AA1H Calculus Notes
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CHAPTER 1

Introduction

This year I will take a slightly different approach to the material. The main reference will be the 1998 Notes, but adjusted as indicated in the current Notes.

The main difference this year is that sequences will be introduced earlier and will play a more central role. One still gets the same results in the end, but I think the main ideas will be easier to understand in this manner.

CHAPTER 2

The Real Number System

Sections 2.1–3

As before.

Section 2.4

Omit this section.

Thus at this stage we assume the real number system satisfies Axioms 1–13. All the usual properies of addition, multiplication, subtraction, division, and of inequalities, are consequences

As noted in Remark 2.6(98), the set of rational numbers also satifies Axioms 1–13, and hence have all these same properties. (However, certain *existence* properties, such as the existence of a number x for which $x^2 = 2$, are not true of the rationals, see Section 5.)

We will introduce the Completeness Axiom after we discuss sequences. It will be in a different, but equivalent, form to that in the 98 Notes. The Completeness Axiom is true for the set of real numbers but the analogous statement is *not* true for the set of rational numbers.

Section 2.5

In the summary, omit everything except the first sentence *"We prove that $\sqrt{2}$ is irrational"*.

Omit everything in Section 2.5 from Theorem 2.12 onwards, leaving only Theorem 2.11 which shows that $\sqrt{2}$ is irrational. More precisely, what Theorem 2.11 proves is that *if* there is a number whose square is 2 then that number cannot be rational.

(One cannot prove from Axioms 1–13 that there *is* a positive number whose square is 2. The reason is that Axioms 1–13 also hold for the rational numbers, and so if we could prove this fact then it would also be true in the rational numbers, i.e. there would be a a positive *rational* number whose square is 2. But this would contradict Theorem 2.11.)

Sections 2.6–10

As before. Note that the last three sections are marked ★, i.e. extra non-examinable material to put the other material in a broader context.

CHAPTER 3

LIMITS

Omit this chapter. We will return to it later, after first treating sequences.

CHAPTER 4

Sequences

This replaces the chapter from the 98 Notes.

The reference for this chapter is [Adams, Section 10.1], but we do considerably more material than this. Another reference is Chapter 2 of *Fundamentals of Analysis* by Michael Reed; this book will also be the text for the second year course "Analysis and its applications" beginning next year.

4.1. Examples of sequences

We introduce the idea of a sequence and give a few examples.

• A sequence is an infinite list of numbers with a first, but no last, element. Simple examples are

$$1, 2, 1, 3, 1, 4, \ldots$$
$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$
$$1, -1, 1, -1, 1, \ldots$$

• A sequence can be written in the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

• More precisely, a sequence is a function f whose domain is the set of natural numbers, where in the above example $f(n) = a_n$. We often just write (a_n) or $(a_n)_{n\geq 1}$ to represent the sequence.

• If the pattern is clear, we may just write the first few terms, as in the first three examples above.

• The general term a_n may instead be given by a formula, such as

$$a_n = \left(1 + \frac{1}{n}\right)^n,$$

which gives the sequence

$$1 + 1, \left(1 + \frac{1}{2}\right)^2, \left(1 + \frac{1}{3}\right)^3, \ldots.$$

• A sequence may be given by a method for calculating each element of the sequence in terms of preceding elements. One example is the *Fibonacci sequence*

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2} \text{ if } n \geq 3.$$

Hence the sequence is

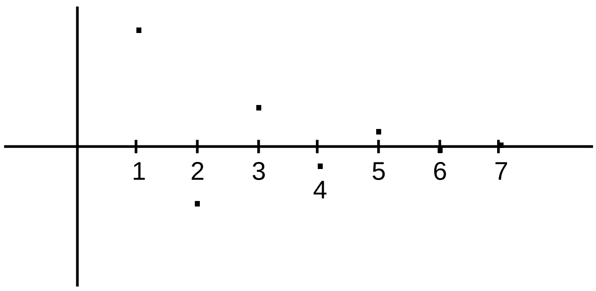
$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

• Sometimes it is convenient to write a sequence in the form

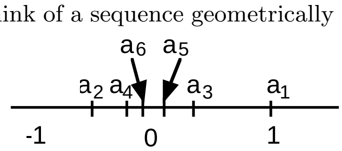
$$a_k, a_{k+1}, a_{k+2}, \ldots$$

where k is some other integer than 1 (e.g. 0).

• One can represent a sequence by its graph. For example the sequence $\big((-1)^{n+1}/n^2\big)$, i.e. $(1, -1/4, 1/9, -1/16, \ldots)$ has graph



where the vertical scale is somewhat distorted. However, this is not usually useful. It is often more helpful to think of a sequence geometrically in terms of points on the real line.



4.2. Limit of sequences

We discuss the idea of the limit of a sequence and give the precise definition. We give some examples.

We are interested in the behaviour of sequences $(a_n)_{n\geq 1}$ for large n. The idea is that "the sequence $(a_n)_{n\geq 1}$ converges to the limit a" if the terms of the sequence get closer and closer to a as n gets larger and larger. This is not exactly what we want, as the terms of the sequence

$$a_1 = 1,\ a_2 = 1\frac{1}{2}, a_3 = 1\frac{2}{3}, a_4 = 1\frac{3}{4}, \ldots$$

get closer and closer to 2, but also to 3 (for example), as n gets larger and larger. Also, the sequences

$$1,\ 1+\frac{1}{2},\ 1-10^{-3},\ 1+\frac{1}{4},\ 1-10^{-5}, 1+\frac{1}{6},\ 1-10^{-7}, \ldots,$$
$$1,\ 1+\frac{1}{2},\ 1+10^{-3},\ 1+\frac{1}{4},\ 1+10^{-5}, 1+\frac{1}{6},\ 1+10^{-7}, \ldots,$$

both converge to 1, but the distance $|a_n - 1|$ between a_n and 1 is not a decreasing function in either case.

