is injective. Consider the map $[\phi] : C \to \Pi_\phi^\infty C$ given by $x \mapsto (x, \phi(x), \phi^2(x), \ldots)$. Then we have the following commutative diagram:

$$\begin{tikzcd} & & C \arrow[dll, bend right, "g"'] \arrow[dl, "f \circ [\phi]"'] \arrow[d, "[\phi]"] \\ W(B) \arrow[r, "w"] \arrow[d, "F"] & \Pi_\phi^\infty B \arrow[d, "F_w"] & \Pi_\phi^\infty C \arrow[l, "f"'] \arrow[d, "F_w"] \\ W(B) \arrow[r, "w"] & \Pi_\phi^\infty B & \Pi_\phi^\infty C \arrow[l, "f"'] \end{tikzcd}$$

Thus the map $f \circ [\phi] : C \to \Pi_\phi^\infty B$ factors through $W(B)$ as our universal map $g : C \to W(B)$.

Let us now give an inductive description of the map g. Write

$$g(x) = (x_0, x_1, \cdots) \in W(B).$$

Then from the above diagram $w \circ g = f \circ [\phi]$. Therefore the vector $(x_0, x_1, \ldots)$ is the unique solution to the system of equations

$$x_0^{\hat{q}^n} + \pi x_1^{\hat{q}^{n-1}} + \cdots + \pi^n x_n = f(\phi^n(x)), \tag{3.6}$$

for $n \geq 0$. For example, we have $x_0 = f(x)$ and $x_1 = f(\delta(x))$.

Now consider the case where B itself has a π-derivation, $C = B$, and $f = \mathbb{1}$. For any $x \in B$, let us write $x^{(n)} := \delta^n(x)$, or simply $x' = \delta(x)$, $x'' = \delta^2(x)$ and so on.

Proposition 3.2. *We have $x_0 = x$, $x_1 = x'$ and $x_2 = x'' + \pi^{\hat{q}-2}(x')^{\hat{q}}$.*

Proof. As stated above, equalities $x_0 = x$ and $x_1 = x'$ follow immediately from (3.6). For $n = 2$, we have

$$\begin{aligned} x_0^{\hat{q}^2} + \pi x_1^{\hat{q}} + \pi^2 x_2 &= \phi^2(x) \\ &= \phi(x^{\hat{q}} + \pi x') \\ &= \phi(x)^{\hat{q}} + \pi\phi(x') \\ &= x^{\hat{q}^2} + \pi^{\hat{q}}(x')^{\hat{q}} + \pi((x')^{\hat{q}} + \pi x'') \end{aligned}$$

And therefore we have $x_2 = x'' + \pi^{\hat{q}-2}(x')^{\hat{q}}$. □

4. A-module schemes, Jet Spaces and preliminieries

An A-module scheme over $S = \operatorname{Spf} R$ is by definition a pair (E, φ_E), where E is a commutative group object in the category of S-schemes and $\varphi_E : A \to \operatorname{End}(E/S)$ is a ring map. (Here and below, by a scheme over the formal scheme S, we mean a formal scheme formed from a compatible family of schemes over the schemes $\operatorname{Spec} R/\mathfrak{p}^n R$.) Then the tangent space T_0E at the identity has two A-modules structures: one coming by restriction of the usual R-module structure to A, and the other coming from differentiating φ_E. We will say that (E, φ_E) is *strict* if these two A-module structures coincide, and *admissible* if it is both strict and isomorphic to the additive group $\hat{\mathbb{G}}_{\mathrm{a}} = \hat{\mathbb{G}}_{\mathrm{a}/S}$ as a group scheme. (Note that it is best practice to require only the isomorphism with $\hat{\mathbb{G}}_{\mathrm{a}}$ to exist locally on S. So below, our Drinfeld modules would more properly be called *coordinatized* Drinfeld modules.)

A *Drinfeld module* (E, φ_E) *of rank* r is an admissible A-module scheme over S such that for each non-zero $a \in A$, the group scheme $\ker(\varphi_E(a))$ is finite of degree $|a|^r = q^{-r\operatorname{ord}_\infty(a)}$ over S.

Proposition 4.1. *If f is an endomorphism of the $\mathbb{F}_q$-module scheme $\hat{\mathbb{G}}_{\mathrm{a}/S}$ over S, then it is of the form*

$$f(x) = \sum_{i=0}^{\infty} a_i x^{q^i},$$

where f is a restricted power series, meaning $a_i \to 0$ π-adically as $i \to \infty$.

Proof. Let $f \in \operatorname{Hom}(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}})$ be an additive endomorphism of $\hat{\mathbb{G}}_{\mathrm{a}}$. Then f is given a restricted power series $\sum_i b_i x^i$ such that $b_i \to 0$ as $i \to \infty$. Since f is additive, we have $b_i = 0$ unless i is a power of p. Second, because f is $\mathbb{F}_q$-linear, we have $\sum_i b_{p^i}(cx)^{p^i} = c\sum_i b_{p^i} x^{p^i}$ for all $c \in \mathbb{F}_q$. Considering the case where c is a generator of $\mathbb{F}_q{}^*$, we see this implies $b_{p^i} = 0$ unless p^i is a power of q. □

Let $R\{\tau\}\hat{}$ be the subring of $R\{\{\tau\}\}$ consisting of (twisted) restricted power series. Then by proposition 4.1, the $\mathbb{F}_q$-linear morphisms between two admissible A-module schemes E_1 and E_2 over $\operatorname{Spf} R$ are given in coordinates by elements in $R\{\tau\}\hat{}$ where τ acts as $\tau(x) = x^q$:

$$\operatorname{Hom}_{\mathbb{F}_q}(E_1, E_2) = R\{\tau\}\hat{}. \tag{4.1}$$

4.1. **Prolongation sequences and jet spaces.** Let X and Y be schemes over $S = \operatorname{Spf} R$. We say a pair (u, δ) is a *prolongation*, and write $Y \overset{(u,\delta)}{\to} X$, if $u : Y \to X$ is a map of schemes over S and $\delta : \mathcal{O}_X \to u_*\mathcal{O}_Y$ is a π-derivation making the following diagram commute:

$$\begin{tikzcd} R \arrow[r] & u_*\mathcal{O}_Y \\ R \arrow[u, "\delta"] \arrow[r] & \mathcal{O}_X \arrow[u, "\delta"'] \end{tikzcd}$$

Following [Bui3], a *prolongation sequence* is a sequence of prolongations

$$\operatorname{Spf} R \xleftarrow{(u,\delta)} T^0 \xleftarrow{(u,\delta)} T^1 \xleftarrow{(u,\delta)} \cdots,$$

where each T^n is a scheme over S. We will often use the notation T^* or $\{T_n\}_{n\geq 0}$. Note that if the T^n are flat over $\operatorname{Spf} R$ then having a π-derivation δ is equivalent to having lifts of Frobenius $\phi : T^{n+1} \to T^n$.

Prolongation sequences form a category $\mathcal{C}_{S^*}$, where a morphism $f : T^* \to U^*$ is a family of morphisms $f^n : T^n \to U^n$ commuting with both the u and δ, in the evident sense. This category has a final object S^* given by $S^n = \operatorname{Spf} R$ for all n, where each u is the identity and each δ is the given π-derivation on R.

For any scheme Y over S, for all $n \geq 0$ we define the n-th jet space $J^n X$ (relative to S) as

$$J^n X(Y) := \operatorname{Hom}_S(W_n^*(Y), X)$$

where $W_n^*(Y)$ is defined as in [Bo2]. We will not define $W_n^*(Y)$ in full generality here. Instead, we will define $\operatorname{Hom}_S(W_n^*(Y), X)$ in the affine case, and that will be sufficient for the purposes of this paper. Write $X = \operatorname{Spf} A$ and $Y = \operatorname{Spf} B$. Then $W_n^*(Y) = \operatorname{Spf} W_n(B)$ and $\operatorname{Hom}_S(W_n^* Y, X)$ is $\operatorname{Hom}_R(A, W_n(B))$, the set of R-algebra homomorphisms $A \to W_n(B)$.

Then $J^*X := \{J^n X\}_{n\geq 0}$ forms a prolongation sequence and is called the *canonical prolongation sequence* [Bui3]. By [Bui3], [Bo2], J^*X satisfies the following universal property—for any $T^* \in \mathcal{C}_{S^*}$ and X a scheme over $S = S^0$, we have Best universal property? Replace S^* with any prolongation sequence, or just remove?

!

$$\operatorname{Hom}(S^0, X) = \operatorname{Hom}_{\mathcal{C}_{S^*}}(S^*, J^*X)$$

Let X be a scheme over $S = \operatorname{Spf} R$. Define X^{ϕ^n} by $X^{\phi^n}(B) := X(B^{\phi^n})$ for any R-algebra B. In other words, X^{ϕ^n} is $X \times_{S,\phi^n} S$, the pull-back of X under the map $\phi^n : S \to S$. Next define

$$\Pi_\phi^n X = X \times_S X^\phi \times_S \cdots \times_S X^{\phi^n}.$$

Then for any R-algebra B we have $X(\Pi_\phi^n B) = X(B) \times_S \cdots \times_S X^{\phi^n}(B)$. Thus the ghost map w in theorem 3.1 defines a map of S-schemes

$$w : J^n X \to \Pi_\phi^n X.$$

Note that w is injective when evaluated on points with coordinates in any flat R-algebra.

The operators F and F_w in (3.3) induce maps ϕ and ϕ_w as follows

$$\begin{tikzcd} J^n X \arrow[r, "w"] \arrow[d, "\phi"'] & \Pi^n_\phi X \arrow[d, "\phi_w"] \\ J^{n-1} X \arrow[r, "w"'] & \Pi^{n-1}_\phi X \end{tikzcd} \tag{4.2}$$

where ϕ_w is the left-shift operator given by

$$\phi_w(w_0, \dots, w_n) = (\phi_S(w_1), \dots, \phi_S(w_n)),$$

and where $\phi_S : X^{\phi^i} \to X^{\phi^{i-1}}$ is the composition given in the following diagram:

$$\begin{tikzcd} X^{\phi^i} \arrow[r, "\sim"] & X^{\phi^{i-1}} \times_{S,\phi} S \arrow[r] \arrow[d] & X^{\phi^{i-1}} \arrow[d] \\ & S \arrow[r, "\phi"'] & S. \end{tikzcd} \tag{4.3}$$

Now let E be an A-module scheme over S with action map $A \overset{\varphi_E}{\to} \mathrm{End}_S(E)$. Then the functor it represents takes values in A-modules, and hence so does the functor $B \mapsto E(W_n(B))$. In this way, for each $n \geq 0$, the S-scheme $J^n E$ comes with an A-module structure. We denote it by $\varphi_{J^n E} : A \to \mathrm{End}_S(J^n E)$. Similarly, φ_E induces an A-linear structure $\varphi_{E^{\phi^n}}$ on each E^{ϕ^n}. In this case, it is easy to describe explicitly. It is the componentwise one:

$$\varphi_{\Pi^n_\phi E}(w_0, \dots, w_n) = (\varphi_E(w_0), \dots, \varphi_{E^{\phi^n}}(w_n)).$$

The ghost map $w : J^n E \to \Pi^n_\phi E$ and the truncation map $u : J^n E \to J^{n-1} E$ homomorphisms of A-module schemes over S. This is because they are given by applying the A-module scheme E to the R-algebra maps $w : W_n(B) \to \Pi^n_\phi B$ and $T : W_n(B) \to W_{n-1}(B)$. On the other hand, the Frobenius map $\phi : J^n E \to J^{n-1} E$ is a homomorphisms of A-module schemes lying over the Frobenius endomorphism ϕ of S. In other words, the induced map $J^n E \to (J^{n-1} E)^\phi$ is a homomorphism of A-module schemes over S.

4.2. **Coordinates on jet spaces.** Given an isomorphism of S-schemes $E \to \hat{\mathbb{G}}_{\mathrm{a}}$, we can identify $(J^n E)(B)$ with $W_n(B)$ and hence, using (3.1), with B^{n+1}. In particular, given a coordinate x on an admissible A-module scheme E, this identification provides a canonical system of coordinates $(x_0, \dots, x_n)$ on $J^n E$. We will use these *Witt coordinates* without further comment. We emphasize once again that there are other canonical systems of coordinates on $J^n E$, for instance the *Buium–Joyal* coordinates denoted $x, x', x'', \dots$. They are related by the formulas of proposition 3.2. Each has their own advantages.

4.3. **Character groups.** Given a prolongation sequence T^* we can define its shift T^{*+n} by $(T^{*+n})^j := T^{n+j}$ for all j [Bui3].

$$\mathrm{Spf}\, R \overset{(u,\delta)}{\leftarrow} T^n \overset{(u,\delta)}{\leftarrow} T^{n+1} \dots$$

We define a *δ-morphism of order n* from X to Y to be a morphism $J^{*+n} X \to J^* Y$ of prolongation sequences. We define a *character of order n*, $\Theta : (E, \varphi_E) \to (\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$ to be a δ-morphism of order n from E to $\hat{\mathbb{G}}_{\mathrm{a}}$ which is also a homomorphism of

A-module objects. By the universal property of jet schemes [Bui3], an order n character is equivalent to a homomorphism $\Theta : J^nE \to \hat{\mathbb{G}}_{\mathrm{a}}$ of A-module schemes over S. We denote the group of characters of order n by $\mathbf{X}_n(E)$. So we have

$$\mathbf{X}_n(E) = \mathrm{Hom}_A(J^nE, \hat{\mathbb{G}}_{\mathrm{a}}),$$

which one could take as an alternative definition. Note that $\mathbf{X}_n(E)$ comes with an R-module structure since $\hat{\mathbb{G}}_{\mathrm{a}}$ is an R-module scheme over S. Also the inverse system $J^{n+1}E \stackrel{u}{\to} J^nE$ defines a directed system

$$\mathbf{X}_n(E) \stackrel{u^*}{\to} \mathbf{X}_{n+1}(E) \stackrel{u^*}{\to} \cdots$$

via pull back. Each morphism u^* is injective because each u has a section (typically not A-linear). We then define $\mathbf{X}_\infty(E)$ to be the direct limit $\lim_n \mathbf{X}_n(E)$.

Similarly, pre-composing with the Frobenius map $\phi : J^{n+1}E \to J^nE$ induces a Frobenius operator $\phi : \mathbf{X}^n(E) \to \mathbf{X}^{n+1}(E)$. However since $\phi : J^{n+1}E \to J^nE$ is not a morphism over Spf R but instead lies over the Frobenius endomorphism ϕ of Spf R, some care is required. Consider the relative Frobenius morphism ϕ_R, defined to be the unique morphism making the following diagram commute:

$$\begin{tikzcd}
J^{n+1}E \arrow[dr, dotted, "\phi_R"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right] & & \\
& J^nE \times_{(\mathrm{Spf}\,R),\phi} \mathrm{Spf}\,R \arrow[r] \arrow[d] & J^nE \arrow[d] \\
& \mathrm{Spf}\,R \arrow[r, "\phi"'] & \mathrm{Spf}\,R
\end{tikzcd}$$

Then ϕ_R is a morphism of A-module formal schemes over Spf R. Now given a δ-character $\Theta : J^nE \to \hat{\mathbb{G}}_{\mathrm{a}}$, define $\phi^*\Theta$ to be the composition

$$J^{n+1}E \xrightarrow{\phi_R} J^nE \times_{(\mathrm{Spf}\,R),\phi} \mathrm{Spf}\,R \xrightarrow{\Theta\times\mathbb{1}} \hat{\mathbb{G}}_{\mathrm{a}} \times_{(\mathrm{Spf}\,R),\phi} \mathrm{Spf}\,R \xrightarrow{\iota} \hat{\mathbb{G}}_{\mathrm{a}} \tag{4.4}$$

where ι is the isomorphism of A-module schemes over S coming from the fact that $\hat{\mathbb{G}}_{\mathrm{a}}$ descends to $\hat{A}$ as an A-module scheme. For any R-algebra B, the induced morphism on B-points is

$$E(W_{n+1}(B)) \xrightarrow{E(F)} E(W_n(B)^\phi) \xrightarrow{\Theta_B^\phi} B^\phi \xrightarrow{b\mapsto b} B.$$

Note that this composition $E(W_{n+1}(B)) \to B$ is indeed a morphism of A-modules because identity map $B^\phi \to B$ is A-linear, which is true because ϕ restricted to $\hat{A}$ is the identity.

Thus we have an additive map $\mathbf{X}_n(E) \to \mathbf{X}_{n+1}(E)$ given by $\Theta \mapsto \phi^*\Theta$. Note that this map is not R-linear. However, the map

$$\phi^* : \mathbf{X}_n(E) \longrightarrow \mathbf{X}_{n+1}(E)^\phi, \quad \Theta \mapsto \phi^*\Theta$$

is R-linear, where $\mathbf{X}_{n+1}(E)^\phi$ denotes the abelian group $\mathbf{X}_{n+1}(E)$ with R-module structure defined by the law $r \cdot \Theta := \phi(r)\Theta$. Taking direct limits in n, we obtain an R-linear map

$$\mathbf{X}_\infty(E) \longrightarrow \mathbf{X}_\infty(E)^\phi, \quad \Theta \mapsto \phi^*\Theta.$$

In this way, $\mathbf{X}_\infty(E)$ is a left module over the twisted polynomial ring $R\{\phi^*\}$ with commutation law $\phi^*r = \phi(r)\phi^*$.

5. A-linearity and Integral Extensions

The purpose of this section is to prove the corollary 5.2 below.

Theorem 5.1. *Let B be a sub-$\mathbb{F}_q$-algebra of A which is a Dedekind domain over which the prime $\mathfrak{p} \subset A$ is unramified. Let (E, φ) be a admissible B-module. Then φ extends to an admissible A-module structure on E in at most one way.*

We make a few remarks. First, this theorem is true without the assumption that $\mathfrak{p}$ is unramified. But because the unramified case is all we need and its proof is much shorter; so we will consider only it. Also note that if E is a Drinfeld module, this theorem follows immediately from basic facts in [D76], section 2. Indeed, $\mathrm{End}_B(E, \varphi)$ is an order in a finite extension of the fraction field of A, which implies that the tangent-space map $\mathrm{End}_B(E, \varphi) \to R$ must be injective; therefore the characteristic map $\theta : A \to R$ can factor through $\mathrm{End}_B(E, \varphi)$ in at most one way. However, we will need to apply the theorem to kernels of the projections $J^1E \to E$, which are admissible A-modules but not Drinfeld modules.

Observe that by transport of structure we have the following:

Corollary 5.2. *Let B and A be as above. Then any B-linear isomorphism between admissible A-modules is in fact A-linear.*

Let us begin by letting $\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}$ denote the formal completion of $\hat{\mathbb{G}}_{\mathrm{a}}$ along the identity section $\mathrm{Spf}\, R \to \hat{\mathbb{G}}_{\mathrm{a}}$. Thus we have $\mathbb{G}_{\mathrm{a}}^{\mathrm{for}} = \mathrm{Spf}\, R[[x]]$, where $R[[x]]$ has the (π, x)-adic topology. We want to extend the A-action on $\hat{\mathbb{G}}_{\mathrm{a}}^{\mathrm{for}}$ to an action of $\hat{A}$:

$$\hat{A} \to \mathrm{End}_{\mathbb{F}_q}(\hat{\mathbb{G}}_{\mathrm{a}}^{\mathrm{for}}/S). \tag{5.1}$$

Recall that $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}})$ agrees with the non-commutative power-series ring $R\{\{\tau\}\}$, with commutation law $\tau b = b^q \tau$ for $b \in R$. Therefore for any $a \in A$, we can write

$$\varphi(a) = \sum_j \alpha_j \tau^j$$

where $\alpha_j \in R$. Each α_j can be thought of as a function of $a \in A$. To construct (5.1) it is enough to prove that these functions are $\mathfrak{p}$-adically continuous, which also implies that such an extension to a continuous $\hat{A}$-action is unique.

Proposition 5.3. *If $a \in \mathfrak{p}^n$, then $\alpha_j \in \mathfrak{p}^{n-j}R$.*

Proof. Clearly, it is true for $n = 0$. Now assume it is true for some given n. Suppose $a \in \mathfrak{p}^{n+1}$ and write $a = \pi b$, where $b \in \mathfrak{p}^n$. Let $\varphi(b) = \sum_j \beta_j \tau^j$ and $\varphi(\pi) = \sum_k \gamma_k \tau^k$. Then we have

$$\sum_j \alpha_j \tau^j = \varphi(a) = \varphi(\pi)\varphi(b) = \sum_k \gamma_k \tau^k \sum_j \beta_j \tau^j = \sum_{k,j} \gamma_k \beta_j^{q^k} \tau^{j+k}$$

and hence $\alpha_j = \sum_{k=0}^{j} \gamma_k \beta_{j-k}^{q^k}$. So to show $\alpha_j \in \mathfrak{p}^{n+1-j}R$, it suffices to show

$$\gamma_k \beta_{j-k}^{q^k} \in \mathfrak{p}^{n+1-j}R, \text{ for } 0 \le k \le j \le n+1.$$

By induction we have $\beta_{j-k} \in \mathfrak{p}^{n-(j-k)}R$ and hence $\gamma_k \beta_{j-k}^{q^k} \in \mathfrak{p}^{(n-(j-k))q^k}R$. Since we have $(n-(j-k))q^k \ge n-j+1$ for $k \ge 1$, we then have $\gamma_k \beta_{j-k}^{q^k} \in \mathfrak{p}^{n-j+1}R$. For $k = 0$, because φ is a strict module structure, we have $\gamma_0 = \pi$ and hence $\gamma_0 \beta_j \in \pi \mathfrak{p}^{n-j}R = \mathfrak{p}^{1+n-j}R$. □

We now consider a local analogue of the setting of theorem 5.1. Let $\hat{B}$ denote a sub-$\mathbb{F}_q$-algebra of $\hat{A}$ which is a complete discrete valuation ring with maximal ideal $\mathfrak{q} = \mathfrak{p} \cap \hat{B}$ and such that the extension $\hat{A}/\hat{B}$ is finite and unramified. (Despite the notation, $\hat{B}$ is not yet the completion of any global object B.)

Theorem 5.4. *Let (E, φ) be a admissible $\hat{B}$-module. Then φ extends to an admissible $\hat{A}$-module structure on E in at most one way.*

Proof. Let φ' be an extension of φ to an admissible $\hat{A}$-module structure. Since $\hat{A}/\hat{B}$ is unramified we can write $\hat{A} = \hat{B}[\zeta]$, where $\zeta \in \hat{A}$ satisfies $\zeta^{\hat{q}-1} = 1$. So to show that φ' is uniquely determined, it is enough to show $\varphi'(\zeta)$ is uniquely determined.

Since φ' extends φ, we know that φ' is a morphism of $\hat{B}$-module schemes. In particular, it is a morphism of $\mathbb{F}_q$-module schemes, and so we have $\varphi'(\zeta) \in R\{\{\tau\}\}$, where $\tau = x^q$. Further, since (E, φ') is admissible, we can write $\varphi'(\zeta) = \zeta + b$, where $b = \sum_i b_i \tau^i \in R\{\{\tau\}\}\tau$ and $b_0 = 0$.

To prove that φ' is uniquely determined, we will show that necessarily $b = 0$. By induction, it is enough to show $b_r = 0$ assuming $b_i = 0$ for $i \leq r-1$. Then we have

$$\zeta + b = \varphi'(\zeta) = \varphi'(\zeta^{\hat{q}}) = \varphi'(\zeta)^{\hat{q}} = (\zeta + b)^{\hat{q}}$$

Now expand $(\zeta + b)^{\hat{q}}$ modulo terms of degree $r+1$ and higher:

$$\begin{aligned}(\zeta + b_r\tau^r)^{\hat{q}} &\equiv \zeta^{\hat{q}} + \sum_{j=0}^{\hat{q}-1} \zeta^{\hat{q}-j-1}(b_r\tau^r)\zeta^j \\ &\equiv \zeta + b_r\Big(\sum_{j=0}^{\hat{q}-1} \zeta^{\hat{q}-j-1}\zeta^{jq^r}\Big)\tau^r\end{aligned}$$

and hence

$$b_r = b_r\zeta^{-1}\sum_{j=0}^{\hat{q}-1} \zeta^{j(q^r-1)}.$$

Now sum the geometric series on the right side. If $\zeta^{q^r-1} = 1$, then it sums to 0, and hence we can conclude $b_r = 0$, as intended. Otherwise, we have

$$b_r = b_r\zeta^{-1}\frac{\zeta^{\hat{q}(q^r-1)} - 1}{\zeta^{q^r} - 1} = b_r\zeta^{-1},$$

which implies $b_r = 0$ in this case as well. $\square$

Proof. (theorem 5.1) It is enough to show that if $\varphi, \varphi' : A \to \mathrm{End}(\mathbb{G}_{\mathrm{a}}/R)$ are two A-module structures that agree when restricted to B, then we have $\varphi = \varphi'$.

Let $\hat{B}$ denote the completion of B with respect to $B \cap \mathfrak{p}$. Then we have the following diagram:

$$\begin{tikzcd} \hat{B} \arrow[r, "\hat{j}"] & \hat{A} \arrow[r, shift left, "\varphi^{\mathrm{for}}"] \arrow[r, shift right, "\varphi'^{\mathrm{for}}"'] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R) \\ B \arrow[u] \arrow[r, "j"] & A \arrow[u] \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi'"'] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}/R). \arrow[u, "i"'] \end{tikzcd}$$

By assumption, we have $\varphi \circ j = \varphi' \circ j$ and hence $\varphi^{\mathrm{for}} \circ \hat{j} = \varphi'^{\mathrm{for}} \circ \hat{j}$. The equality $\varphi^{\mathrm{for}} = \varphi'^{\mathrm{for}}$ then follows from theorem 5.4. Finally since i is injective, we have $\varphi = \varphi'$. $\square$

6. Kernel of $u : J^1E \to E$

Let (E, φ_E) be an admissible A-module scheme over $S = \operatorname{Spf} R$. By equation (4.1), we can write

$$\varphi(t) = \sum a_i \tau^i \tag{6.1}$$

with $a_i \in R$, $a_i \to 0$, and $a_0 = \pi$. Let N^n denote the kernel of the projection $u : J^nE \to E$. Thus we have a short exact sequence of A-module schemes over S:

$$0 \to N^n \to J^nE \xrightarrow{u} E \to 0$$

We will show in this section that, when $q \geq 3$, there is an isomorphism $(N^1, \varphi_{N^1}) \to (\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$ of A-module schemes, where $\hat{\mathbb{G}}_{\mathrm{a}}$ denotes the tautological A-module with the A-action is given by the usual multiplication of scalars: $\varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}(a) = a\tau^0$.

This result has some interest on its own, but our primary interest in it will come in the next section, where we will use it to understand the group $\operatorname{Hom}_A(N^1, \hat{\mathbb{G}}_{\mathrm{a}})$ of A-module homomorphisms from N^1 to $\hat{\mathbb{G}}_{\mathrm{a}}$.

Lemma 6.1. *The R-module map $R \to \operatorname{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}})$ defined by $b \mapsto b\tau^0$ is an isomorphism.*

Proof. Let $\Psi \in \operatorname{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}})$. Write $\Psi = \sum_{i=0}^{\infty} b_i \tau^i$ with $b_i \in R$. Then we will show $b_i = 0$ for $i \geq 1$.

For all $a \in A$, we have

$$\begin{aligned} \Psi \circ \varphi(a) &= \varphi(a) \circ \Psi \\ \sum b_i \tau^i \circ \theta(a)\tau^0 &= \theta(a)\tau^0 \circ \sum b_i \tau^i \\ \sum \theta(a)^{q^i} b_i \tau^i &= \sum \theta(a) b_i \tau^i \end{aligned}$$

So we have $\theta(a)^{q^i} b_i = \theta(a) b_i$ for all i. For $a = \pi$, this means that b_i is $(\pi - \pi^{q^i})$-torsion element of R. But R is π-torsion free and $1 - \pi^{q^i - 1}$ is a unit, for $i \geq 1$. Therefore $b_i = 0$ for $i \geq 1$. $\square$

Lemma 6.2. *If $q \geq 3$, then $q^i - q^{i-j} - j - 1 \geq 0$ for all $j = 1, \ldots, i$.*

Proof. Consider $f(x) = q^i - q^{i-x} - x - 1$, for $1 \leq x \leq i$. Then $f(1) \geq 0$ since $q \geq 3$. Now $f'(x) = q^{i-x} \ln q - 1$. Since $\ln q > 1$ for $q \geq 3$, we have $f'(x) \geq 0$ for all $1 \leq x \leq i$ and hence $f(x) \geq 0$ for all $1 \leq x \leq i$ and we are done. $\square$

Consider the subset $S^\dagger \subset R\{\tau\}^\wedge$ defined by

$$S^\dagger := \Big\{ \sum_{i \geq 0} b_i \tau^i \in R\{\tau\}^\wedge \mid v(b_i) \geq i, \text{ for all } i \text{ and } b_0 \in R^* \Big\}. \tag{6.2}$$

Here, and below, we write $v(b) \geq i$ to mean simply $b \in \mathfrak{p}^i R$.

Proposition 6.3. *$S^\dagger$ is a group under composition.*

Proof. The fact that $S^\dagger$ is a submonoid of $R\{\tau\}^\wedge$ under composition follows immediately from the law $b\tau^i \circ c\tau^j = bc^{q^i}\tau^{i+j}$ and linearity. Indeed if $v(b) \geq i$ and $v(c) \geq j$, then $v(bc^{q^i}) \geq i+j$.

Now let us show that any element $f = \sum b_i\tau^i \in S^\dagger$ has an inverse under composition. Let $g = \sum_{n=0}^{\infty} c_n\tau^n$, where $c_0 = b_0^{-1}$ and we define inductively $c_n = -b_0^{-q^n}(c_0b_n + c_1b_{n-1}^{q} + \cdots + c_{n-1}b_1^{q^{n-1}})$. Then it is easy to check that $g \circ f = \mathbb{1}$. Take $n \geq 1$ and assume $v(c_i) \geq i$ for all $i = 0, \ldots, n-1$. Then it is enough to show $v(c_n) \geq n$. We have $v(c_n) \geq \min\{v(c_ib_{n-i}^{q^i}) \,|\, i = 0, \ldots, n-1\}$. Now

$$\begin{aligned} v(c_ib_{n-i}^{q^i}) &= v(c_i) + q^i v(b_{n-i}) \\ &= i + q^i(n-i) \\ &\geq i + (n-i) = n. \end{aligned}$$

Therefore the left inverse g of f lies in $S^\dagger$.

Now consider $g' = \sum_{n=0}^{\infty} d_n\tau^n \in R\{\{\tau\}\}$, where $d_0 = b_0^{-1}$ and we inductively define $d_n = -b_0^{-1}(b_1d_{n-1}^{q^1} + b_2d_{n-2}^{q^2} + \cdots + b_nd_0^{q^n})$. Then as above, one can easily check that $f \circ g' = \mathbb{1}$ and hence it is a right inverse of f in $R\{\{\tau\}\}$. But using the associativity property of $R\{\{\tau\}\}$ we get $g' = (g \circ f) \circ g' = g \circ (f \circ g') = g$ and hence g is both a left and right inverse of f in $S^\dagger$. $\square$

Theorem 6.4. *Suppose $q \geq 3$ and $v(a_i) \geq q^i - 1$, for all $i \geq 1$. Then there exists a unique A-linear homomorphism $f : E \to \hat{\mathbb{G}}_{\mathrm{a}}$, written $f = \sum_{i=0}^{\infty} b_i\tau^i$ in coordinates, such that $v(b_i) \geq i$ and $b_0 = 1$. Moreover, f is an isomorphism of A-module schemes over S.*

Proof. Consider $B := \mathbb{F}_q[t] \subseteq A$. Then A is unramified over B at $\mathfrak{p}$. So by corollary 5.2, it is sufficient to construct a B-linear isomorphism $f : E \to \hat{\mathbb{G}}_{\mathrm{a}}$. In other words, without loss of generality, we may assume $A = \mathbb{F}_q[t]$.

Define $f = \sum_{i=0}^{\infty} b_i\tau^i$, $b_i \in R$, where $b_0 = 1$ and inductively

$$b_i = \pi^{-1}(1 - \pi^{q^i-1})^{-1}\sum_{j=1}^{i} b_{i-j}a_j^{q^{i-j}}. \tag{6.3}$$

Then it is easy to see that the map f satisfies $\varphi(t) \circ f = f \circ \varphi(t)$, which implies $\varphi(b) \circ f = f \circ \varphi(b)$ for all $b \in B$. It is also the unique A-linear map $E \to \hat{\mathbb{G}}_{\mathrm{a}}$ with constant term 1.

It remains to show $v(b_i) \geq i$. For $i = 0$, it is clear. For $i \geq 1$, we may assume by induction that $v(b_j) \geq j$ for all $j = 1, \ldots, i-1$. By (6.3), we have $v(b_i) \geq \min\{v(b_{i-j}a_j^{q^{i-j}}) - 1 \,|\, j = 1, \ldots, i\}$. Now

$$\begin{aligned} v(b_{i-j}a_j^{q^{i-j}}) - 1 &\geq v(b_{i-j}) + v(a_j^{q^{i-j}}) - 1 \\ &\geq i - j + q^{i-j}(q^j - 1) - 1 \\ &= i - j + q^i - q^{i-j} - 1 \\ &\geq i, \text{ by lemma 6.2.} \end{aligned}$$

Therefore we have $v(b_i) \geq i$.