A more precise version of what we mean by "the sequence $(a_n)_{n\geq 1}$ converges to the limit a" is that beyond some term in the sequence, all terms are within .1 of a; beyond a further point in the sequence all terms are within .01 of a; beyond a further point in the sequence all terms are within .001 of a; etc.

For example, we may have

$$|a_n - a| \leq .1 \text{ if } n \geq 50,$$
$$|a_n - a| \leq .01 \text{ if } n \geq 300,$$
$$|a_n - a| \leq .001 \text{ if } n \geq 780,$$

etc.

More precisely:

> *no matter which* positive number is chosen (let's call it ε), there is *always* an integer (let's call it N, or $N(\varepsilon)$ to emphasise that it may depend on ε), such that the Nth and all later members of the sequence are within ε of a.

In the above situation we could choose N as follows:

ε	N
.1	50
.01	300
.001	780

For example, according to this criterion, the sequence $1, -1/4, 1/9, -1/16, \ldots$ converges to 0, the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ converges to 0, the sequence $1, -1, 1, -1, 1, \ldots$ does not converge (it "oscillates back and forth between ± 1"), the sequence $1, 2, 1, 3, 1, 4, \ldots$ does not converge and the sequence $1, 4, 9, 16, \ldots$ also does not converge (we sometimes say that it "diverges to $+\infty$").

Often it is not clear whether or not a sequence converges. For example, it is not immediately clear if the sequences $\left((1+\frac{1}{n})^n\right)_{n\geq 1}$, $\left(n \sin \frac{1}{n}\right)_{n\geq 1}$, or the sequence given by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}a_n + 2$ for $n \geq 1$, will converge. For this and more complicated examples we need a precise definition of convergence. Also, in order to prove general theorems about convergent sequences, we need to have a precise definition.

The following definition makes our previous idea very precise.

Definition 4.1. We say that *the sequence* (a_n) *converges to a limit* a, and write

$$\lim a_n = a, \quad \text{or} \quad a_n \to a,$$

if for every positive number ε there exists an integer N such that

$$n \geq N \quad \text{implies} \quad |a_n - a| \leq \varepsilon. \tag{4.1}$$

Note that $|a_n - a| \leq \varepsilon$ is equivalent to $a_n \in [a - \varepsilon, a - \varepsilon]$. Thus another way of expressing the definition is "for every interval $[a-\varepsilon, a+\varepsilon]$ (provided $\varepsilon > 0$) there is an integer N such that all members of the sequence from the Nth onwards belong to this interval".

We sometimes say a_n *converges to* a *as* n *approaches* ∞ and write

$$\lim_{n\to\infty} a_n = a \qquad \text{or} \qquad a_n \to a \quad \text{as} \quad n \to \infty.$$

(Note that ∞ is *not* a number, and the symbol ∞ by itself here has no meaning, just as $\to$ has no meaning by itself.)

Example 4.2. Show that the sequence given by $a_n = 1 + \frac{1}{n^2}$ converges to 1 according to the definition.

Solution. Let $\varepsilon > 0$ be given.

We want to find N such that (4.1) is true with $a = 1$. But

$$|a_n - 1| = \frac{1}{n^2}.$$

Since

$$\frac{1}{n^2} \leq \varepsilon \quad \text{if} \quad n^2 \geq \frac{1}{\varepsilon},$$

i.e.

$$\text{if} \quad n \geq \frac{1}{\sqrt{\varepsilon}},$$

we can take N to be any integer $\geq 1/\sqrt{\varepsilon}$, e.g. take

$$N = \left[\frac{1}{\sqrt{\varepsilon}}\right] + 1,$$

where [ ] denotes "the integer part of". □

Thus if $\varepsilon = .1$ we can take any integer $N \geq 1/\sqrt{.1}$, for example $N = 4$ (or anything larger). If $\varepsilon = .01$ we can take $N = 10$ (or anything larger). If $\varepsilon = .001$ we can take $N = 32$ (or anything larger). But the above proof works of course for *any* $\varepsilon > 0$.

EXAMPLE 4.3. We previously mentioned the sequence given by $a_1 = 1$, and $a_{n+1} = \frac{1}{2}a_n + 2$ for $n \geq 1$.

The first few terms are

$$1, 2.5, 3.25, 3.625, 3.8125, 3.90625, 3.953125, 3.9765625, \ldots.$$

It seems reasonable that the sequence is converging to 4. One way to prove this is as follows.

PROOF. Let $\varepsilon > 0$ be given.

We want to find N such that[1]

$$n \geq N \quad \Rightarrow \quad |a_n - 4| \leq \varepsilon. \tag{4.2}$$

We have a formula for a_{n+1} in terms of a_n, and we first use this to get a formula for $|a_{n+1} - 4|$ in terms of $|a_n - 4|$. Thus

$$|a_{n+1} - 4| = \left|\frac{1}{2}a_n + 2 - 4\right| = \left|\frac{1}{2}a_n - 2\right| = \left|\frac{1}{2}(a_n - 4)\right| = \frac{1}{2}|a_n - 4|.$$

Thus $|a_1 - 4| = 3$, $|a_2 - 4| = 3/2$, $|a_3 - 4| = 3/2^2$, $|a_4 - 4| = 3/2^3$, ... In general[2] $|a_n - 4| = 3/2^{n-1}$.

It follows that

$$|a_n - 4| \leq \varepsilon \quad \text{for those } n \text{ such that} \quad \frac{3}{2^{n-1}} \leq \varepsilon.$$

This last inequality is equivalent to $2^{n-1}/3 \geq 1/\varepsilon$, i.e. $2^{n-1} \geq 3/\varepsilon$, i.e. $(n-1)\ln 2 \geq \ln(3/\varepsilon)$, i.e. $n \geq 1 + \ln(3/\varepsilon)/\ln 2$.