In particular f is a restricted power series and hence defines a map between $\hat{\mathfrak{p}}$-formal schemes $f : E \to \hat{\mathbb{G}}_{\mathrm{a}}$ which is A-linear. By proposition 6.3, there exists a linear map $g : \hat{\mathbb{G}}_{\mathrm{a}} \to E$ such that $f \circ g = g \circ f = \mathbb{1}$. Then g is also A-linear for formal reasons. Indeed, for any $a \in A$, we have $f(g(\varphi(a)x)) = \varphi(a)x = f(\varphi(a)g(x))$. Since f is injective, we must have $g(\varphi(a)x) = \varphi(a)g(x)$ which shows the A-linearity g and we are done. □

Corollary 6.5. *The derivative map* $\mathrm{Hom}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) \to R$*, given in coordinates by* $\sum_i b_i\tau^i \mapsto b_0$*, is injective. In particular, if* R *is a discrete valuation ring, then* $\mathrm{Hom}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ *is free of rank* 0 *or* 1*.*

Proof. If $f \in \mathrm{Hom}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and $f = \sum_{i=0}^{\infty} b_i\tau^i$, it is sufficient to show that for all $i \geq 1$, the elements b_i are uniquely determined by b_0. But a morphism $f : E \to \hat{\mathbb{G}}_{\mathrm{a}}$ satisfies $\varphi(t) \circ f = f \circ \varphi(t)$ if and only if equation (6.3) is satisfied. In particular, any such morphism is determined by the value of b_0. □

7. Characters of N^n—upper bounds

We continue to let E denote the admissible A-module scheme over S of (6.1).

Lemma 7.1. *For all* $n \geq 0$*,* $\phi^n(x) = \pi^n x^{(n)} + O(n-1)$*, where* $O(n-1)$ *are elements of order less than equal to* $n-1$*.*

Proof. For $n = 0$, it is clear. For $n \geq 1$, we have by induction

$$\begin{aligned}\phi^n(x) &= \phi(\pi^{n-1}x^{(n-1)} + O(n-2)) \\ &= \pi^{n-1}\phi(x^{(n-1)}) + O(n-1) \\ &= \pi^{n-1}\pi\delta(x^{(n)}) + (x^{(n-1)})^{\hat{q}} + O(n-1) \\ &= \pi^n x^{(n)} + O(n-1).\end{aligned}$$

□

Theorem 7.2. *Assume* $q \geq 3$*. For any* $n \geq 0$*, let* H^n *denote the kernel of the projection* $u : J^{n+1}E \to J^nE$*. Then there is a unique* A*-linear isomorphism* $\vartheta_n : H^n \to \hat{\mathbb{G}}_{\mathrm{a}}$ *of the form* $\vartheta_n(x) = x + b_1x^q + b_2x^{q^2} + \cdots$ *in coordinates such that* $v(b_i) \geq i$ *for all* $i \geq 1$*.*

Proof. First observe that we have

$$\begin{aligned}\varphi_E(t)\phi^n(x) &= \phi^n(\varphi_E(t)) \\ &= \phi^n(\pi)\phi^n(x) + \phi^n(a_1)\phi^n(x)^q + \cdots + \phi^n(a_r)\phi^n(x)^{q^r}.\end{aligned}$$

Second, the subscheme H^n is defined by setting the $x, x', \ldots, x^{(n-1)}$ coordinates to 0. Combining these two observations and lemma 7.1, we obtain

$$\pi^n\varphi_E(t)x^{(n)} = \pi\pi^n x^{(n)} + \phi^n(a_1)(\pi^n x^{(n)})^q + \cdots + \phi^n(a_r)(\pi^n x^{(n)})^{q^r}$$

and hence

$$\varphi_E(t)x^{(n)} = \pi x^{(n)} + \phi^n(a_1)\pi^{n(q-1)}(x^{(n)})^q + \cdots + \phi^n(a_r)\pi^{n(q^r-1)}(x^{(n)})^{q^r}.$$

But then by theorem 6.4, there is a unique isomorphism $(H^n, \varphi_{H^n}) \to (\hat{\mathbb{G}}_{\mathrm{a}}, \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}})$ of the kind desired. □

Using ϑ_n, we can identify the short exact sequence

$$0 \to H^n \to N^n \to N^{n-1} \to 0$$

with a short exact sequence

$$0 \to \hat{\mathbb{G}}_{\mathrm{a}} \to N^n \to N^{n-1} \to 0. \tag{7.1}$$

Now consider the corresponding long exact sequence

$$0 \to \mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}}) \to \cdots .$$

By lemma 6.1, $\mathrm{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}})$ is canonically a sub-R-module of R. Therefore we have a filtration of R-modules

$$\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \supseteq \mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \supseteq \cdots \supseteq \mathrm{Hom}_A(N^0, \hat{\mathbb{G}}_{\mathrm{a}}) = 0,$$

and each associated graded module is canonically a submodule of R.

In particular, we have the following:

Proposition 7.3. *If R is a discrete valuation ring, then* $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ *is a free R-module of rank at most n.*

8. The Lateral Frobenius and characters of N^n

Now we will construct a family of important operators which we call the *lateral Frobenius* operators. That is, for all n, we will construct maps $\mathfrak{f} : N^{n+1} \to N^n$ which are lifts of Frobenius relative to the projections $u : N^{n+1} \to N^n$ and hence make the system $\{N^n\}_{n=0}^{\infty}$ into a prolongation sequence. Do note that *a priori* the A-modules N^n do not form a prolongation sequence to start with.

Let N^{∞} denote the inverse limit the projection maps $u : N^{n+1} \to N^n$. Then the maps $\mathfrak{f}$ induce a lift of Frobenius on N^{∞}. Similarly on $J^{\infty}E = \lim_n J^nE$, the maps ϕ induce a lift of Frobenius. Now for all $n \geq 1$, the inclusion $N^n \hookrightarrow J^nE$ is a closed immersion and hence induces a closed immersion of schemes $N^{\infty} \hookrightarrow J^{\infty}E$. But $\mathfrak{f}$ is not obtained by restricting ϕ to N^{∞}. In fact, ϕ does not even preserve N^{∞}. So $\mathfrak{f}$ is an interesting operator which is distinct from ϕ, although it does satisfy a certain relation with ϕ which we will explain below.

Here we would also like to remark that the lateral Frobenius can also be constructed in the mixed-characteristic setting of p-jet spaces of arbitrary schemes. But it is much more involved than for Drinfeld modules, and the authors will present that theory in a subsequent note.

Let $F : W_n \to W_{n-1}$ and $V : W_{n-1} \to W_n$ denote the Frobenius and Verschiebung maps of 3.3. Let us arrange them in the following diagram, although it does not commute.

$$\begin{tikzcd}
W_n \arrow[r, "V"] \arrow[d, "F"'] & W_{n+1} \arrow[d, "F"] \\
W_{n-1} \arrow[r, "V"] & W_n \arrow[d, "F"] \\
 & W_{n-1}
\end{tikzcd}$$

Rather the following is true

$$FFV = FVF. \tag{8.1}$$

Indeed, the operator FV is multiplication by $\pi = \theta(t)$, and F is a morphism of A-algebras.

We can re-express this in terms of jet spaces using the natural identifications $J^nE \simeq W_n$ and $N^n \simeq W_{n-1}$. For jet spaces, let us switch to the notation $i := V$, $\phi := F$ for the right column, and $\mathfrak{f} := F$ for the left column. Then the diagram above becomes the following:

$$\begin{tikzcd}
N^{n+1} \arrow[r, "i"] \arrow[d, "\mathfrak{f}"'] & J^{n+1}E \arrow[d, "\phi"] \\
N^n \arrow[r, "i"] & J^nE \arrow[d, "\phi"] \\
 & J^{n-1}E
\end{tikzcd}$$

Note again that it is not commutative. However rewriting (8.1) in the above notation, we do have

$$\phi^{\circ 2} \circ i = \phi \circ i \circ \mathfrak{f}. \tag{8.2}$$

We emphasize that when we use the notation N^n, the A-module structure will always be understood to be the one that makes i an A-linear morphism. It should not be confused with the A-module structure coming by transport of structure from the isomorphism $N^n \simeq W_{n-1} = J^{n-1}E$ of group schemes.

We also emphasize that while i is a morphism of S-schemes, the vertical arrows ϕ and $\mathfrak{f}$ in the diagram above lie over the Frobenius endomorphism ϕ of S, rather than the identity morphism.

Lemma 8.1. *For any torsion-free R-algebra B, the map $FV : W_n(B) \to W_n(B)$ is injective.*

Proof. Since B is torsion free, the ghost map $W_n(B) \to B \times \cdots \times B$ is injective, and hence $W_n(B)$ is torsion free. The result then follows because FV is multiplication by π. □

Proposition 8.2. *The morphism $\mathfrak{f} : N^n \to N^{n-1}$ is A-linear.*

Proof. Since both ϕ and i are A-linear morphisms, so are ϕi and $\phi^2 i$. Therefore for all $a \in A$, we have

$$\phi i(\mathfrak{f}(ax)) = \phi^2 i(ax) = a\phi^2 i(x) = a\phi i(\mathfrak{f}(x)) = \phi i(a\mathfrak{f}(x))$$

Thus the two morphisms $N^{n+1} \to N^n$, given by $x \mapsto a\mathfrak{f}(x)$ and by $x \mapsto \mathfrak{f}(ax)$, become equal after application of ϕi. We can interpret the morphisms as two elements of $N^n(B)$, where B is the algebra representing the functor N^{n+1}, which become equal after applying ϕi. But since B is torsion free, lemma 8.1 implies these two elements must be equal. □

For $0 \leq i \leq k-1$, let us abusively write $\mathfrak{f}^{\circ i}$ for the following composition

$$\mathfrak{f}^{\circ i} : N^n \overset{\overbrace{\mathfrak{f} \circ \cdots \circ \mathfrak{f}}^{i\text{-times}}}{\to} N^{n-i} \overset{u}{\to} N^{n-k}.$$

Then for all $1 \leq i \leq n$, we define $\Psi_i \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ as

$$\Psi_i = \vartheta_1 \circ \mathfrak{f}^{\circ i-1} \tag{8.3}$$

where ϑ_1 is as in theorem 7.2. Clearly, the maps Ψ_i are A-linear since each one of the maps above is. Finally, given a character $\Psi \in \mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}})$, we will write $\mathfrak{f}^*\Psi = \Psi \circ \mathfrak{f}$.

The points of $J^n E$ contained in N^n are those with Witt coordinates of the form $(0, x_1, x_2, \ldots, x_n)$. We will use the abbreviated coordinates $(x_1, \ldots, x_n)$ on N^n instead.

Lemma 8.3. *For all $i = 1, \ldots, n$, we have*

$$\Psi_i(x_1, \ldots, x_n) \equiv x_1^{\hat{q}^{i-1}} \bmod \pi.$$

Proof. Since $\mathfrak{f}$ is identified with the Frobenius map $F : W_n \to W_{n-1}$, it reduces modulo π to the $\hat{q}$-th power of the projection map. Therefore, we have

$$\Psi_i(x_1, \ldots, x_n) = \vartheta_1 \circ \mathfrak{f}^{\circ(i-1)}(x_1, \ldots, x_n) \equiv \vartheta_1(x_1^{\hat{q}^{i-1}}) \bmod \pi,$$

and hence is equivalent to $x_1^{\hat{q}^{i-1}}$ modulo π, by the defining property of ϑ_1 in theorem 7.2. □

Proposition 8.4. *If R is a discrete valuation ring, then the elements $\Psi_1, \ldots, \Psi_n$ form an R-basis for $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$, if $q \geq 3$.*

Proof. By proposition 7.3, the R-module $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ is free of rank at most n. So to show the elements $\Psi_1, \ldots, \Psi_n$ form a basis, it is enough by Nakayama's lemma to show they are linearly independent modulo π. But by lemma 8.3, we have $\Psi_i \equiv x_1^{\hat{q}^{i-1}} \bmod \pi$, and so the Ψ_i map to linearly independent elements of $R/\pi R \otimes_R \mathcal{O}(N^n)$. Thus they are linearly independent in $R/\pi R \otimes_R \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$.
! improve this? □

Remark. Another interpretation of the main results of this section is as follows. First, there is a canonical isomorphism $N^n \to J^{n-1}(\hat{\mathbb{G}}_{\mathrm{a}})$ of A-module schemes. Second, any A-module homomorphism $J^{n-1}(\hat{\mathbb{G}}_{\mathrm{a}}) \to \hat{\mathbb{G}}_{\mathrm{a}}$ factors uniquely through the ghost map $J^{n-1}(\hat{\mathbb{G}}_{\mathrm{a}}) \to \hat{\mathbb{G}}_{\mathrm{a}}^n$. It then follows that the character group $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ is canonically identified with R^n.

9. $\mathbf{X}_\infty(E)$

We now assume further that R is a discrete valuation ring and E is a Drinfeld module over $\mathrm{Spf}\, R$. Let r denote the rank of E. We continue to write $\varphi_E(t) = a_0\tau^0 + a_1\tau^1 + \cdots + a_r\tau^r$, where $a_0 = \pi$, $a_i \in R$ for all i, and $a_r \in R^*$.

Given such a Drinfeld module, one of the important δ-arithmetic objects that one can attach to it is the group of all δ-characters of E to $\hat{\mathbb{G}}_{\mathrm{a}}$, denoted $\mathbf{X}_\infty(E)$. In the case of elliptic curves, Buium has shown that this group contains important arithmetic data as analogues of Manin maps in differential algebra and has found diophantine applications on Heegner points on modular curves [BP2].

In this section and the next, we will determine the structure of $\mathbf{X}_\infty(E)$. In the case of elliptic curves, it falls in two distinct cases as to when the elliptic curve admits a canonical lift and when not. A similar story happens in our case when E is a Drinfeld module of rank 2, which one might consider the closest analogue of an elliptic curve. However, when the rank exceeds 2, the behavior of $\mathbf{X}_\infty(E)$ offers much more interesting cases which leads us to introduce the concept of the *splitting order* m of a Drinfeld module E. This natural number is always less than or equal to the rank of E and when the rank equals 2, the notion coincides with the canonical lift property of Drinfeld modules.

We would like to point out here that our structure result for $\mathbf{X}_\infty(E)$ is is an integral version of that of [Bui2]. Buium shows that $\mathbf{X}_\infty(E) \otimes_R K$ is generated by a single element as a $K\{\phi^*\}$-module where $K = R[\frac{1}{p}]$. But here we show that the module $\mathbf{X}_\infty(E)$ itself is generated by a single element as a $R\{\phi^*\}$-module. Although our result is for Drinfeld modules over function rings in positive characteristic, our methods work in the elliptic curves over p-adic rings setting and hence this stronger result can be achieved in that case too.

The following theorem should be viewed as an analogue of the fact that an elliptic curve has no non-zero homomorphism of $\mathbb{Z}$-module schemes to $\mathbb{G}_{\mathrm{a}}$. In our case, we show that no Drinfeld module admits a non-zero homomorphism of A-module schemes to $\hat{\mathbb{G}}_{\mathrm{a}}$.

Theorem 9.1. *We have* $\mathbf{X}_0(E) = \{0\}$.

Proof. Any character $f = \sum_{i\geq 0} b_i\tau^i \in \mathbf{X}_0(E)$ satisfies the following chain of equalities:

$$\begin{aligned} \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}(t)\circ f &= f\circ\varphi_E(t) \\ \theta(t)\tau^0\circ\sum_{i\geq 0} b_i\tau^i &= \sum_{i\geq 0} b_i\tau^i\circ\sum_j a_j\tau^j \\ \sum_{i\geq 0}\theta(t)b_i\tau^i &= \sum_{i\geq 0}\Big(\sum_{j=0}^{r} b_{i-j}a_j^{q^{i-j}}\Big)\tau^i \end{aligned}$$

Comparing the coefficients of τ^i for $i > r$, we have

$$b_i(1-\theta(t)^{q^i-1})\theta(t) = a_r^{q^{i-r}}b_{i-r} + a_{r-1}^{q^{i-r+1}}b_{i-r+1} + \cdots + a_1^{q^{i-1}}b_{i-1} \tag{9.1}$$

Suppose f is nonzero. There there exists an N such that $b_{N-r} \neq 0$ and $v(b_{N-r}) < v(b_i)$ for all $i \geq N-r+1$. Then the valuation of the right-hand side of equation (9.1) for $i = N$ becomes $v(a_r^{q^{i-r}}b_{N-r}) = v(b_{N-r})$, since $v(a_r) = 0$. But then taking the valuation of both sides of (9.1), we have

$$v(b_N) = v(b_{N-r}) - 1 < v(b_{N-r})$$

and $N \geq N-r+1$, which is a contradiction to the hypothesis above. Therefore f must be 0. □

As a consequence the short exact sequence of A-module schemes over S

$$0 \to N^n \xrightarrow{i} J^nE \to E \to 0, \tag{9.2}$$

induces an exact sequence

$$0 \to \mathbf{X}_n(E) \xrightarrow{i^*} \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \xrightarrow{\partial} \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}), \tag{9.3}$$

which we will use repeatedly.

The following result is the analogue of Buium's in the mixed-characteristic setting.

Theorem 9.2. *Let (E, φ_E) be a Drinfeld module of rank r.*

(1) *$\mathbf{X}_r(E)$ is nonzero.*
(2) *We have*

$$\mathbf{X}_1(E) \simeq \begin{cases} R, & \text{if } E \text{ has a lift of Frobenius,} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Proof. (1): Consider the exact sequence (9.3). By proposition 8.4, the R-module $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ is free of rank n. But also $\mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ is free of rank $r-1$, by [Ge1]. Therefore when $n = r$, the kernel $\mathbf{X}_n(E)$ is nonzero.

(2) Now consider $\mathbf{X}_1(E)$. It is contained in $\mathrm{Hom}_A(N^1, \hat{\mathbb{G}}_{\mathrm{a}})$, which is free of rank 1, and the quotient is contained in $\mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$, which is torsion free. Therefore $\mathbf{X}_1(E)$ is either $\{0\}$ or all of $\mathrm{Hom}_A(N^1, \hat{\mathbb{G}}_{\mathrm{a}}) \simeq R$.

Let $\mathbb{1}$ denote the identity map in $\mathrm{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}})$. Then its image $\partial(\mathbb{1})$ in $\mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ is the class of the extension (9.2). Therefore we have the equivalences $\mathbf{X}_1(E) \simeq R \iff i^*$ is an isomorphism $\iff \partial(\mathbb{1}) = 0 \iff$ (9.2) is split $\iff E$ has a lift of Frobenius. □

Define the *splitting order* of the Drinfeld module E to be the integer m such that $\mathbf{X}_m(E) \neq \{0\}$ and $\mathbf{X}_{m-1}(E) = \{0\}$. We also say that E *splits at order* m. By the theorems above, we have $1 \leq m \leq r$ and additionally $m = 1$ if and only if E is a canonical lift.

9.1. **Splitting of $J^n(E)$.** The exact sequence (9.3) is split by the Teichmüller section $v : E \to J^n E$, as defined in section 3. We emphasize that v is only a morphism of $\mathbb{F}_q$-modules schemes and is not a morphism of A-module schemes. Nevertheless, it induces an isomorphism

$$J^n(E) \xrightarrow{\sim} E \times N^n$$

of $\mathbb{F}_q$-module schemes. Therefore for any character $\Theta \in \mathbf{X}_n(E)$, we can write $\Theta = g \oplus \Psi$ or

$$\Theta(x_0, \ldots, x_n) = g(x_0) + \Psi(x_1, \ldots, x_n), \tag{9.4}$$

where $\Psi = i^*\Theta \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ and $g = v^*\Theta$. Note that because v is only $\mathbb{F}_q$-linear, g is also only $\mathbb{F}_q$-linear. But it can still be expressed an additive restricted power series.

Lemma 9.3. *For any R-algebra B, consider the exact sequence for all $n \geq 1$*

$$0 \to W_{n-1}(B) \xrightarrow{V} W_n(B) \xrightarrow{T^n} B \to 0$$

Then there exists a map $g : B \to W_n(B)$ *such that*

$$\begin{tikzcd} W_n(B) \arrow[r, "T^n"] \arrow[d, "FV-VF"'] & B \arrow[dl, "g"] \\ W_n(B) & \end{tikzcd}$$

commutes. It is of the form $g(x) = (\pi x, c_1 x^{\hat{q}}, c_2 x^{\hat{q}^2}, \dots)$, *for some elements* $c_j \in R$.

Proof. For any $y \in W_{n-1}(B)$, we have

$$(FV - VF)(Vy) = FVVy - VFVy = \pi Vy - V(\pi y) = 0.$$

So such a function g exists.

To conclude that $g(x)$ is of the given form, we use a homogeneity argument. Let $(z_0, z_1, \dots)$ denote the ghost components of $(x_0, x_1, \dots)$. If interpret each x_j as an indeterminate of degree $\hat{q}^j$, then each z_j is a homogenous polynomial in the $x_0, \dots, x_j$ of degree $\hat{q}^j$ and with coefficients in A: $z_1 = x_0^{\hat{q}} + \pi x_1$, and so on. Solving for x_j in terms of $z_0, \dots, z_j$, we see that x_j is a homogenous polynomial in the $z_0, \dots, z_j$ with coefficients in $A[1/\pi]$.

Now let $(y_0, y_1, \dots)$ denote $(FV - VF)(x_0, x_1, \dots)$, where $y_j \in R[x_0, \dots, x_j]$. Then the ghost components of $(y_0, y_1, \dots)$ are $(\pi z_0, 0, 0, \dots) = (\pi x_0, 0, 0, \dots)$. It follows that $y_0 = \pi x_0$. Further, by the above, y_j is an element of $R[x_0, \dots, x_j]$ but also a homogeneous polynomial in πx_0 of degree $\hat{q}^j$ and with coefficients in $A[1/\pi]$. Therefore it is of the form $c_j x_0^{\hat{q}^j}$ for some $c_j \in R$. $\square$

Proposition 9.4. *Let* Θ *be a character in* $\mathbf{X}_n(E)$.

(1) *We have*

$$i^* \phi^* \Theta = \mathfrak{f}^*(i^* \Theta) + \gamma \Psi_1,$$

where $\gamma = \pi g'(0)$ *and where* $g'(x_0)$ *denotes the usual derivative of the polynomial* $g(x_0) \in R[x_0]$ *of equation (*9.4*).*

(2) *For* $n \geq 1$, *we have*

$$i^*(\phi^{\circ n})^* \Theta = (\mathfrak{f}^{n-1})^* i^* \phi^* \Theta.$$

Proof. (1): By lemma 9.3, we have

$$(\phi \circ i - i \circ \mathfrak{f})(x_1, \dots, x_{n+1}) = (\pi x_1, c_1 x_1^{\hat{q}}, c_2 x_1^{\hat{q}^2}, \dots),$$

where $c_j \in R$. Therefore we have

$$\begin{aligned} ((i^*\phi^* - \mathfrak{f}^* i^*)\Theta)(x_1, \dots, x_{n+1}) &= \Theta(\pi x_1, c_1 x_1^{\hat{q}}, \dots) \\ &= g(\pi x_1) + \Psi(c_1 x_1^{\hat{q}}, \dots), \end{aligned} \tag{9.5}$$

where g and Ψ are as in equation (9.4). In particular, the character $(i^*\phi^* - \mathfrak{f}^* i^*)\Theta$ depends only on x_1. Therefore it is of the form $\gamma \Psi_1$, for some $\gamma \in R$. Further since by theorem 7.2 we have $\Psi_1'(0) = 1$, the coefficient γ is simply the linear coefficient of $(i^*\phi^* - \mathfrak{f}^* i^*)\Theta$, which by (9.5) is $\pi g'(0)$.

(2): This is another way of expressing $\phi^{\circ n} \circ i = \phi \circ i \circ \mathfrak{f}^{\circ(n-1)}$, which follows from (8.2) by induction. $\square$

9.2. **Frobenius and the filtration by order.** We would like to fix a notational convention here. Let $u : J^nE \to J^{n'}E$ denote the canonical projection map for any $n' < n$, given in Witt coordinates by $u(x_0, \dots, x_n) = (x_0, \dots, x_{n'})$.

Consider the following morphism of exact sequences of A-modules

$$\begin{tikzcd} 0 \arrow[r] & N^n \arrow[r, "i"] \arrow[d, two heads, "u"'] & J^nE \arrow[r, "u"] \arrow[d, two heads, "u"'] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N^{n-1} \arrow[r, "i"] & J^{n-1}E \arrow[r, "u"] & E \arrow[r] & 0 \end{tikzcd}$$

Since $\mathbf{X}_0(E) = \{0\}$ by theorem 9.1, applying $\mathrm{Hom}_A(-, \hat{\mathbb{G}}_{\mathrm{a}})$ to the above, we obtain the following morphism of exact sequences of R-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{X}_n(E) \arrow[r, "i^*"] & \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\partial"] & \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) \\ 0 \arrow[r] & \mathbf{X}_{n-1}(E) \arrow[r, "i^*"] \arrow[u, hook, "u^*"] & \mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\partial"] \arrow[u, hook, "u^*"] & \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[u, equal] \end{tikzcd}$$

Proposition 9.5. *For any $n \geq 0$, the diagram*

$$\begin{tikzcd} \mathbf{X}_n(E)/\mathbf{X}_{n-1}(E) \arrow[r, hook, "\phi^*"] \arrow[d, hook, "i^*"] & \mathbf{X}_{n+1}(E)/\mathbf{X}_n(E) \arrow[d, hook, "i^*"] \\ \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})/\mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\mathfrak{f}^*", "\sim"'] & \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_{\mathrm{a}})/\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \end{tikzcd}$$

is commutative. The morphisms i^* and ϕ^* are injective, and $\mathfrak{f}^*$ is bijective.

In fact, we will show in corollary 10.9 that all the morphisms in the diagram of proposition 9.5 are isomorphisms.

Proof. For $n \geq 1$, commutativity of the diagram follows from proposition 9.4; for $n = 0$, it follows from theorem 9.1.

The maps i^* are injective because the projections $J^nE \to J^{n-1}E$ and $N^n \to N^{n-1}$ have the same kernel, and $\mathfrak{f}^*$ is an isomorphism by proposition 8.4. It follows that ϕ^* is an injection. $\square$

9.3. **The character Θ_m.** Recall the exact sequence (9.3)

$$0 \to \mathbf{X}_n(E) \xrightarrow{i^*} \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \xrightarrow{\partial} \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$$

Let m denote the splitting order of E. Then for all $n < m$, the map

$$\partial : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$$

is injective since $\mathbf{X}_n(E) = \{0\}$. But at $n = m$, we have $\mathbf{X}_m(E) \neq \{0\}$, and so there is a nonzero character $\Psi \in \mathrm{Hom}_A(N^m, \hat{\mathbb{G}}_{\mathrm{a}})$ in the kernel of ∂. Write $\Psi = \tilde{\lambda}_0\Psi_m - \tilde{\lambda}_1\Psi_{m-1} - \cdots - \tilde{\lambda}_{m-1}\Psi_1$, where $\tilde{\lambda}_i \in R$ for all $i = 0, \dots, m-1$. Then we necessarily have $\lambda_0 \neq 0$ since $\mathbf{X}_{m-1} = \{0\}$. Therefore we have

$$\partial\Psi_m = \lambda_1\partial\Psi_{m-1} + \cdots + \lambda_{m-1}\partial\Psi_1 \in \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})_K \tag{9.6}$$

where $\lambda_{m-i} = \tilde{\lambda}_{m-i}/\tilde{\lambda}_0$ for all $i = 1, \dots, m-1$. This implies that the character

$$\Psi_m - \lambda_1\Psi_{m-1} - \cdots - \lambda_{m-1}\Psi_1$$

is in $\ker(\partial)$ and hence by the main exact sequence (9.3), there exists a unique $\Theta_m \in \mathbf{X}_m(E)_K$ such that

$$i^*\Theta_m = \Psi_m - \lambda_1\Psi_{m-1} - \cdots - \lambda_{m-1}\Psi_1 \tag{9.7}$$

It then follows immediately that Θ_m is a K-linear basis for $\mathbf{X}_m(E)_K$, say by propositions 8.4 and 9.5. (We will show in corollary 10.9 that Θ_m actually lies in the group $\mathbf{X}_m(E)$ of integral characters, and is in fact an integral basis for it.)

Proposition 9.6. *Let m denote the splitting order of E. Then for any $j \geq 0$, the character $i^*(\phi^*)^j\Theta_m$ agrees with Ψ_{m+j} modulo rational characters of lower order, and the elements $\Theta_m, \phi^*\Theta_m, \cdots, \phi^{n-m^*}\Theta_m$ are a basis of the K-vector space $\mathbf{X}_n(E)_K$.*

Proof. By 9.5, each element $\phi^{i^*}\Theta_m$ lies in $\mathbf{X}_{m+i}(E)$ but not in $\mathbf{X}_{m+i-1}(E)$. Therefore such elements are linearly independent. At the same time, by the diagram above, each $\mathbf{X}_{m+i}(E)/\mathbf{X}_{m+i-1}(E)$ has rank at most 1. Thus the rank of $\mathbf{X}_n(E)$ is at most $n - m + 1$, and so the elements in question form a spanning set. □

Do note that this result will be improved to an integral version in theorem 10.10.

10. Ext Groups and de Rham cohomology

We will prove theorem 1.1 in this section. We continue with the notation from the previous section. In particular, R is a discrete valuation ring.

We will briefly describe our strategy in the next few lines. Recall from (9.7) the equality

$$i^*\Theta_m = \Psi_m - \lambda_1\Psi_{m-1} - \cdots - \lambda_{m-1}\Psi_1$$

where $\lambda_j \in K$. *A priori*, the elements λ_j need not belong to R, but we prove in theorem 10.8 that they actually do. This will imply that $i^*\Theta_m$ lies in $\mathrm{Hom}_A(N^m, \hat{\mathbb{G}}_{\mathrm{a}})$ and $\ker(\partial)$, and hence by the exact sequence (9.3), we have $\Theta_m \in \mathbf{X}_m(E)$—that is, the character Θ_m is integral. From there, it is an easy consequence that $\mathbf{X}_n(E)$ is generated by $\Theta_m, \ldots, \Theta_m^{\phi^{n-m}}$ as an R-module.

So the key result to prove is theorem 10.8. But it will require some preparation before we can present the proof. For all $n \geq 1$, we will define maps from $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ to $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ which is also interpreted as the de Rham cohomology from associated to the Drinfeld module E. These maps are obtained by push-outs of J^nE by $\Psi \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$. To give an idea, do note that, for every $n \geq 1$, there are canonical elements $E^*_\Psi \in \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$ group where the E^*_Ψ is a push-out of J^nE by Ψ as follows

$$\begin{tikzcd}
0 \arrow[r] & N^n \arrow[r, "i"] \arrow[d, "\Psi"] & J^nE \arrow[r, "\pi"] \arrow[d, "g_\Psi"] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r] & E^*_\Psi \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

as $E^*_\Psi \in \mathrm{Ext}_A(E, \hat{\mathbb{G}}_{\mathrm{a}})$. It leads to a very interesting theory of δ-modular forms over the moduli space of Drinfeld modules and will be studied in a subsequent paper. And similar to previous cases, the main principles carry over to the case of elliptic curves or abelian schemes as well.

Now we introduce the theory of extensions of A-module group schemes. Given an extension $\eta_C \in \mathrm{Ext}(G,T)$ and $f : T \to T'$ where G, T and T' are A-modules and f is an A-linear map we have the following diagram of the push-forward extension f_*C.

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T' \arrow[r] & f_*C \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

The class of f_*C is obtained as follows—the class of η_C is represented by a linear (not necessarily A-linear) function $\eta_C : G \to T$. Then η_{f_*C} is represented by the class $\eta_{f_*C} = [f \circ \eta_C] \in \mathrm{Ext}(E, T')$. In terms of the action of $t \in A$, $\varphi_C(t)$ is given by $\begin{pmatrix} \varphi_G(t) & 0 \\ \eta_C & \varphi_T(t) \end{pmatrix}$ where $\eta_C : G \to T$. Then $\varphi_{f_*C}(t)$ is given by

$$\begin{pmatrix} \varphi_G(t) & 0 \\ f(\eta_C) & \varphi_{T'}(t) \end{pmatrix} \tag{10.1}$$

Now consider the exact sequence

$$0 \to N^n \xrightarrow{i} J^nE \xrightarrow{\pi} E \to 0 \tag{10.2}$$

Given a $\Psi \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ consider the push out

$$\begin{tikzcd} 0 \arrow[r] & N^n \arrow[r, "i"] \arrow[d, "\Psi"'] & J^nE \arrow[r, "\pi"] \arrow[d, "g_\Psi"'] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r, "i"] & E^*_\Psi \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

where $E^*_\Psi = \frac{J^nE \times \hat{\mathbb{G}}_{\mathrm{a}}}{\Gamma(N^n)}$ and $\Gamma(N^n) = \{(i(z), \Psi(z)) |\ z \in N^n\} \subset J^nE \times N^n$ and $g_\Psi(x) = [x, 0] \in E^*_\Psi$.