Hence (4.2) is true for

$$N = 1 + \left[\frac{\ln \frac{3}{\varepsilon}}{\ln 2}\right].$$

□

You may object that we used ln, the natural logarithm, in the previous example, but we have not yet shown how to define logarithms and establish their properties from the axioms. This is a valid criticism. But in order to have interesting examples, we will often do this sort of thing.

However, we will not do it when we are establishing the underlying theory. In particular, the development of the theory will not depend on the examples.

EXAMPLE 4.4. Show that $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any real number c and any $p > 0$.

Solution. (See [Adams, Example 4, page 522]). Let $\varepsilon > 0$ be given. Then

$$\left|\frac{c}{n^p}\right| \leq \varepsilon \quad \text{if} \quad n^p \geq \frac{|c|}{\varepsilon}, \quad \text{i.e. if} \quad n \geq \left(\frac{|c|}{\varepsilon}\right)^{1/p}.$$

Thus we can take any integer $N \geq \left(\frac{|c|}{\varepsilon}\right)^{1/p}$, and it then follows that

$$\left|\frac{c}{n^p}\right| \leq \varepsilon \quad \text{if} \quad n \geq N.$$

This implies the required limit exists and equals zero. □

REMARK 4.5. Have a look again at (4.1) and compare the four statements

$$\begin{aligned} n \geq N &\quad \text{implies} \quad |a_n - L| \leq \varepsilon, \\ n \geq N &\quad \text{implies} \quad |a_n - L| < \varepsilon, \\ n > N &\quad \text{implies} \quad |a_n - L| \leq \varepsilon, \\ n > N &\quad \text{implies} \quad |a_n - L| < \varepsilon. \end{aligned}$$

[1] We will often write "⇒" for "implies".

[2] This could easily be proved by induction, but it is not necessary to do so.

These statements are certainly *not* equivalent. However, the following four statements *are* equivalent!

for every positive number ε there exists an integer N such that
$$n \geq N \quad \text{implies} \quad |a_n - a| \leq \varepsilon,$$
for every positive number ε there exists an integer N such that
$$n \geq N \quad \text{implies} \quad |a_n - a| < \varepsilon,$$
for every positive number ε there exists an integer N such that
$$n > N \quad \text{implies} \quad |a_n - a| \leq \varepsilon,$$
for every positive number ε there exists an integer N such that
$$n > N \quad \text{implies} \quad |a_n - a| < \varepsilon.$$

Do you see why? I will discuss this in class. Which version does Adams use?

If the sequence (a_n) does not converge, we say that it *diverges*. This may happen in various ways. There are three examples in the first paragraph on page 4.

There is a special case that is important. This is as in the following definition where for each real number K, $a_n \geq K$ for all sufficiently large n. Note that of all the examples in the first paragraph on page 4, this only occurs for the last of these examples.

DEFINITION 4.6. The sequence (a_n) *diverges to* $+\infty$ if for each real number K there is an integer N such that
$$a_n \geq K \quad \text{for all} \quad n \geq N.$$
We write $a_n \to \infty$ or $\lim a_n = \infty$.

Similarly, (a_n) *diverges to* $-\infty$ if for each real number K there is an integer N such that
$$a_n \leq K \quad \text{for all} \quad n \leq N.$$
We write $a_n \to -\infty$ or $\lim a_n = -\infty$.

(In the case $a_n \to \infty$, think of K as large and positive. In the case $a_n \to -\infty$, think of K as large and negative.)

Note, by the way, that we *never* say "(a_n) converges to ∞" or "(a_n) converges to $-\infty$".

EXAMPLE 4.7. Show from the definition that $2^n \to \infty$.

PROOF. Let K be any (positive) real number.

We want to show that for all sufficiently large n, $2^n \geq K$. But this inequality is equivalent to $n \ln 2 \geq \ln K$, which is equivalent to $n \geq \ln K / \ln 2$. Hence if we let $N = 1 + \left[\frac{\ln K}{\ln 2}\right]$, this last inequality is true whenever $n \geq N$. Hence
$$n \geq N \Rightarrow 2^n \geq K.$$
This means $2^n \to \infty$. □

Note that if $a_n \to \infty$ then $1/a_n \to 0$ (assuming $a_n \neq 0$ in order that $1/a_n$ is defined). This follows from the definitions, because $|1/a_n| \leq \varepsilon$ is equivalent to $|a_n| \geq 1/\varepsilon$. But the latter is true for all sufficiently large n since $a_n \to \infty$, and so the former is also true for all sufficiently large n.

4.3. The Archimedean Axiom

If you look at the 1998 Calculus Notes, Section 2.4, you will see that there is another axiom, the *Completeness Axiom*, as well as the Algebraic and Order axioms for the ral numbers.

In these Notes, we take a different approach, and instead of the Completeness Axiom we introduce *two* axioms:

- The *Archimedean Axiom*,
- The *Cauchy Completeness Axiom*.

The second will be discussed in a later section and the first will be discussed here.

It can be shown that no more axioms are necessary. More precisely, it can be shown that any two models of *all* the axioms are essentially the same (more precisely, they are "isomorphic"). If you are adventurous, this is discussed in the last Chapter of the book *Calculus* by Michael Spivak.

It can also be proved that these two axioms together are equivalent to the Completeness Axiom used in the 1998 Notes. More precisely, using just the algebraic and order axioms, one can prove that

1. (Archimedean Axiom + Cauchy Completeness Axiom) $\Rightarrow$ Completeness Axiom,
2. Completeness Axiom $\Rightarrow$ (Archimedean Axiom + Cauchy Completeness Axiom).

If you are very adventurous, see the book Kripke, *Introduction to Analysis*, p38 Q's 9 &10, where the equivalence is essentially set as an (advanced) exercise.

Here then is the Archimedean Axiom:

Axiom (**Archimedean Axiom**). For every real number x there is a natural number n such that $|x| \le n$.

Here is a simple consequence. If $|x| \le n$, then by properties of inequalities, $|x| < n+1$, $|x| < n+2$, etc. In particular, we can find natural numbers *strictly* greater than $|x|$.