The Teichmüller section $v : E \to J^n(E)$ is an $\mathbb{F}_q$-linear splitting of the sequence (10.2). The induced retraction

$$\rho = \mathbb{1} - v \circ \pi : J^n(E) \to N^n$$

is given in coordinates simply by $\rho : (x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$. Let us denote by s_{Witt} the morphism on Lie algebras induced by ρ. Thus we have the following split exact sequence of R-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}\, N^n \arrow[r, shift left, "Di"] & \mathrm{Lie}\, J^nE \arrow[l, shift left, "s_{\mathrm{Witt}}"] \arrow[r, "D\pi"] & \mathrm{Lie}(E) \arrow[r] & 0 \end{tikzcd}$$

Let s_Ψ denote the induced splitting of the push out extension

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}\, \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r, shift left] & \mathrm{Lie}(E^*_\Psi) \arrow[l, shift left, "s_\Psi"] \arrow[r] & \mathrm{Lie}(E) \arrow[r] & 0 \end{tikzcd}$$

It is given explicitly by $\tilde{s}_\Psi : \mathrm{Lie}\, J^nE \times \mathrm{Lie}\, \hat{\mathbb{G}}_{\mathrm{a}} \to \mathrm{Lie}\, \hat{\mathbb{G}}_{\mathrm{a}}$

$$\tilde{s}_\Psi(x, y) := D\Psi(s_{\mathrm{Witt}}(x)) + y$$

and

$$s_\Psi : \mathrm{Lie}(E^*_\Psi) = \frac{\mathrm{Lie}\, J^nE \times \mathrm{Lie}\, \hat{\mathbb{G}}_{\mathrm{a}}}{\mathrm{Lie}\, \Gamma(N^n)} \to \mathrm{Lie}\, \hat{\mathbb{G}}_{\mathrm{a}}$$

This induces the following morphism of exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{X}_n(E) \arrow[r] \arrow[d] & \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] \arrow[d, "\Psi\mapsto(E^*_\Psi, s_\Psi)"'] & \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Lie}(E)^* \arrow[r] & \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & 0
\end{tikzcd}
\tag{10.3}
$$

Proposition 10.1. *Let Θ be a character in $\mathbf{X}_n(E)$, and put $\Psi = i^*\Theta \in \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})$ and $g = v^*\Theta : E \to \hat{\mathbb{G}}_{\mathrm{a}}$, as in equation (9.4).*

1. *The map $\mathbf{X}_n(E) \to \mathrm{Lie}(E)^*$ of (10.3) sends Θ to $-Dg$.*
2. *Let $\tilde{\Theta} = \phi^*\Theta$, and put $\Psi = i^*\tilde{\Theta}$ and $g = v^*\tilde{\Theta}$. Then we have $\tilde{g}(x) = g(x^{\hat{q}})$ and $\tilde{\Psi}(y) = \Psi(\rho(\phi(i(y)))) + g(\pi y_1)$.*

Proof. (1): Let us recall in explicit terms how the map is given. For the split extension $E \times \hat{\mathbb{G}}_{\mathrm{a}}$, the retractions $\mathrm{Lie}(E) \times \mathrm{Lie}\,\hat{\mathbb{G}}_{\mathrm{a}} = \mathrm{Lie}(E \times \hat{\mathbb{G}}_{\mathrm{a}}) \to \mathrm{Lie}\,\hat{\mathbb{G}}_{\mathrm{a}}$ are in bijection with maps $\mathrm{Lie}(E) \to \mathrm{Lie}\,\hat{\mathbb{G}}_{\mathrm{a}}$, a retraction s corresponding to map $x \mapsto s(x, 0)$. Therefore to determine the image of Θ, we need to identify E^*_Ψ with a split extension and then apply this map to s_Ψ.

A trivialization of the extension E^*_Ψ is given by the map

$$\frac{J^n E \times \hat{\mathbb{G}}_{\mathrm{a}}}{\Gamma(N^n)} = E^*_\Psi \xrightarrow{\sim} E \times \hat{\mathbb{G}}_{\mathrm{a}}$$

defined by $[a, b] \mapsto (\pi(a), \theta(a) + b)$. The inverse isomorphism H is then given by the expression

$$H(x, y) = [v(x), y - \Theta(v(x))],$$

and so the composition $E \to E \times \hat{\mathbb{G}}_{\mathrm{a}} \to E^*_\Psi \to \hat{\mathbb{G}}_{\mathrm{a}}$ is simply $-\Theta \circ v = -g$, which induces the map $-Dg$ on the Lie algebras.

(2): We have

$$
\begin{aligned}
\tilde{\Theta}(x) &= \Theta(\phi(x)) \\
&= \Psi(\rho(\phi(t))) + g(x_0^{\hat{q}} + \pi x_1) \\
&= \big(\Psi(\rho(\phi(t)) + g(\pi x_1)\big) + g(x_0^{\hat{q}}).
\end{aligned}
$$

In other words, we have $\tilde{\Psi}(\rho(x)) = \Psi(\rho(\phi(x)) + g(\pi x_1)$ and $\tilde{g}(x_0) = g(x_0^{\hat{q}})$. Setting $x = i(y)$, we obtain the desired result. □

Proposition 10.2. *If $\Psi \in i^*\phi^*(\mathbf{X}_n(E))$, then the class $(E^*_\Psi, s_\Psi) \in \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ is zero.*

Proof. We know from diagram (10.3) that E^*_Ψ is a trivial extension since $\tilde{\Psi}$ lies in $i^*\mathbf{X}_{n+1}(E)$. Now as in part (2) of proposition 10.1, we have, in the notation of that proposition, $\tilde{g}(x_0) = g(x_0^{\hat{q}})$ and hence $D\tilde{g} = 0$. Therefore by part (1) of that proposition, the class in $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ is zero. □

10.1. **The crystal $\mathbf{H}(E)$.** The ϕ-linear map $\phi^* : \mathbf{X}_{n-1}(E) \to \mathbf{X}_n(E)$ induces a linear map $\mathbf{X}_{n-1}(E)^\phi \to \mathbf{X}_n(E)$, which we will abusively also denote ϕ^*. We then define

$$\mathbf{H}_n(E) = \frac{\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a})}{i^*\phi^*(\mathbf{X}_{n-1}(E)^\phi)}$$

Then $u : N^{n+1} \to N^n$ induces $u^* : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a}) \to \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_\mathrm{a})$. And since $u^* i^* \phi^*(\mathbf{X}_n(E)) = i^* u^* \phi^*(\mathbf{X}_n(E)) = i^*\phi^* u^*(\mathbf{X}_n(E)) \subset i^*\phi^*(\mathbf{X}_{n+1}(E))$, it also induces a map $u^* : \mathbf{H}_n(E) \to \mathbf{H}_{n+1}(E)$. Define $\mathbf{H}(E) = \lim_\to \mathbf{H}_n(E)$.

Similarly, $\mathfrak{f} : N^{n+1} \to N^n$ induces $\mathfrak{f}^* : \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a}) \to \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_\mathrm{a})$, which descends to a ϕ-linear morphism of R-modules

$$\mathfrak{f}^* : \mathbf{H}_n(E) \to \mathbf{H}_{n+1}(E)$$

because we have $\mathfrak{f}^* i^* \phi^*(\mathbf{X}_{n-1}(E)) = i^*\phi^*\phi^*(\mathbf{X}_{n-1}(E) \subset i^*\phi^*\mathbf{X}_n(E)$. This then induces a ϕ-linear endomorphism $\mathfrak{f}^* : \mathbf{H}(E) \to \mathbf{H}(E)$. Finally, let $\mathbf{I}_n(E)$ denote the image of $\partial : \mathrm{Hom}(N^n, \hat{\mathbb{G}}_\mathrm{a}) \to \mathrm{Ext}_A(E, \hat{\mathbb{G}}_\mathrm{a})$. So $\mathrm{Hom}(N^n, \hat{\mathbb{G}}_\mathrm{a})/\mathbf{X}_n(E) \simeq \mathbf{I}_n(E)$. Then u induces maps $u^* : \mathbf{I}_n(E) \to \mathbf{I}_{n+1}(E)$, and we put $\mathbf{I}(E) = \lim_\to \mathbf{I}_n(E)$.

Proposition 10.3. *The morphism*

$$u^* : \mathbf{H}_n(E) \otimes K \to \mathbf{H}_{n+1}(E) \otimes K$$

is injective. For $n \geq m$, it is an isomorphism.

Proof. Consider the following diagram of exact sequences:

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & K\langle \phi^{\circ(n-m)^*}\Theta\rangle^\phi \arrow[u] \arrow[r, "i^*\phi^*"] & K\langle \Psi_{n+1}\rangle \arrow[u] & & \\
0 \arrow[r] & bX_n(E)^\phi_K \arrow[u] \arrow[r, "i^*\phi^*"] & \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_\mathrm{a})_K \arrow[u] \arrow[r] & \mathbf{H}_{n+1}(E)_K \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{X}_{n-1}(E)^\phi_K \arrow[u, "u^*"] \arrow[r, "i^*\phi^*"] & \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a})_K \arrow[u, "u^*"] \arrow[r] & \mathbf{H}_n(E)_K \arrow[u, "u^*"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The cokernels of the two maps u^* are of the displayed form by propositions 8.4 and 9.6. If $n < m$, the expression $K\langle \phi^{\circ(n-m)^*}\Theta\rangle$ is understood to be zero. The map $i^*\phi^* : K\langle \phi^{\circ(n-m)^*}\Theta\rangle^\phi \to K\langle \Psi_{n+1}\rangle$ is injective, by proposition 9.5. Therefore the map $u^* : \mathbf{H}_n(E)_K \to \mathbf{H}_{n+1}(E)_K$ is also injective. It is an isomorphism if $n \geq m$, because $K\langle \phi^{\circ(n-m)^*}\Theta\rangle$ is 1-dimensional and hence the map

$$i^*\phi^* : K\langle \phi^{\circ(n-m)^*}\Theta\rangle^\phi \to K\langle \Psi_{n+1}\rangle$$

is an isomorphism. □

Corollary 10.4. *We have*

$$\mathbf{H}_n(E)\otimes K \simeq \begin{cases} K\langle \Psi_1,\ldots,\Psi_n\rangle, & \text{if } n\le m\\ K\langle \Psi_1,\ldots,\Psi_m\rangle, & \text{if } n\ge m\end{cases}$$

Do note that we will promote this to an integral result in (10.7). But before we get there, we will need some preparation.

Proposition 10.5. *We have*

$$\mathbf{I}_n(E)\otimes K \simeq \begin{cases} K\langle \Psi_1,\ldots,\Psi_n\rangle, & \text{if } n\le m-1\\ K\langle \Psi_1,\ldots,\Psi_{m-1}\rangle, & \text{if } n\ge m-1\end{cases}$$

Proof. The case $n\le m-1$ is clear. So suppose $n\ge m-1$. Then $\mathrm{Hom}_A(N^j,\hat{\mathbb{G}}_{\mathrm{a}})\otimes K$ has basis $\Psi_1,\ldots,\Psi_j$, and $\mathbf{X}_n(E)\otimes K$ has basis $\Theta_m,\ldots,(\phi^{n-m})^*\Theta_m$. Since each $(\phi^j)^*\Theta_m$ equals Ψ_{m+j} plus lower order terms, $K\langle\Psi_1,\ldots,\Psi_{m-1}\rangle$ is a complement to the subspace $X_n(E)$ of $\mathrm{Hom}_A(N^n,\hat{\mathbb{G}}_{\mathrm{a}})$. Therefore the map ∂ from $K\langle\Psi_1,\ldots,\Psi_{m-1}\rangle$ to the quotient $I_n(E)$ is an isomorphism. ☐

Finally the morphism $\mathrm{Hom}_A(N^n,\hat{\mathbb{G}}_{\mathrm{a}})\to\mathrm{Ext}^\sharp(E,\hat{\mathbb{G}}_{\mathrm{a}})$ of diagram (10.3) vanishes on $\phi^*(\mathbf{X}_{n-1}(E))$, by proposition 10.2, and hence induces a morphism of exact sequences

(10.4)
$$\begin{tikzcd} 0 \arrow[r] & \frac{\mathbf{X}_n(E)}{\phi^*(\mathbf{X}_{n-1}(E)^\phi)} \arrow[r] \arrow[d, "\Upsilon"'] & \mathbf{H}_n(E) \arrow[r] \arrow[d, "\Phi"] & \mathbf{I}_n(E) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathrm{Lie}(E)^* \arrow[r] & \mathrm{Ext}^\sharp(E,\hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & \mathrm{Ext}(E,\hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r] & 0 \end{tikzcd}$$

where as in the introduction, $\mathbf{I}_n(E)$ denotes the image of $\partial:\mathrm{Hom}(N^n,\hat{\mathbb{G}}_{\mathrm{a}})\to\mathrm{Ext}_A(E,\hat{\mathbb{G}}_{\mathrm{a}})$.

Proposition 10.6. *The map $\Phi:\mathbf{H}_n(E)\otimes K\to\mathrm{Ext}^\sharp(E,\hat{\mathbb{G}}_{\mathrm{a}})\otimes K$ is injective if and only if $\gamma\neq 0$.*

Proof. It is enough to show that Υ is injective if and only if $\gamma\neq 0$. By proposition 9.6, the class of Θ_m is a K-linear basis for $\frac{\mathbf{X}_n(E)}{\phi^*(\mathbf{X}_{n-1}(E)^\phi)}\otimes K$, and so it is enough to show Φ is injective if and only if $\Upsilon(\Theta_m)\neq 0$. As in (9.4), write $\Theta_m=\Psi+g$. Then by proposition 10.1, it is enough to show $g'(0)\neq 0$ if and only if $\gamma\neq 0$. But this holds because by proposition 9.4, we have $\gamma=\pi g'(0)$. ☐

Lemma 10.7. *Consider the ϕ-linear endomorphism F of K^m with matrix*

$$\begin{pmatrix} 0 & 0 & \ldots & & 0 & \mu_m\\ 1 & 0 & & & 0 & \mu_{m-1}\\ 0 & 1 & & & 0 & \mu_{m-2}\\ \vdots & & \ddots & \ddots & \vdots & \vdots\\ & & & & & \\ 0 & 0 & & & 1 & \mu_1 \end{pmatrix},$$

for some given $\mu_1,\ldots,\mu_m\in K$. If K^m admits an R-lattice which is stable under F, then we have $\mu_1,\ldots,\mu_m\in R$.

Proof. We use Dieudonné–Manin theory. Without loss of generality, we may assume that $R/\pi R$ is algebraically closed. Let P denote the polynomial $F^m - \mu_1 F^{m-1} - \cdots - \mu_m$ in the twisted polynomial ring $K\{F\}$. Then by (B.1.5) of [Lau] (page 257), there exists an integer $r \geq 1$ and elements $\beta_1, \ldots, \beta_m \in K(\pi^{1/r})$ such that we have

$$P = (F - \beta_1) \cdots (F - \beta_m)$$

in the ring $K(\pi^{1/r})\{F\}$ with commutation law $F\pi^{1/r} = \pi^{1/r}F$. (Note that the results of [Lau] are stated under the assumption that the residue field of R is an algebraic closure of $\mathbb{F}_p$, but they hold if it is any algebraically closed field of characteristic p.) Since $R = K \cap R[\pi^{1/r}]$, it is enough to show $\mu_i \in R[\pi^{1/r}]$. Therefore, by replacing $R[\pi^{1/r}]$ with R, it is enough to assume that P factors as above where in addition all β_i lie in K.

Now fix i, and let us show $\beta_i \in R$. Assume $\beta_i \neq 0$, the case $\beta_i = 0$ being immediate. Because the (left) $K\{F\}$-module K^m has an F-stable integral lattice M, every quotient of K^m also has a F-stable integral lattice, namely the image of M. By (B.1.9) of [Lau] (page 260), for each i, the $K\{F\}$-module K^m has a quotient (in fact, a summand) isomorphic to $N = K\{F\}/K\{F\}(F - \pi^{v(\beta_i)})$. Therefore N also has a F-stable integral lattice. But this can happen only if $v(\beta_i) \geq 0$, because F sends the basis element $1 \in N$ to $\pi^{v(\beta_i)} \in N$. □

Theorem 10.8. *If E splits at m, then we have $\lambda_1 \ldots, \lambda_{m-1} \in R$.*

Proof. We will prove the cases when $\gamma = 0$ and $\gamma \neq 0$ separately.

Case $\gamma = 0$ When $\gamma = 0$ we have $\mathfrak{f}^* i^* = i^* \phi^*$, and hence for all $n \geq 1$, this induces a ϕ-linear map $\mathfrak{f} : \mathbf{I}_{n-1}(E) \to \mathbf{I}_n(E)$ as follows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{X}_n(E) \arrow[r, "i^*"] & \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\partial"] & \mathbf{I}_n(E) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{X}_{n-1}(E) \arrow[r, "i^*"] \arrow[u, "\phi"'] & \mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\partial"] \arrow[u, "\mathfrak{f}"'] & \mathbf{I}_{n-1}(E) \arrow[r] \arrow[u, "\mathfrak{f}"'] & 0
\end{tikzcd}$$

Let $\mathbf{I}(E) = \lim_{\to} \mathbf{I}_n(E) \subseteq \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$. Then by proposition 10.5, the vector space $\mathbf{I}(E)_K$ has a K-basis $\partial\Psi_1, \ldots, \partial\Psi_{m-1}$, and with respect to this basis, the ϕ-linear endomorphism $\mathfrak{f}$ has matrix

$$\Gamma_0 = \begin{pmatrix} 0 & 0 & \cdots & & 0 & \lambda_{m-1} \\ 1 & 0 & & & 0 & \lambda_{m-2} \\ 0 & 1 & & & 0 & \lambda_{m-3} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ & & & & & \\ 0 & 0 & & & 1 & \lambda_1 \end{pmatrix}$$

Since $\mathbf{I}(E)$ is contained in $\mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$, it is a finitely generated free R-module and hence an integral lattice in $\mathbf{I}(E)_K$. But then lemma 10.7 implies $\lambda_1, \ldots, \lambda_{m-1} \in R$.

Case $\gamma \neq 0$ Let $\mathbf{H}(E) = \lim_{\to} \mathbf{H}_n(E)$. Let us consider the matrix Γ of the ϕ-linear endomorphism $\mathfrak{f}$ of $\mathbf{H}(E)_K$ with respect to the K-basis $\Psi_1, \ldots, \Psi_m$ given by corollary 10.4. Let $\gamma \in R$ be as in proposition 9.4. Then we have

$$\begin{aligned} i^*\phi^*\Theta_m &= \mathfrak{f}^* i^*\Theta_m + \gamma\Psi_1 \\ &= \mathfrak{f}^*(\Psi_m - \lambda_1\Psi_{m-1} - \cdots - \lambda_{m-1}\Psi_1) + \gamma\Psi_1 \\ &= \mathfrak{f}^*(\Psi_m) - \phi(\lambda_1)\Psi_m - \cdots - \phi(\lambda_{m-1})\Psi_2 + \gamma\Psi_1. \end{aligned}$$

Therefore we have

$$\mathfrak{f}^*(\Psi_m) \equiv \phi(\lambda_1)\Psi_m + \cdots + \phi(\lambda_{m-1})\Psi_2 - \gamma\Psi_1 \bmod i^*\phi^*(X_m^\phi)$$

and hence

$$\Gamma = \begin{pmatrix} 0 & 0 & \dots & & 0 & -\gamma \\ 1 & 0 & & & 0 & \phi(\lambda_{m-1}) \\ 0 & 1 & & & 0 & \phi(\lambda_{m-2}) \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 0 & \phi(\lambda_2) \\ 0 & 0 & & & 1 & \phi(\lambda_1) \end{pmatrix}$$

We will now apply lemma 10.7 to the operator $\mathfrak{f}^*$ on $\mathbf{H}(E)_K$, but to do this we need to produce an integral lattice M. Consider the commutative square

$$\begin{tikzcd} \mathbf{H}(E) \arrow[r, "\Phi"] \arrow[d] & \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[d, "j"] \\ \mathbf{H}(E)_K \arrow[r, "\Phi_K"] & \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})_K. \end{tikzcd}$$

Let M denote the image of $\mathbf{H}(E)$ in $\mathbf{H}(E)_K$. It is clearly stable under $\mathfrak{f}^*$. But also the maps Φ_K and j are injective, by proposition 10.6 and because $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}}) \simeq R^r$; so M agrees with the image of $\mathbf{H}(E)$ in $\mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and is therefore finitely generated.

We can then apply lemma 10.7 and deduce $\phi(\lambda_{m-1}), \ldots, \phi(\lambda_1) \in R$. This implies $\lambda_{m-1}, \ldots, \lambda_1 \in R$, since $R/\pi R$ is a field and hence the Frobenius map on it is injective. □

Corollary 10.9. (1) *The element $\Theta_m \in \mathbf{X}_m(E)_K$ lies in $\mathbf{X}_m(E)$.*
(2) *For $n \geq m$, all the maps in the diagram*

$$\begin{tikzcd} \mathbf{X}_n(E)/\mathbf{X}_{n-1}(E) \arrow[r, "\phi^*"] \arrow[d, "i^*"] & \mathbf{X}_{n+1}(E)/\mathbf{X}_n(E) \arrow[d, "i^*"] \\ \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})/\mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}}) \arrow[r, "\mathfrak{f}^*"] & \mathrm{Hom}_A(N^{n+1}, \hat{\mathbb{G}}_{\mathrm{a}})/\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}}) \end{tikzcd}$$

are isomorphisms.

Proof. (1): By theorem 10.8, the element $i^*\Theta_m$ of $\mathrm{Hom}_A(N^m, \hat{\mathbb{G}}_{\mathrm{a}})_K$ actually lies in $\mathrm{Hom}_A(N^m, \hat{\mathbb{G}}_{\mathrm{a}})$, and therefore by the exact sequence (9.3) we have $\Theta_m \in \mathbf{X}_m(E)$.

(2): By proposition 9.5, we know $\mathfrak{f}^*$ is an isomorphism.

By proposition 9.5, the maps i^* are injective for all $n \geq m$. So to show they are isomorphisms, it is enough to show they are surjective. The R-linear generator Ψ_m of $\mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_{\mathrm{a}})/\mathrm{Hom}_A(N^{n-1}, \hat{\mathbb{G}}_{\mathrm{a}})$ is the image of Θ_m, which by part (1), lies in

$\mathbf{X}_m(E)$. Therefore i^* is surjective for $n = m$. Then because $\mathfrak{f}^*$ is an isomorphism, it follows by induction that i^* is surjective for all $n \geq m$.

Finally, ϕ^* is an isomorphism because all the other morphisms in the diagram are. □

We knew before that $i^*(\phi^j)^*\Theta_m$ agrees with Ψ_{m+j} plus lower order rational characters, but the corollary above implies that these lower order characters are in fact integral.

Theorem 10.10. *Let E be a Drinfeld module that splits at m.*

(1) *For any $n \geq m$, the composition*

$$\mathbf{X}_n(E) \longrightarrow \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a}) \longrightarrow \mathrm{Hom}_A(N^n, \hat{\mathbb{G}}_\mathrm{a})/\mathrm{Hom}_A(N^{m-1}, \hat{\mathbb{G}}_\mathrm{a}) \tag{10.5}$$

is an isomorphism of R-modules.

(2) *$\mathbf{X}_n(E)$ is freely generated as an R-module by $\Theta_m, \dots, (\phi^*)^{n-m}\Theta_m$.*

Proof. (i): By corollary 10.9, the induced morphism on each graded piece is an isomorphism. It follows that the map in question is also an isomorphism.

(ii): This follows formally from (i) and the fact, which follows from 10.9, that the map (10.5) sends any $(\phi^*)^j\Theta_m$ to Ψ_{m+j} plus lower order terms. □

10.2. **$\mathbf{H}(E)$ and de Rham cohomology.** Collecting the results above, we have isomorphisms

$$R\langle \Psi_1, \dots, \Psi_{m-1}\rangle = \mathrm{Hom}_A(N^{m-1}, \hat{\mathbb{G}}_\mathrm{a}) \xrightarrow{\sim} \mathbf{I}_n(E)$$
$$R\langle \Psi_1, \dots, \Psi_{m}\rangle = \mathrm{Hom}_A(N^{m}, \hat{\mathbb{G}}_\mathrm{a}) \xrightarrow{\sim} \mathbf{H}_n(E)$$

for $n \geq m$, and hence in the limit

$$R\langle \Psi_1, \dots, \Psi_{m-1}\rangle \xrightarrow{\sim} \mathbf{I}(E) \tag{10.6}$$

$$R\langle \Psi_1, \dots, \Psi_{m}\rangle \xrightarrow{\sim} \mathbf{H}(E) \tag{10.7}$$

And so the K-linear bases of $K\otimes\mathbf{I}(E)$ and $K\otimes\mathbf{H}(E)$—the ones respect to which the action of $\mathfrak{f}^*$ is described by the matrices Γ_0 and Γ in the proof of theorem 10.8—are in fact R-linear bases of $\mathbf{I}(E)$ and $\mathbf{H}(E)$.

We also have isomorphisms for $n \geq m$

$$R\langle\Theta_m\rangle = \mathbf{X}_m(E) \xrightarrow{\sim} \mathbf{X}_n(E)/\phi^*(\mathbf{X}_{n-1}(E)^\phi).$$

Combining these, we have the following map between exact sequences of R-modules, as in (10.4):

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{X}_m(E) \arrow[r] \arrow[d, "\Upsilon"'] & \mathbf{H}(E) \arrow[r] \arrow[d, "\Phi"] & \mathbf{I}(E) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{Lie}(E)^* \arrow[r] & \mathrm{Ext}^\sharp(E, \hat{\mathbb{G}}_\mathrm{a}) \arrow[r] & \mathrm{Ext}(E, \hat{\mathbb{G}}_\mathrm{a}) \arrow[r] & 0
\end{tikzcd}$$

where Υ sends Θ_m to γ/π (in coordinates). It follows that Φ is injective if and only if $\gamma \neq 0$.

11. Computation of λ_1 and γ in the rank 2 case

Theorem 11.1. *Let $A = \mathbb{F}_q[t]$ with $q \geq 3$, let $\pi \in A$ be an irreducible polynomial of degree ℓ, and let E be a Drinfeld module over R of the form*

$$\varphi_E(t)(x) = \pi x + a_1 x^q + a_2 x^{q^2}. \tag{11.1}$$

Then we have

$$\lambda_1 \equiv (-1)^\ell w^{\frac{q^{\ell-1}(q^\ell-1)}{q-1}} \left(1 - a_1' w^{q^{\ell-1}} + a_2' w^{q^{\ell-1}+q^\ell}\right)^{q^\ell-1} \mod \pi,$$

where $w = a_1 a_2^{-1}$, and

$$\gamma = \pi\lambda_1/a_1 \mod \pi^2.$$

Observe that when $\varphi_E(t)(x)$ is of the form $\pi x + a x^q + x^{q^2}$, which is always true after changing the coordinate x (perhaps passing to a cover of S), we have the simplified forms

$$\lambda_1 \equiv (-1)^\ell a^{\frac{q^{\ell-1}(q^\ell-1)}{q-1}} \left(1 - a' a^{q^{\ell-1}}\right)^{q^\ell-1} \mod \pi, \tag{11.2}$$

$$\gamma = \pi\lambda_1/a \mod \pi^2. \tag{11.3}$$

Proof. Let $\vartheta_1 : N^1 \to \hat{\mathbb{G}}_{\mathrm{a}}$ be the isomorphism defined in theorem 6.4. Then $\vartheta_1 \equiv \tau^0 \bmod \pi$. Also ϑ_1 induces the isomorphism $(\vartheta_1)_* : \mathrm{Ext}(E, N^1) \to \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$. In order to determine the action of A on J^1E and J^2E we need to determine how t acts on the coordinates x' and x''. Now we note that $J^nE \simeq W_n$ can be endowed with the δ-coordinates (denoted $[z, z', z'', \dots]$) or the Witt coordinates (denoted $(z_0, z_1, z_2, \dots)$) and they are related by the following in J^2E by proposition 3.2

$$[z, z', z''] = (z, z', z'' + \pi^{\hat{q}-2}(z')^{\hat{q}}) \tag{11.4}$$

Taking π-derivatives of both sides of equation (11.1) using the formula

$$\delta(a x^{q^j}) = a' x^{\hat{q} q^i} + \phi(a) \pi^{q^i - 1} (x')^{q^i},$$

we obtain

$$\begin{aligned} \varphi(t)(x') = {} & \pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}} \\ & + \pi x' + \phi(a_1)\pi^{q-1}(x')^q + \phi(a_2)\pi^{q^2-1}(x')^{q^2} \end{aligned} \tag{11.5}$$

and

$$\begin{aligned} \varphi(t)(x'') = {} & \pi'' x^{\hat{q}^2} + a_1'' x^{q\hat{q}^2} + a_2'' x^{q^2\hat{q}^2} \\ & + \{\text{terms with } x' \text{ and } x''\} \end{aligned} \tag{11.6}$$

Then the A-action $\varphi_{J^1E} : A \to \mathrm{End}(J^1E)$ is given in Witt coordinates by the 2×2 matrix

$$\varphi_{J^1E}(t) = \begin{pmatrix} \varphi_E(t) & 0 \\ \eta_{J^1E} & \varphi_{N^1}(t) \end{pmatrix}$$

where $\eta_{J^1E} = \pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}}$. And by (11.6) and (11.4), the A-action $A \to \mathrm{End}(J^2E)$ is given by the $(1+2) \times (1+2)$ block matrix

$$\varphi_{J^2E}(t) = \begin{pmatrix} \varphi_E(t) & 0 \\ \eta_{J^2E} & \varphi_{N^2}(t) \end{pmatrix}$$

where (using 11.4) η_{J^2E} is the column vector

$$\eta_{J^2E} = \begin{pmatrix} \pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}} \\ \Delta(\pi) x^{\hat{q}^2} + \Delta(a_1) x^{q\hat{q}^2} + \Delta(a_2) x^{q^2\hat{q}^2} \end{pmatrix}$$

and where $\Delta(y) = y'' + \pi^{\hat{q}-2}(y')^{\hat{q}}$.

Now we will consider two cases—

(1): Consider $\eta_{\Psi_{1*}(J^1E)} \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ which is the image of Ψ_1 under the connecting morphism $\mathrm{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}}) \xrightarrow{\partial} \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and $\Psi_1 : N^1 \to \hat{\mathbb{G}}_{\mathrm{a}}$ is the isomorphism defined in theorem 6.4 and satisfies $\Psi_1 = \vartheta_1 \circ \mathfrak{f}^{\circ 0} = \tau^0 \bmod \pi$ where $\mathfrak{f}^{\circ 0} = \mathbb{1}$.

$$\begin{tikzcd} 0 \arrow[r] & N^1 \arrow[r] \arrow[d, "\Psi_1"'] & J^1E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r] & f_*(J^1E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

where $\eta_{J^1E} = [\pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}}] \in \mathrm{Ext}(E, N^1)$ Hence

$$\begin{aligned} \eta_{\Psi_{1*}(J^1E)} &= [\pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}}] \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \\ \partial(\Psi_1) &\equiv [x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}}] \bmod \pi. \end{aligned}$$

(2): Now consider $\eta_{\Psi_{2*}(J^2E)} \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ obtained as

$$\begin{tikzcd} 0 \arrow[r] & N^2 \arrow[r] \arrow[d, "\Psi_2"'] & J^2E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r] & f_*(J^2E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

Now we have

$$\eta_{J^2E} = \left[\begin{pmatrix} \pi' x^{\hat{q}} + a_1' x^{q\hat{q}} + a_2' x^{q^2\hat{q}} \\ \Delta(\pi) x^{\hat{q}^2} + \Delta(a_1) x^{q\hat{q}^2} + \Delta(a_2) x^{q^2\hat{q}^2} \end{pmatrix} \right] \in \mathrm{Ext}(E, N^2)$$

Let $\Xi(y) = (y')^{\hat{q}} + \pi\Delta(y)$. Then applying $\Psi_2 = \vartheta_1 \circ \mathfrak{f}$ and $\mathfrak{f}(z_1, z_2) = z_1^{\hat{q}} + \pi z_2$, we have

$$\begin{aligned} \partial(\Psi_2) = \eta_{\Psi_{2*}(J^2E)} &= [\vartheta_1(\Xi(\pi) x^{\hat{q}^2} + \Xi(a_1) x^{q\hat{q}^2} + \Xi(a_2) x^{q^2\hat{q}^2})] \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \\ \partial(\Psi_2) &\equiv [\Xi(\pi) x^{\hat{q}^2} + \Xi(a_1) x^{q\hat{q}^2} + \Xi(a_2) x^{q^2\hat{q}^2}] \bmod \pi \\ &\equiv [(\pi')^{\hat{q}} x^{\hat{q}^2} + (a_1')^{\hat{q}} x^{q\hat{q}^2} + (a_2')^{\hat{q}} x^{q^2\hat{q}^2}] \bmod \pi \\ &\equiv [x^{\hat{q}^2} + (a_1')^{\hat{q}} x^{q\hat{q}^2} + (a_2')^{\hat{q}} x^{q^2\hat{q}^2}] \bmod \pi. \end{aligned}$$

Recall that the map $R\{\tau\}\hat{} \to \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ given by $\eta \mapsto [\eta]$ is surjective and the kernel consists of the inner derivations, which is to say all η of the form

$$\pi\alpha - \alpha \circ \varphi_E(t),$$

for some $\alpha \in R\{\tau\}\hat{}$. Let us now work out these relations explicitly for $\alpha = \tau^0, \tau^1, \tau^2$. If $\alpha = \tau^j$, with $j \geq 0$, we get the relation

$$\begin{aligned}\pi\tau^j &= \tau^j(\pi\tau^0 + a_1\tau^1 + a_2\tau^2)\\ \tau^{j+2} &= a_2^{-q^j}[(\pi - \pi^{q^j})\tau^j - a_1^{q^j}\tau^{j+1}]\\ \tau^{j+2} &\equiv -(a_1a_2^{-1})^{q^j}\tau^{j+1} \bmod \pi\end{aligned}$$

and hence we have by induction the relations

$$\tau^{i+1} \equiv (-1)^i w^{\frac{q^i-1}{q-1}}\tau^1 \bmod \pi \tag{11.7}$$

where $w = a_1a_2^{-1}$, for all $i \geq 0$.