While the axiom may seem obvious, and indeed it is clearly true for the model of the real numbers which we have in our mind, it does not actually follow from the algebraic and order axioms, as I discuss below in a starred comment.

One *difference* between the Archimedean Axiom and the Algebraic and Order Axioms is as follows. The Algebraic and Order Axioms are *either* of the form

1. for all real numbers, some elementary property (such as $a+b = b+a$, or $a+0 = a$, or $a < b \Rightarrow a+c < b+c$) is true, where "elementary" means that the property does not involve any further "quantifiers" (i.e. expressions of the form "for all" or "there exists"); *or* are one of the
2. additive or multiplicative inverse axioms, in which case the real number which is asserted to exist is in fact unique.

On the other hand, the natural number n asserted to exist by the Archimedean Axiom for each real number x is certainly not unique. If $|x| \le n$ is true, then n may be replaced by any larger natural number. Moreover, the set $\mathbb{N}$ of natural numbers (which is needed in the statement of the Archimedean Axiom) also involves a certain level of complexity in its (precise) definition; see the last exercise in Chapter 2 of Spivak.

There is one immediate consequence of the Archimedean Axiom which is quite important.

Corollary 4.8. *For every real number $\varepsilon > 0$ there is a natural number N such that $1/N \le \varepsilon$.*

Proof. Suppose $\varepsilon > 0$. From the Archimedean Axiom there is a natural number N such that $N \ge 1/\varepsilon$. From properties of inequalities it follows that $1/N \le \varepsilon$. $\square$

Consider the ε and N as in the Corollary. If $n \ge N$ then $1/n \le 1/N$ (by properties of inequalities). Since $1/N \le \varepsilon$ it follows that

$$n \ge N \Rightarrow 1/n \le \varepsilon.$$

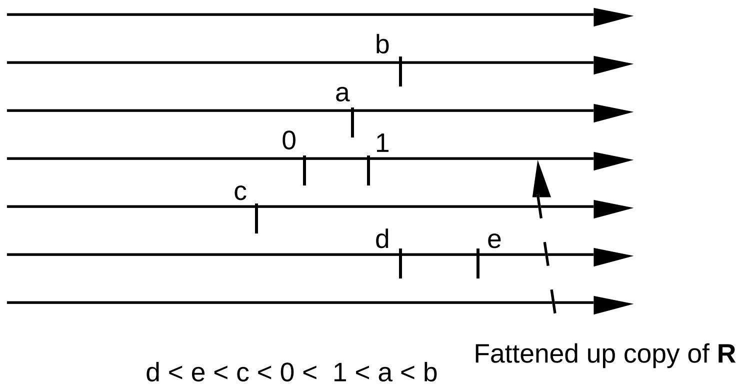
Since $\varepsilon > 0$ was an arbitrary positive number, this is the same as asserting $1/n \to 0$. Thus we have used the Archimedean Axiom to prove $1/n \to 0$.

($\bigstar$ By essentially arguing in the reverse direction we can also prove the converse; namely that the statement $1/n \to 0$ implies the Archimedean Axiom. Thus if we assume just the algebraic and order axioms, the statement $1/n \to 0$ is in fact *equivalent* to the Archimedean Axiom.)

So now I am forced to confess that I cheated in the previous section.[3] In each of Examples 4.2–4.4, 4.7 there was a hidden application of the Archimedean Axiom somewhere in the last three lines of the solution. *Find the hidden application for Easter.*

★ The fact that the Archimedean Axiom is necessary, in the sense that it does not follow from the previous axioms, is a consequence of the fact that there are models of the algebraic and order axioms which do not satisfy the Archimedean Axiom. They are sometimes called the *Hyper-reals*!

Part of any such model looks like a “fattened up” copy of $\mathbb{R}$, in the sense that it contains a copy of $\mathbb{R}$ together with “infinitesimals” squeezed between each real a and all reals greater than a. This part is followed and preceded by infinitely many “copies” of itself, and between any two copies there are infinitely many other copies. See the following crude diagram.



A "number" on any line is less than any number to the right , and less than any any number on any higher line.
Between any two lines there is an infinite number of other lines.

Remark 4.9. *The main point to remember from this section is that in specific examples where we have to show that for any real number of a given form (such as $\frac{1}{\sqrt{\varepsilon}}$ in Example 4.2) there is always an integer N at least as large (N will depend of course on the particular real number $\frac{1}{\sqrt{\varepsilon}}$), the Archimedean Axiom is probably needed. You should be aware of this, but after a few more examples we will adopt our previous cavalier attitude and not explicitly note when the axiom is needed.*

4.4. Properties of limits

We prove that limits of sequences behave as we expect under addition, subtraction, multiplication and division, and we prove the Squeeze Theorem.

It is not usually not very efficient to use the definition of a limit in order to prove that a sequence converges. Instead, we prove a number of theorems which will make things much easier.

The first theorem shows that if two sequences converge, then so does their sum, and moreover the limit of the new sequence is just the sum of the limits of the original sequences. The theorem may be written more briefly as:

$$\textit{if } a_n \to a \textit{ and } b_n \to b \textit{ then } a_n + b_n \to a + b\ .$$

Note that the theorem has two claims; first that $(a_n + b_n)$ *is* convergent, and second that the actual limit is $a + b$.

The result is not very surprising, since if a_n is getting close to a and b_n is getting close to b then we expect that $a_n + b_n$ is getting close to $a + b$. So you may consider

[3] I promise not to do it again, and I only cheated in the examples, not in definitions or theorems.

the theorem as a partial justification that Definition 4.1 does indeed capture the informal notion of a limit.

Notice in the (rather subtle and elegant) proof how the definition of a limit is used three times; once to get information from the fact $a_n \to a$, once to get information from the fact $b_n \to b$, and finally to deduce that $a_n + b_n \to a + b$.