Therefore writing $\hat{q} = q^\ell$, we have

$$\begin{aligned}\partial(\Psi_1) &\equiv x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}}\\ &\equiv x^{q^\ell} + a_1'x^{q^{\ell+1}} + a_2'x^{q^{\ell+2}}\\ &\equiv \tau^\ell + a_1'\tau^{\ell+1} + a_2'\tau^{\ell+2}\\ &\equiv (-1)^{\ell+1}w^{1+\cdots+q^{\ell-2}}(1 - a_1'w^{q^{\ell-1}} + a_2'w^{q^{\ell-1}+q^\ell})\tau^1\end{aligned}$$

and

$$\begin{aligned}\partial(\Psi_2) &\equiv x^{\hat{q}^2} + (a_1')^{\hat{q}}x^{q\hat{q}^2} + (a_2')^{\hat{q}}x^{q^2\hat{q}^2}\\ &\equiv \tau^{2\ell} + (a_1')^{q^\ell}\tau^{2\ell+1} + (a_2')^{q^\ell}\tau^{2\ell+2}\\ &\equiv (-1)^{2\ell+1}w^{1+\cdots+q^{2\ell-2}}\big(1 - (a_1')^{q^\ell}w^{q^{2\ell-1}} + (a_2')^{q^\ell}w^{q^{2\ell-1}+q^{2\ell}}\big)\tau^1\\ &\equiv (-1)^{2\ell+1}w^{1+\cdots+q^{2\ell-2}}\big(1 - a_1'w^{q^{\ell-1}} + a_2'w^{q^{\ell-1}+q^\ell}\big)^{q^\ell}\tau^1.\end{aligned}$$

and hence

$$\begin{aligned}\lambda_1 = \frac{\partial(\Psi_2)}{\partial(\Psi_1)} &\equiv (-1)^\ell w^{q^{\ell-1}+\cdots+q^{2\ell-2}}\big(1 - a_1'w^{q^{\ell-1}} + a_2'w^{q^{\ell-1}+q^\ell}\big)^{q^\ell-1} \bmod \pi\\ &\equiv (-1)^\ell w^{q^{\ell-1}(1+\cdots+q^{\ell-1})}\big(1 - a_1'w^{q^{\ell-1}} + a_2'w^{q^{\ell-1}+q^\ell}\big)^{q^\ell-1} \bmod \pi\\ &\equiv (-1)^\ell w^{\frac{q^{\ell-1}(q^\ell-1)}{q-1}}\big(1 - a_1'w^{q^{\ell-1}} + a_2'w^{q^{\ell-1}+q^\ell}\big)^{q^\ell-1} \bmod \pi\end{aligned}$$

Now we determine γ. Write $g = \sum_i \alpha_i\tau^i$. Then from proposition 9.4, we know $\gamma = \pi\alpha_0$. Now we will compute α_0. Let $(z_0, z_1, z_2) := \varphi_{J^2E}(t)(x, 0, 0)$. Then

$$\begin{aligned}\Theta_2(\varphi_{J^2E}(t)(x,0,0)) &= \Psi_2(z_1, z_2) - \lambda_1\Psi_1(z_1) + g(z_0)\\ &= \vartheta_1(z_1^{\hat{q}} + \pi z_2) - \lambda_1\vartheta_1(z_1) + g(z_0)\\ &\equiv z_1^{\hat{q}} - \lambda_1 z_1 + g(z_0) \bmod \pi\end{aligned}$$

where $z_0 = \pi x + a_1x^q + a_2x^{q^2}$ and $z_1 = \pi'x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}}$. On the other hand from the A-linearity of Θ_2 we have

$$\Theta_2(\varphi_{J^2E}(t)(x,0,0)) = \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}(t)\Theta_2(x,0,0) = \pi\Theta_2(x,0,0) \equiv 0 \bmod \pi$$

and hence $z_1^{\hat{q}} - \lambda_1 z_1 + g(z_0) \equiv 0 \bmod \pi$. Substituting z_0 and z_1 in, we obtain

$$\begin{aligned}0 &\equiv (\pi'x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}})^{\hat{q}} - \lambda_1(\pi'x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}}) + g(\pi x + a_1x^q + a_2x^{q^2})\\ &\equiv (x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}})^{\hat{q}} - \lambda_1(x^{\hat{q}} + a_1'x^{q\hat{q}} + a_2'x^{q^2\hat{q}}) + g(a_1x^q + a_2x^{q^2})\end{aligned}$$

Now substitute $g(x) = \sum_{j\geq 0} \alpha_j x^{q^j}$ into this and consider the coefficient of x^q. If $\hat{q} = q$, we obtain $\lambda_1 \equiv \alpha_0 a_1$ and hence

$$\gamma = \pi\alpha_0 \equiv \pi\lambda_1/a_1 \bmod \pi^2.$$

If $\hat{q} \neq q$, we obtain $\alpha_0 a_1 \equiv 0$ and hence $\gamma \equiv 0 \bmod \pi^2$. □

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12. A-linearity and Integral Extensions

The purpose of this section is to prove the corollary 12.2 below.

!

Theorem 12.1. *Let B be a subring of A which is a finitely generated Dedekind domain. hypotheses clear? Let (E, φ) be a admissible B-module. Then φ extends to an admissible A-module structure on E in at most one way.*

We note that if E is a Drinfeld module, this theorem follows immediately from basic facts in [D76], section 2. Indeed, $\mathrm{End}_B(E, \varphi)$ is an order in a finite extension of the fraction field of A, which implies that the tangent-space map $\mathrm{End}_B(E, \varphi) \to R$ must be injective; therefore the characteristic map $\theta : A \to R$ can factor through $\mathrm{End}_B(E, \varphi)$ in at most one way. However, we will need to apply the theorem to kernels of the projections $J^1E \to E$, which are admissible A-modules but not Drinfeld modules.

Observe that by transport of structure we have the following:

Corollary 12.2. *Let B and A be as above. Then any B-linear isomorphim between admissible A-modules is in fact A-linear.*

We emphasize that we will apply this only in the proof of theorem 6.4, where A will be unramified over B at $\mathfrak{p}$; and with this restriction, the proofs below simplify considerably.

Let $\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}$ denote the formal completion of $\hat{\mathbb{G}}_{\mathrm{a}}$ along the identity section $\mathrm{Spf}\, R \to \hat{\mathbb{G}}_{\mathrm{a}}$. Thus we have $\mathbb{G}_{\mathrm{a}}^{\mathrm{for}} = \mathrm{Spf}\, R[[x]]$, where $R[[x]]$ has the (π, x)-adic topology. We want to extend the A-action to a map

$$\hat{A} \to \mathrm{End}(\hat{\mathbb{G}}_{\mathrm{a}}^{\mathrm{for}}). \tag{12.1}$$

Recall that $\mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}})$ agrees with the non-commutative power-series ring $R\{\{\tau\}\}$, with commutation law $\tau b = b^{q_2}\tau$ for $b \in R$. Therefore for any $a \in A$, we can write

$$\varphi(a) = \sum_j \alpha_j \tau^j$$

where $\alpha_j \in R$. Each α_j can be thought of as a function of $a \in A$. To construct (12.1) it is enough to prove that these functions are $\mathfrak{p}$-adically continuous, which also implies that such an extension to a continuous $\hat{A}$-action is unique.

Proposition 12.3. *If $a \in \mathfrak{p}^n$, then $\alpha_j \in \mathfrak{p}^{n-j}R$.*

Proof. Clearly, it is true for $n = 0$. Now assume it is true for some given n. Suppose $a \in \mathfrak{p}^{n+1}$ and write $a = \pi b$, where $b \in \mathfrak{p}^n$. Let $\varphi(b) = \sum_j \beta_j \tau^j$ and $\varphi(\pi) = \sum_k \gamma_k \tau^k$. Then we have

$$\varphi(a) = \varphi(\pi)\varphi(b) = \sum_k \gamma_k \tau^k \sum_j \beta_j \tau^j = \sum_{k,j} \gamma_k \beta_j^{q_2{}^k} \tau^{j+k} = \sum_j \alpha_j \tau^j$$

where $\alpha_j = \sum_{k=0}^{j} \gamma_k \beta_{j-k}^{q_2{}^k}$. So to show $\alpha_j \in \mathfrak{p}^{n+1-j}R$, it suffices to show

$$\gamma_k \beta_{j-k}^{q_2{}^k} \in \mathfrak{p}^{n+1-j}R \text{ when } 0 \le k \le j \le n+1.$$

By induction we have $\beta_j \in \mathfrak{p}^{n-j}R$. Thus for $k \ge 1$, we have $\gamma_k \beta_{j-k}^{q_2{}^k} \in \mathfrak{p}^{(n-(j-k))q_2{}^k}R \subseteq \mathfrak{p}^{n-j+1}R$. If $k = 0$, then because φ is a strict module structure, we have $\gamma_0 = \pi$ and hence $\gamma_0\beta_j \in \pi\mathfrak{p}^{n-j}R = \mathfrak{p}^{1+n-j}R$. $\square$

We now consider a local analogue of the setting of theorem 12.1. Let $\hat{B}$ denote a subring of $\hat{A}$ which is a complete discrete valuation ring with maximal ideal $\mathfrak{q} = \mathfrak{p} \cap \hat{B}$ and such that the extension $\hat{A}/\hat{B}$ is finite. (Despite the notation, $\hat{B}$ is not yet the completion of any global object B.)

Theorem 12.4. *Let (E, φ) be a admissible $\hat{B}$-module. Then φ extends to an admissible $\hat{A}$-module structure on E in at most one way.*

The proof, given below, will consider the case where $\hat{A}$ is of the form $\hat{B}[x]/(f)$ where $f(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0 \in \hat{B}[x]$ is a monic irreducible polynomial of various types. Thus we are led to consider the universal extension of the $\hat{B}$-module structure to a $\hat{B}[x]$-module structure: $\tilde{\varphi} : \hat{B}[x] \to \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R[\lambda])$, where $R[\lambda]$ denotes the polynomial ring $R[\lambda_0, \lambda_1, \dots]$ and $\tilde{\varphi}$ is defined by $\tilde{\varphi}(x) = \sum \lambda_j \tau^j$. The proof will then go by showing that there is at most one specialization

$$t : R[\lambda] \to R \tag{12.2}$$

with the property that the induced action $\hat{B}[x] \to \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R)$ factors through the quotient map $s : \hat{B}[x] \to \hat{A}$, thereby inducing an $\hat{A}$-module structure $\varphi^e : \hat{A} \to \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R)$:

$$\begin{tikzcd}
\hat{A} \arrow[r, dashed, "\varphi^e"] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R) \\
\hat{B}[x] \arrow[u, two heads, "s"] \arrow[r, "\tilde{\varphi}"] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R[\lambda]) \arrow[u, "t^*"'] \\
\hat{B} \arrow[u, hook] \arrow[r, "\varphi"] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R) \arrow[u, hook]
\end{tikzcd} \tag{12.3}$$

Lemma 12.5. *Let $f(x)$ be a polynomial in $\hat{B}[x]$. Then for any $i \geq 0$, the coefficient of τ^i in the power series $\tilde{\varphi}(f(x))$ is of the form $C_i(f(x))\lambda_i + D_i(f(x))$, where $C_i(f(x)) \in R[\lambda_0]$ and $D_i(f(x)) \in R[\lambda_0, \ldots, \lambda_{i-1}]$. More precisely, if $f(x) = \sum_m d_m x^m$, then $C_i(f(x))$ is given by the formula*

$$C_i(f(x)) = \sum_m \sum_{l=0}^{m-1} \theta(d_m)\lambda_0^{{q_2}^i(m-l-1)+l}.$$

Proof. Since the morphism sending $f(x)$ to the coefficient of τ^i in $\tilde{\varphi}(f(x))$ is a $\hat{B}$-linear, it is enough to consider polynomials $f(x)$ of the form x^m, for $m \geq 0$. Then we have $\tilde{\varphi}(x^m) = (\sum_j \lambda_j \tau^j)^m$. If we expand the product on the right-hand side, the term of degree i will be the sum of all terms

$$(\lambda_{j_1}\tau^{j_1}) \cdots (\lambda_{j_m}\tau^{j_m}) = (\lambda_{j_1}\lambda_{j_2}^{{q_2}^{j_1}}\lambda_{j_3}^{{q_2}^{j_1+j_2}} \cdots)\tau^i,$$

for $j_1 + \cdots + j_m = i$. If all the j_k are less than i, then the coefficient of the right-hand side is a monomial in $\lambda_0, \ldots, \lambda_{i-1}$. On the other hand, if say j_{l+1} equals i, then all the other j_k are 0; and so the coefficient is

$$\lambda_0^l \lambda_i (\lambda_0^{{q_2}^i})^{(m-(l+1))} = \lambda_0^{{q_2}^i(m-l-1)+l}\lambda_i.$$

Therefore the sum of these coefficients over all choices $j_1, \ldots, j_m$ will be of the form $C_i(f(x))\lambda_i + D_i(f(x))$, where $C_i(f(x)) = \sum_{l=0}^{m-1} \lambda_0^{{q_2}^i(m-l-1)+l}$ and $D_i(f(x)) \in R[\lambda_0, \ldots, \lambda_{i-1}]$, as required. □

Proposition 12.6. *Suppose $t : R[\lambda] \to R$ is a morphism such that $t(\lambda_0) = \pi$. Let $f(x) \in \hat{B}[x]$ be an Eisenstein polynomial of degree n, and assume $\mathfrak{q}\hat{A} \subseteq \mathfrak{p}^n$. Then for every i, there is a unit $u \in R^*$ such that $t(C_i(f(x))) = u\pi^{n-1}$.*

Proof. Write $f(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_0$. From proposition 12.5, we have

$$C_i(f(x)) = \sum_{l=0}^{n-1} \lambda_0^{{q_2}^i(n-l-1)+l} + \sum_{j=1}^{n} \theta(d_{n-j})C_i(x^{n-j})$$

and hence

$$t(C_i(f(x))) = \sum_{l=0}^{n-1} \pi^{{q_2}^i(n-l-1)+l} + \sum_{j=1}^{n} \theta(d_{n-j})t(C_i(x^{n-j})).$$

Since $\theta(d_j) \in \mathfrak{q}R$, we have $\sum_{j=1}^{n} \theta(d_{n-j})C_i(x^{n-j}) \in \mathfrak{q}R \subseteq \pi^n R$. Therefore we can write

$$t(C_i(f(x))) = \sum_{l=0}^{n-1} \pi^{{q_2}^i(n-l-1)+l} + \pi^n v$$

for some $v \in R$. Now in the sum $\sum_{l=0}^{n-1} \pi^{{q_2}^i(n-l-1)+l}$, the valuation of the l-th term is strictly decreasing as a function of l. Thus the minimum is attained at $l = n-1$, and so we have

$$t(C_i(f(x))) = \pi^{n-1} + \pi^n w$$

for some $w \in R$. We can now take our element u to be $1 + \pi w$, which is invertible because R is π-adically complete. □

Proof. (theorem 12.4) Let K/L denote the extension of fraction fields induced by the inclusion $\hat{B} \to \hat{A}$. This is a finite extension of local fields and hence can be written as a tower of extensions of two types: (i) separable and unramified and (ii) totally ramified (and possibly inseparable).
! need reference or proof Therefore it is enough to assume K/L is one of these two types.

(i): K/L is separable and unramified

Write $\hat{A} = \hat{B}[\zeta]$, where $\zeta^{q_2-1} = 1$. To show φ^e is uniquely determined by φ, it is enough to show $\varphi^e(\zeta)$ is uniquely determined by φ. Since (E, φ^e) is admissible, we can write $\varphi^e(\zeta) = \zeta + b$, where $b \in \hat{A}[[\tau]]\tau$. Since we have $\tau\zeta = \zeta^{q_2}\tau = \zeta\tau$, the element ζ lies in the center of the ring $\mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}})$. Therefore we have

$$\zeta + b = \varphi^e(\zeta) = \varphi^e(\zeta^{q_2}) = \varphi^e(\zeta)^{q_2} = (\zeta + b)^{q_2} = \zeta^{q_2} + b^{q_2} = \zeta + b^{q_2}$$

and hence $b = b^{q_2}$. Since $b \in \tau\hat{A}[[\tau]]$, this is possible only if $b = 0$. This implies $\varphi^e(\zeta) = \zeta$ and in particular that $\varphi^e(\zeta)$ is uniquely determined. Therefore so is φ^e.