By the way, why do we use $\varepsilon/2$ in (4.3) and (4.4), and why is this justifiable by Definition 4.1?

THEOREM 4.10. *Suppose* (a_n) *and* (b_n) *are convergent sequences with limits* a *and* b *respectively. Then* $(a_n + b_n)$ *is a convergent sequence, and its limit is* $a + b$.

PROOF. Let $\varepsilon > 0$ be given.

Since $a_n \to a$ there exists an integer N_1 (by Definition 4.1) such that

$$n \geq N_1 \quad \text{implies} \quad |a_n - a| \leq \varepsilon/2. \tag{4.3}$$

Since $b_n \to b$ there exists an integer N_2 (again by Definition 4.1) such that

$$n \geq N_1 \quad \text{implies} \quad |b_n - b| \leq \varepsilon/2. \tag{4.4}$$

It follows that if $n \geq \max\{N_1, N_2\}$ then

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \quad \text{by the triangle inequality} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by (4.3) and (4.4)} \\ &= \varepsilon. \end{aligned}$$

It follows from Definition 4.1, with $N = \max\{N_1, N_2\}$, that $(a_n + b_n)$ converges and the limit is $a + b$. □

The next easy result is useful in a number of situations. It is certainly not true in general, consider for example the sequence

$$1, 2, 3, 4, \ldots.$$

THEOREM 4.11. *Suppose* $a_n \to a$. *Then the sequence is bounded; i.e. there is a real number* M *such that* $|a_n| \leq M$ *for all* n.

PROOF. From the definition of convergence, taking $\varepsilon = 1$, there is an integer N such that

$$a - 1 \leq a_n \leq a + 1 \quad \text{for all } n \geq N. \tag{4.5}$$

Fix this N. Since the set of terms

$$a_1, a_2, \ldots, a_{N-1}$$

is *finite*, it follows that there exist real numbers M_1 and M_2 such that

$$M_1 \leq a_n \leq M_2 \quad \text{for all } n < N. \tag{4.6}$$

(Just take $M_1 = \min\{a_1, a_2, \ldots, a_{N-1}\}$ and $M_2 = \max\{a_1, a_2, \ldots, a_{N-1}\}$.)

From (4.5) and (4.6),

$$M_1^* \leq a_n \leq M_2^* \quad \text{for } \textit{all } n,$$

where $M_1^* = \min\{a - 1, M_1\}$, $M_2^* = \max\{a + 1, M_2\}$.

The required result follows by taking $M = \max\{|M_1^*|, |M_2^*|\}$. □

The standard properties about products, quotients etc. of convergent sequences can all be similarly established. We state them together in the following theorem (which includes Theorem 4.10). The proofs are ★ material and are at the end of this section. But you should try to understand them, and I will discuss them in class.

THEOREM 4.12. *Suppose*

$$\lim a_n = a, \quad \lim b_n = b,$$

and c is a real number. Then the following limits exist and have the given values.

$$\begin{aligned} \lim a_n \pm b_n &= a \pm b, \\ \lim ca_n &= ca, \\ \lim a_n b_n &= ab, \\ \lim \frac{a_n}{b_n} &= \frac{a}{b}, \qquad \textit{assuming } b \neq 0 \textit{ and } b_n \neq 0 \textit{ for each } n. \end{aligned}$$

EXAMPLE 4.13. Let $a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^2 - (1 + 2^{-n})$.

We can prove directly from the definition of convergence that $\frac{1}{\sqrt{n}} \to 0$ and $2^{-n} \to 0$. It then follows from the previous theorem that $1 + \frac{1}{\sqrt{n}} \to 1$ (since we can think of $1 + \frac{1}{\sqrt{n}}$ as obtained by adding the term 1 from the constant sequence (1) to the term $\frac{1}{\sqrt{n}}$). Applying the theorem again, $\left(1 + \frac{1}{\sqrt{n}}\right)^2 \to 1$. Similarly, $1 + 2^{-n} \to 1$.

Hence (again from the theorem) $a_n \to 0$.

EXAMPLE 4.14. Let $a_n = \frac{2n^2 - 1}{3n^2 - 7n + 1}$.
Write

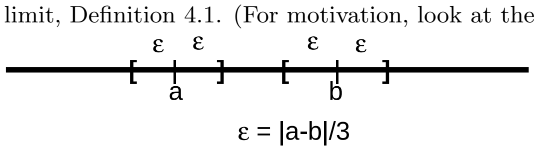
$$\frac{2n^2 - 1}{3n^2 - 7n + 1} = \frac{2 - \frac{1}{n^2}}{3 - \frac{7}{n} + \frac{1}{n^2}}.$$

Since the numerator and denominator converge to 2 and 3 respectively, it follows $a_n \to 0$.

The next theorem says that a sequence cannot have two distinct limits. It is not surprising of course, but note how it does follow from the actual definition of a limit.

THEOREM 4.15. *Suppose $a_n \to a$ and $a_n \to b$. Then $a = b$.*

PROOF. Assume (in order to obtain a contradiction) that $a \neq b$. Take $\varepsilon = |a - b|/3$ in the definition of a limit, Definition 4.1. (For motivation, look at the following diagram).



Since $a_n \to a$, it follows that

$$a_n \in [a - \varepsilon, a + \varepsilon] \tag{4.7}$$

for all sufficiently large n, say for $n \geq N_1$.

Since $a_n \to b$, it follows that

$$a_n \in [b - \varepsilon, b + \varepsilon] \tag{4.8}$$

for all sufficiently large n, say for $n \geq N_2$.

But this implies

$$a_n \in [a - \varepsilon, a + \varepsilon] \quad \text{and} \quad a_n \in [b - \varepsilon, b + \varepsilon]$$

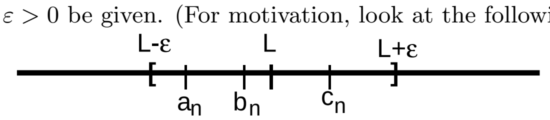
for all $n \geq \max\{N_1, N_2\}$, which is impossible as $\varepsilon = |a - b|/3$.

Thus the assumption $a \neq b$ led to a contradiction and so $a = b$. □

The following theorem says that if a sequence is "squeezed" between two sequences which both converge to the same limit, then the original sequence also converges, and it converges to the same limit.

THEOREM 4.16. *Suppose $a_n \leq b_n \leq c_n$ for all n (or at least for all $n \geq N$ for some N). Suppose $a_n \to L$ and $c_n \to L$ as $n \to \infty$. Then $b_n \to L$ as $n \to \infty$.*

PROOF. Let $\varepsilon > 0$ be given. (For motivation, look at the following diagram).



Since $a_n \to L$ there is some integer N_1 such that

$$n \geq N_1 \Rightarrow a_n \in [L - \varepsilon, L + \varepsilon]. \tag{4.9}$$

Since $c_n \to L$ there is some integer N_2 such that

$$n \geq N_2 \Rightarrow c_n \in [L - \varepsilon, L + \varepsilon]. \tag{4.10}$$

Let N be the larger of N_1 and N_2, i.e. let $N = \max\{N_1, N_2\}$. Then since $a_n \leq b_n \leq c_n$ it follows from (4.9) and (4.10) that

$$n \geq N \Rightarrow b_n \in [L - \varepsilon, L + \varepsilon].$$

But ε was an arbitrary positive number, and so it follows that $b_n \to L$. □

EXAMPLE 4.17. Consider the sequence $3 + (\sin\cos n)/n$.

Since $-1 \leq \sin x \leq 1$, it follows that $3 - 1/n \leq 3 + (\sin\cos n)/n \leq 3 + 1/n$. But $3 - 1/n \to 3$ and $3 + 1/n \to 3$. Hence $3 + (\sin\cos n)/n \to 3$.

We finish this section with the promised proofs of the algebraic properties of limits. Try to understand the ideas, although the material is ★ (well, sort of $\frac{1}{2}$★).

★ PROOF OF THEOREM 4.12.

- We first establish the result for ca_n.

Let $\varepsilon > 0$ be any positive number. We want to show

$$|ca_n - ca| \leq \varepsilon$$

for all sufficiently large n.

Since $a_n \to a$ there exists an integer N such that

$$|a_n - a| \leq \varepsilon/|c| \quad \text{for all} \quad n \geq N.$$

(This assumes $c \neq 0$. But if $c = 0$, then the sequence (ca_n) is the sequence all of whose terms are 0, and this sequence certainly converges to $ca = 0$.) Multiplying both sides of the inequality by $|c|$ we see

$$|ca_n - ca| \leq \varepsilon \quad \text{for all} \quad n \geq N,$$

and so $ca_n \to ca$ by the definition of convergence.

- The result for $a_n - b_n$ now follows easily. Just note that

$$a_n - b_n = a_n + (-1)b_n.$$

But $(-1)b_n \to (-1)b$ by the previous result with $c = -1$, and so the result now follows from Theorem 4.10 about the sum of two sequences.

- The result for a_nb_n uses Theorem 4.11 in the proof.

As usual, let $\varepsilon > 0$ be any positive number. We want to show that

$$|a_nb_n - ab| \leq ab$$

for all $n \geq$ some N.

To see how to choose N, write

$$\begin{aligned} |a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n|\,|b_n - b| + |b|\,|a_n - a|. \end{aligned} \tag{4.11}$$

(This trick of adding and subtracting the same term, here it is a_nb, is often very useful.) We will show that both terms are $\leq \varepsilon/2$ for all sufficiently large n.

For the second term $|b|\,|a_n - a|$, the result is certainly true if $b = 0$, since the term is then 0. If $b \neq 0$, since $a_n \to a$, we can choose N_1 such that

$$|a_n - a| \leq \frac{\varepsilon}{2|b|} \quad \text{for all} \quad n \geq N_1,$$

and so

$$|b|\,|a_n - a| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_1. \tag{4.12}$$

For the first term $|a_n|\,|b_n - b|$, we use Theorem 4.11 to deduce for some M that $|a_n| \leq M$ for all n. By increasing M if necessary take $M \neq 0$. By the same argument as for the second term, we can choose N_2 such that

$$M\,|b_n - b| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_2,$$

and so

$$|a_n|\,|b_n - b| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N_2. \tag{4.13}$$

Putting (4.11), (4.12) and (4.13) together, it follows that if $n \geq N$, where $N = \max\{N_1, N_2\}$, then

$$|a_n b_n - ab| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves $a_n b_n \to ab$.

- We can prove that $a_n/b_n \to a/b$ by first showing $1/b_n \to 1/b$ and then using the previous result about products of sequences.

We first prove the *Claim*: *there is some number $K > 0$ such that $|b_n| > K$ for all n.* The proof is similar to that in Theorem (4.11), and goes as follows:

Since $b \neq 0$ we can choose $\varepsilon = |b|/2$ (> 0) in the definition of convergence and deduce that for some integer N_1,

$$n \geq N_1 \Rightarrow |b_n - b| \leq |b|/2,$$

and so in particular

$$n \geq N_1 \Rightarrow |b_n| \geq |b|/2.$$

Next let

$$c = \min\{|b_1|, \ldots, |b_{N-1}|\}.$$

Then

$$n < N_1 \Rightarrow |b_n| \geq c,$$

where $c > 0$ since $|b_1|, \ldots, |b_{N-1}| > 0$.[4]

Putting the previous results together, and letting $K = \min\{c, |b|/2\} > 0$, we see

$$|b_n| \geq K \tag{4.14}$$

for *all* n. This establishes the *Claim*.

We now proceed with the proof that $1/b_n \to 1/b$. For this let $\varepsilon > 0$ be any positive number.