(ii): K/L is totally ramified

Then we have $\hat{A} \simeq \hat{B}[x]/(f)$, where f is an Eisenstein need reference! polynomial such that π corresponds to the coset of x. So it is sufficient to show that $\varphi^e \circ s(x)$ is uniquely determined in $\mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R)$. Since we have *!*

$$\varphi^e \circ s(x) = t \circ \tilde{\varphi}(x) = t(\sum_j \lambda_j \tau^j) = \sum_j t(\lambda_j)\tau^j,$$

it is sufficient to show that the elements $t(\lambda_j) \in R$ are uniquely determined.

We do this by induction on j. For $j = 0$, it is true because $t(\lambda_0) = s \circ \theta(x) = \pi$, since (E, φ^e) is an admissible A-module. For $j \geq 1$, we may assume $t(\lambda_j)$ is uniquely determined for $j = 0, \dots, i-1$. Then by lemma 12.5, the element $t(\lambda_i)$ satisfies

$$t(C_i(f(x)))t(\lambda_i) + t(D_i(f(x))) = 0.$$

By proposition 12.6, we know that $t(C_i(f(x)))$ is of the form $u\pi^{n-1}$, where $u \in R^*$. Since R is flat, π is not a zero divisor and hence neither is $u\pi^{n-1} = t(C_i(f(x)))$. Therefore $t(\lambda_i)$ is the unique solution in R to the equation above. □

Proof. (theorem 12.1) It is enough to show that if $\varphi, \varphi' : A \to \mathrm{End}(\mathbb{G}_{\mathrm{a}}/R)$ are two A-module structures that agree when restricted to B, then we have $\varphi = \varphi'$.

Let $\hat{B}$ denote the completion of B with respect to $B \cap \mathfrak{p}$. Then we have the following diagram:

$$\begin{tikzcd}
\hat{B} \arrow[r, "\hat{j}"] & \hat{A} \arrow[r, shift left, "\varphi^{\mathrm{for}}"] \arrow[r, shift right, "\varphi'^{\mathrm{for}}"'] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}^{\mathrm{for}}/R) \\
B \arrow[u] \arrow[r, "j"] & A \arrow[u] \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\varphi'"'] & \mathrm{End}(\mathbb{G}_{\mathrm{a}}/R). \arrow[u, "i"']
\end{tikzcd}$$

By assumption, we have $\varphi \circ j = \varphi' \circ j$ and hence $\varphi^{\mathrm{for}} \circ \hat{j} = \varphi'^{\mathrm{for}} \circ \hat{j}$. The equality $\varphi^{\mathrm{for}} = \varphi'^{\mathrm{for}}$ then follows from theorem 12.4. Finally since i is injective, we have $\varphi = \varphi'$. □

13. Computation of λ_1 and γ in the rank 2 case

Theorem 13.1. *Let $A = \mathbb{F}_{q_2}[t]$ with $q_2 \geq 3$, and let E be the Drinfeld module over R of the form*

$$\varphi_E(t)(x) = \pi x + a_1 x^{q_2} + a_2 x^{q_2^2}. \tag{13.1}$$

Let $g = v^*\Theta_2$, and write $\alpha_0 = g'(0)$. Then we have

$$\begin{aligned}
\alpha_0 &\equiv -a_2^{-1}(1 - a_1'(a_1 a_2^{-1}) + a_2'(a_1 a_2^{-1})^{q_2+1})^{q_2-1} \mod \pi \\
\lambda_1 &\equiv a_1 \alpha_0 \mod \pi \\
\gamma &\equiv \pi \alpha_0 \mod \pi^2.
\end{aligned}$$

Observe that when $\varphi_E(t)(x)$ is of the form $\pi x + a x^{q_2} + x^{q_2^2}$, which is always true after changing the coordinate x (perhaps passing to a cover of S), we have the simplified forms

$$\alpha_0 \equiv -(1 - aa')^{q_2-1} \mod \pi \tag{13.2}$$

$$\lambda_1 \equiv -a(1 - aa')^{q_2-1} \mod \pi \tag{13.3}$$

$$\gamma \equiv -\pi(1 - aa')^{q_2-1} \mod \pi^2 \tag{13.4}$$

Proof. Let $\vartheta_1 : N^1 \to \hat{\mathbb{G}}_{\mathrm{a}}$ be the isomorphism defined in theorem 6.4. Then $\vartheta_1 \equiv \tau^0 \mod \pi$. Also ϑ_1 induces the isomorphism $(\vartheta_1)_* : \mathrm{Ext}(E, N^1) \to \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$. In order to determine the action of A on J^1E and J^2E we need to determine how t acts on the coordinates x' and x''. Now we note that $J^nE \simeq W_n$ can be endowed with the δ-coordinates (denoted $[z, z', z'', \dots]$) or the Witt coordinates (denoted $(z_0, z_1, z_2, \dots)$) and they are related by the following in J^2E by proposition 3.2

$$[z,z',z''] = (z,z',z''+\pi^{q_2-2}(z')^{q_2}) \tag{13.5}$$

By taking π-derivatives on both sides of equation (13.1) we get

$$\varphi(t)(x') = \pi' x^{q_2} + a_1' x^{q_2^2} + a_2' x^{q_2^3} + \pi x' + \phi(a_1)\pi^{q_2-1}(x')^{q_2} + \phi(a_2)\pi^{q_2^2-1}(x')^{q_2^2}$$
$$\varphi(t)(x'') = \pi'' x^{q_2^2} + a_1'' x^{q_2^3} + a_2'' x^{q_2^4} + \{\text{terms with } x' \text{ and } x''\} \tag{13.6}$$

Then the A-action $\varphi_{J^1E} : A \to \mathrm{End}(J^1E)$ is given by the 2×2 matrix

$$\varphi_{J^1E}(t) = \begin{pmatrix} \varphi_E(t) & 0 \\ \eta_{J^1E} & \varphi_{N^1}(t) \end{pmatrix}$$

where $\eta_{J^1E} = \pi'\tau + a_1'\tau^2 + a_2'\tau^3$. And by (13.6) and (13.5), the A-action $A \to \mathrm{End}(J^2E)$ is given by the $(1+2)\times(1+2)$ block matrix

$$\varphi_{J^2E}(t) = \begin{pmatrix} \varphi_E(t) & 0 \\ \eta_{J^2E} & \varphi_{N^2}(t) \end{pmatrix}$$

where

$$\eta_{J^2E} = \begin{pmatrix} \pi' x^{q_2} + a_1' x^{q_2^2} + a_2' x^{q_2^3} \\ \Delta(\pi) x^{q_2^2} + \Delta(a_1) x^{q_2^3} + \Delta(a_2) x^{q_2^4} \end{pmatrix}$$

and where $\Delta(z) = z'' + \pi^{q_2-2}(z')^{q_2}$.

Given an extension $\eta_C \in \mathrm{Ext}(G,T)$ and $f : T \to T'$ where G, T and T' are A-modules and f is an A-linear map we have the following diagram of the push-out extension f_*C.

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T' \arrow[r] & f_*C \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

The class of f_*C is obtained as follows- the class of η_C is represented by a linear (not necessarily A-linear) function $\eta_C : G \to T$. Then η_{f_*C} is represented by the class $\eta_{f_*C} = [f \circ \eta_C] \in \mathrm{Ext}(E,T')$. In terms of the action of $t \in A$, $\varphi_C(t)$ is given by $\begin{pmatrix} \varphi_G(t) & 0 \\ \eta_C & \varphi_T(t) \end{pmatrix}$ where $\eta_C : G \to T$. Then $\varphi_{f_*C}(t)$ is given by

$$\begin{pmatrix} \varphi_G(t) & 0 \\ f(\eta_C) & \varphi_{T'}(t) \end{pmatrix}$$

Now we will consider two cases—

(1): Consider $\eta_{\Psi_{1*}(J^1E)} \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ which is the image of Ψ_1 under the connecting morphism $\mathrm{Hom}_A(\hat{\mathbb{G}}_{\mathrm{a}}, \hat{\mathbb{G}}_{\mathrm{a}}) \xrightarrow{\partial} \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ and $\Psi_1 : N^1 \to \hat{\mathbb{G}}_{\mathrm{a}}$ is the isomorphism defined in theorem 6.4 and satisfies $\Psi_1 = \vartheta_1 \circ \mathfrak{f}^{\circ 0} = \tau^0 \bmod \pi$ where $\mathfrak{f}^{\circ 0} = \mathbb{1}$.

$$\begin{tikzcd} 0 \arrow[r] & N^1 \arrow[r] \arrow[d, "\Psi_1"] & J^1E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r] & f_*(J^1E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

where $\eta_{J^1E} = [\pi' x^{q_2} + a_1' x^{q_2^2} + a_2' x^{q_2^3}] \in \mathrm{Ext}(E, N^1)$. Hence

$$\begin{aligned} \eta_{\Psi_{1*}(J^1E)} &= [\vartheta_1(\pi' x^{q_2} + a_1' x^{q_2^2} + a_2' x^{q_2^3})] \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \\ \partial(\Psi_1) &\equiv [x^{q_2} + a_1' x^{q_2^2} + a_2' x^{q_2^3}] \bmod \pi. \end{aligned}$$

(2): Now consider $\eta_{\Psi_{2*}(J^2E)} \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ obtained as

$$\begin{tikzcd} 0 \arrow[r] & N^2 \arrow[r] \arrow[d, "\Psi_2"'] & J^2E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \hat{\mathbb{G}}_{\mathrm{a}} \arrow[r] & f_*(J^2E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

Now we have

$$\eta_{J^2E} = [(\pi' x^{q_2} + a_1' x^{q_2{}^2} + a_2' x^{q_2{}^3}, \Delta(\pi) x^{q_2{}^2} + \Delta(a_1) x^{q_2{}^3} + \Delta(a_2) x^{q_2{}^4})^{\mathrm{T}}] \\ \in \mathrm{Ext}(E, N^2).$$

Let $\Xi(z) = (z')^{q_2} + \pi\Delta(z)$. Then applying $\Psi_2 = \vartheta_1 \circ \mathfrak{f}$ and $\mathfrak{f}(z_1, z_2) = z^{q_2}{}_1 + \pi z_2$, we have

$$\begin{aligned} \partial(\Psi_2) = \eta_{\Psi_{2*}(J^2E)} =& [\vartheta_1(\Xi(\pi) x^{q_2{}^2} + \Xi(a_1) x^{q_2{}^3} + \Xi(a_2) x^{q_2{}^4})] \in \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}}) \\ \partial(\Psi_2) \equiv& [\Xi(\pi) x^{q_2{}^2} + \Xi(a_1) x^{q_2{}^3} + \Xi(a_2) x^{q_2{}^4}] \bmod \pi. \end{aligned}$$

Recall that the map $R\{\tau\}\hat{} \to \mathrm{Ext}(E, \hat{\mathbb{G}}_{\mathrm{a}})$ given by $\eta \mapsto [\eta]$ is surjective and the kernel consists of all η of the form

$$\pi\alpha - \alpha \circ \varphi_E(t),$$

for some $\alpha \in R\{\tau\}\hat{}$ (i.e., the inner derivations). Let us now work out these relations explicitly for $\alpha = \tau^0, \tau^1, \tau^2$.

If $\alpha = \tau^0$ we get the relation

$$\begin{aligned} \pi\tau^0 &= \tau^0(\pi\tau^0 + a_1\tau^1 + a_2\tau^2) \\ \tau^2 &= -(a_1 a_2^{-1})\tau^1 \end{aligned} \tag{13.7}$$

If $\alpha = \tau^1$ we get

$$\begin{aligned} \pi\tau^1 &= \tau^1(\pi\tau^0 + a_1\tau^1 + a_2\tau^2) \\ \tau^3 &= a_2^{-q_2}[(\pi - \pi^{q_2})\tau^1 - a_1^{q_2}\tau^2] \\ \tau^3 &\equiv -(a_1 a_2^{-1})^{q_2}\tau^2 \bmod \pi \\ \tau^3 &\equiv -(a_1 a_2^{-1})^{q_2+1}\tau^1 \bmod \pi \end{aligned} \tag{13.8}$$

If $\alpha = \tau^2$ we get

$$\begin{aligned} \pi\tau^2 &= \tau^2(\pi\tau^0 + a_1\tau^1 + a_2\tau^2) \\ \tau^4 &= a_2^{-q_2{}^2}[(\pi - \pi^{q_2{}^2})\tau^2 - a_1^{q_2{}^2}\tau^3] \\ \tau^4 &\equiv (a_1 a_2^{-1})^{q_2{}^2}\tau^3 \bmod \pi \\ \tau^4 &\equiv (a_1 a_2^{-1})^{q_2{}^2+q_2+1}\tau^1 \bmod \pi \end{aligned} \tag{13.9}$$

Therefore from (13.7), (13.8), (13.9) we get

$$\begin{aligned} \partial(\Psi_1) &\equiv (1 - a_1'(a_1 a_2^{-1}) + a_2'(a_1 a_2^{-1})^{q_2+1})\tau^1 \bmod \pi \\ \partial(\Psi_2) &\equiv -(a_1 a_2^{-1})(1 - a_1'(a_1 a_2^{-1}) + a_2'(a_1 a_2^{-1})^{q_2+1})^{q_2}\tau^1 \bmod \pi \end{aligned}$$

and hence

$$\lambda_1 = \frac{\partial(\Psi_2)}{\partial(\Psi_1)} \equiv -(a_1 a_2^{-1})(1 - a_1'(a_1 a_2^{-1}) + a_2'(a_1 a_2^{-1})^{q_2+1})^{q_2-1} \bmod \pi$$

Now we determine γ and α_0. Write

$$g = \sum_i \alpha_i \tau^i. \tag{13.10}$$

Then from proposition 9.4, we know $\gamma = \pi\alpha_0$. Now we will compute α_0. Let $(z_0, z_1, z_2) := \varphi_{J^2E}(t)(x, 0, 0)$. Then

$$\begin{aligned}\Theta_2(\varphi_{J^2E}(t)(x,0,0)) &= \Psi_2(z_1, z_2) - \lambda_1\Psi_1(z_1) + g(z_0)\\ &= \vartheta_1(z_1^{q_2} + \pi z_2) - \lambda_1\vartheta_1(z_1) + g(z_0)\\ &\equiv z_1^{q_2} - \lambda_1 z_1 + g(z_0) \bmod \pi\end{aligned}$$

where $z_0 = \pi x + a_1 x^{q_2} + a_2 x^{{q_2}^2}$ and $z_1 = \pi' x^{q_2} + a_1' x^{{q_2}^2} + a_2' x^{{q_2}^3}$. On the other hand from the A-linearity of Θ_2 we have

$$\Theta_2(\varphi_{J^2E}(t)(x,0,0)) = \varphi_{\hat{\mathbb{G}}_{\mathrm{a}}}(t)\Theta_2(x,0,0) = \pi\Theta_2(x,0,0) \equiv 0 \bmod \pi$$

and hence $z_1^{q_2} - \lambda_1 z_1 + g(z_0) \equiv 0 \bmod \pi$. Substituting z_0 and z_1 in, we obtain

$$\begin{aligned}(a_2')^{q_2} x^{{q_2}^4} + ((a_1')^{q_2} - \lambda_1 a_2')x^{{q_2}^3} &+ (1 - \lambda_1 a_1')x^{{q_2}^2} - \lambda_1 x^{q_2}\\ &+ g(a_1 x^{q_2} + a_2 x^{{q_2}^2}) \equiv 0 \bmod \pi.\end{aligned} \tag{13.11}$$

Now write $g(x) = \sum_{j\geq 0} \alpha_j x^{{q_2}^j}$. Substituting this into (13.11) and considering the coefficient of x^{q_2}, we obtain $\lambda_1 = \alpha_0 a_1$ and hence $\gamma = \pi\alpha_0 = \pi\lambda_1/a_1$. □

It is possible to determine all the coefficients α_j in (13.10) modulo π as we did α_0. One finds

$$\alpha_1 \equiv a_2^{-q_2}(\lambda_1 a_2' - \alpha_2 a_1^{{q_2}^2} - (a_1')^{q_2}), \qquad \alpha_2 \equiv (a_2')^{q_2} a_2^{-{q_2}^2}, \qquad \alpha_j \equiv 0 \text{ for } j \geq 3.$$

If $a_2 = 1$ and $a_1 = a$, these simplify to

$$\alpha_1 \equiv -(a')^{q_2}, \qquad \alpha_j \equiv 0 \text{ for } j \geq 2.$$