In order to see how to choose N in the definition of convergence, we compute

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n|\,|b|} \leq \frac{|b - b_n|}{K\,|b|}, \tag{4.15}$$

where for the inequality we have replaced $|b_n|$ by the smaller (but still positive) number K in (4.14).

Since $b_n \to b$ we can find an integer N such that for all $n \geq N$,

$$|b - b_n| \leq K\,|b|\,\varepsilon.$$

[4] This is an important point. If we have an *infinite* set of numbers, such as $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$, all of which are > 0, then there may not be a minimum member of the set. In fact for this example, if $c \leq 1/n$ for all n, we have to take $c = 0$ (or $c < 0$), not $c > 0$.

It follows from (4.15) that if $n \geq N$ then

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $1/b_n \to 1/b$.

Since $a_n \to a$, it now follows from the result for products that $a_n/b_n \to a/b$. □

4.5. Cauchy sequences

A sequence converges iff it is Cauchy. This gives a criterion for convergence which does not require knowledge of the limit. We deduce that a bounded monotone sequence is convergent.

Is there a method (or criterion) for telling if a sequence (a_n) converges, if we do not know the actual limit? Definition 4.1 does not help, as it involves the limit a. But there is indeed such a criterion, and it is due to Cauchy,

Loosely speaking, the Cauchy criterion says that if we go far enough out in the sequence then we can make the members of the sequence as close to each other as we like. *The previous sentence is vague and ambiguous, and you should never ever say or write anything like that in a mathematical argument.* What I mean is that for each $\varepsilon > 0$ there is an N such that any two members of the sequence from the Nth onwards are within ε of each another. More precisely:

Definition 4.18. A sequence (a_n) is a *Cauchy sequence* if for each number $\varepsilon > 0$ there exists an integer N such that

$$|a_m - a_n| \leq \varepsilon \quad \text{whenever} \quad m \geq N \text{ and } n \geq N.$$

If the condition in the above definition is true, that is if the sequence (a_n) is Cauchy, we sometimes write:

$$|a_m - a_n| \to 0 \quad \text{as} \quad m, n \to \infty.$$

Example 4.19. Show that the sequence $a_n = \frac{2n-1}{n+3}$ is Cauchy.

Solution. Let $\varepsilon > 0$ be given.

In order to find N as in the definition, we compute

$$\begin{aligned}
|a_m - a_n| &= \left|\frac{2m-1}{m+3} - \frac{2n-1}{n+3}\right| \\
&= \left|\frac{7(m-n)}{(m+3)(n+3)}\right| \\
&\leq \left|\frac{7m}{(m+3)(n+3)}\right| + \left|\frac{7n}{(m+3)(n+3)}\right| \\
&\leq \frac{7}{n+3} + \frac{7}{m+3} \quad \text{since } \frac{m}{m+3} \leq 1 \text{ and } \frac{n}{n+3} \leq 1 \\
&\leq \frac{7}{n} + \frac{7}{m}.
\end{aligned}$$

Next we note that $\frac{7}{n} \leq \frac{\varepsilon}{2}$ provided $n \geq \frac{14}{\varepsilon}$, i.e. provided $n \geq N$ where $N = \left[\frac{14}{\varepsilon}\right] + 1$. Similarly, $\frac{7}{m} \leq \frac{\varepsilon}{2}$ provided $m \geq N$.

It follows that

$$|a_m - a_n| \leq \varepsilon \quad \text{whenever} \quad m, n \geq N.$$

Hence the sequence is Cauchy. □

It is important to realise that the definition requires more than just that *consecutive* members of the sequence from the Nth onwards are within ε of each other.

For example, it is not too hard to see from the "graph" of the function $\sqrt{x}$ that the following statement is *true*:

for each number $\varepsilon > 0$ there exists N such that

$$n \geq N \quad \textit{implies} \quad |\sqrt{n+1} - \sqrt{n}| \leq \varepsilon.$$

(The main point is that the slope of the graph is $1/\bigl(2\sqrt{x}\bigr)$, and so is very small when x is very large.) The algebraic proof is in footnote[5].

On the other hand, it is clear that the next statement is *false*:
for each number $\varepsilon > 0$ there exists N such that

$$m, n \geq N \quad \textit{implies} \quad |\sqrt{m} - \sqrt{n}| \leq \varepsilon.$$

The algebraic proof is in footnote[6].

The major fact in this section is:

$$\textit{a sequence converges iff it is Cauchy.} \tag{4.16}$$

One direction of (4.16), the fact that a convergent sequence is Cauchy, is easy to prove.

Theorem 4.20. *If a sequence converges then it is Cauchy.*

Proof. Suppose $a_n \to a$. Let $\varepsilon > 0$ be given.
In order to find N as in the definition of a Cauchy sequence, we write

$$|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a_n - a|. \tag{4.17}$$

Since $a_n \to a$ there exists an integer N such that $|a_n - a| \leq \varepsilon/2$ whenever $n \geq N$ (and what is the same thing, $|a_m - a| \leq \varepsilon/2$ whenever $m \geq N$). Hence

$$|a_m - a_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever} \quad m, n \geq N.$$

This proves that (a_n) is Cauchy. □

It turns out that the other direction of (4.16), that every Cauchy sequence is convergent, does not follow from the axioms so far, and so must be assumed as an extra (and final!) axiom.

Axiom (**Cauchy Completeness Axiom**). If a sequence is Cauchy then it is convergent.

The Cauchy Completeness Axiom says, informally, that any sequence which is "trying to converge", really does have something to converge to, and so is indeed convergent in the sense of Definition 4.1. That is, there are no "gaps" in the real numbers.

★ The analogous statement is not true for the rational numbers. That is, if a sequence of rational numbers is Cauchy, then it is not necessarily true that the sequence will converge to a *rational* number. (Of course, it *will* converge to a real number — this is just what the axiom says — but the limit may be irrational.)

---

[5] To show that $\left|\sqrt{n+1} - \sqrt{n}\right| \leq \varepsilon$ for all sufficiently large n, we compute

$$\begin{aligned}
\sqrt{n+1} - \sqrt{n} &= \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \left(\sqrt{n+1} - \sqrt{n}\right) \\
&= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&\leq \frac{1}{\sqrt{n}}.
\end{aligned}$$

It follows that $\sqrt{n+1} - \sqrt{n} \leq \varepsilon$ if $\frac{1}{\sqrt{n}} \leq \varepsilon$, i.e. if $n \geq \frac{1}{\varepsilon^2}$.

[6] It is very important to realise that the statement is false means there is some ("bad") $\varepsilon > 0$ for which there is *no* N such that

$$m, n \geq N \quad \text{implies} \quad |\sqrt{m} - \sqrt{n}| \leq \varepsilon.$$

Thus we have to find just one "bad" ε.

But $\varepsilon = 1$ is bad since there is certainly no N such that

$$m, n \geq N \quad \text{implies} \quad |\sqrt{m} - \sqrt{n}| \leq 1.$$

Thus we have shown the required statement is false! (In fact, in this case any $\varepsilon > 0$ will be "bad".)

For example, take any sequence (a_n) of *rational* numbers which converges to $\sqrt{2}$ (we saw in Section 2.5 that $\sqrt{2}$ cannot be rational). One example of such a sequence is obtained from the decimal expansion of $\sqrt{2} = 1.414213562\ldots$; thus we can take the sequence

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ \ldots\ . \tag{4.18}$$

This sequence of rational numbers converges, and so is Cauchy by Theorem 4.20. But it does not converge to a *rational* number. Thus the analogue of the Cauchy Completeness Axiom is not true in the rational numbers.

In particular, we now see that the Cauchy Completeness Axiom cannot follow from the other algebraic, order and Archimedean axioms. The analogues of all these other axioms hold for the rational numbers. But the analogue of the Cauchy Completeness Axiom does not hold for the rational numbers.

We have now come to the end of the list of axioms. They are the *algebraic, order, Archimedean and Cauchy Completeness axioms*. (To compare the approach here with that in the 1998 Calculus Notes, go back and reread the first four paragraphs in Section 4.3.)

Throughout the rest of the course, we will rarely indicate when we are using the algebraic, order or Archimedean axioms, but we will usually remark if we are using the Cauchy Completeness axiom.

We next show that if a sequence is bounded and increasing (or decreasing) then it is convergent. We do this by proving that such a sequence is Cauchy, and then we use the Cauchy Completeness Axiom to show that the sequence is convergent.

First we give a definition.

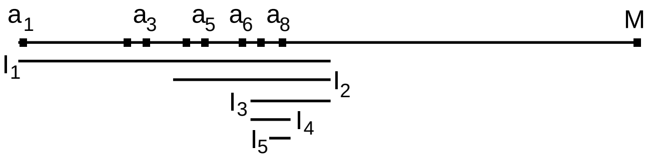
DEFINITION 4.21. A sequence $(a_n)_{n\geq 1}$ is *monotone increasing* if $a_n \leq a_{n+1}$ for all $n \geq 1$. It is is *monotone decreasing* if $a_n \geq a_{n+1}$ for all $n \geq 1$.

Thus the sequences $1, 1, 2, 2, 3, 3, \ldots$ and $1/2, 2/3, 3/4, 4/5, \ldots$ are both monotone increasing.

THEOREM 4.22. *If a sequence is monotone increasing (or decreasing) and bounded, then it is convergent.*

PROOF. Suppose the sequence (a_n) is monotone increasing and bounded. (A similar proof will apply if it is decreasing.) In particular, suppose $a_n \leq M$ for all n.

To motivate the argument look at the following diagram:



Let I be the interval $[a_1, M]$.

Divide I into two closed intervals of equal length. Either all members of the sequence are in the left interval, or at least one member of the sequence is in the right interval (and in the second case, since the sequence is monotone increasing, *all* later members of the sequence are also in the right interval). In *either* case, all members of the sequence after some term are in one of the two subintervals. Choose this subinterval and denote it by I_1. Thus

$$a_n \in I_1 \quad \text{for all } n \geq N_1 \text{ (say)}.$$

(In the diagram, $N_1 = 1$ and I_1 is the left interval; this implies there are no members of the sequence beyond the right endpoint of I_1. There is not enough information just in the diagram to tell us this fact.)

Similarly divide I_1 into two closed intervals of equal length and choose I_2 to be that subinterval which eventually contains all members of the sequence, i.e.

$$a_n \in I_2 \quad \text{for all } n \geq N_2 \text{ (say)}.$$

(In the diagram, $N_2 = 4$ and I_2 is the right interval.)

Similarly divide I_2 into two closed intervals of equal length and choose I_3 to be that subinterval which eventually contains all members of the sequence, i.e.

$$a_n \in I_3 \quad \text{for all } n \geq N_3 \text{ (say)}.$$

(In the diagram, $N_3 = 7$ and I_3 is the right interval.)

etc.

In order to show that (a_n) is Cauchy, let $\varepsilon > 0$ be given.

Since the length[7] of I_k is $(M - a)/2^k$, it follows that there is an integer k such that the length of I_k is $\leq \varepsilon$ [8]. But

$$a_m, a_n \in I_k \quad \text{if} \quad m, n \geq N_k,$$

and so

$$|a_m - a_n| \leq \varepsilon \quad \text{if} \quad m, n \geq N_k.$$

It follows that the sequence (a_n) is Cauchy (take N in Definition 4.18 to equal N_k). Hence (a_n) converges by the Cauchy Completeness Axiom. □

EXAMPLE 4.23. Show the sequence $\sqrt[n]{n}$ is monotone decreasing and bounded below, and hence is convergent.

SOLUTION. As remarked before, in order to have interesting examples we will occasionally use material that has not been rigorously established. We do that here.

So for $x > 0$, let[9]

$$f(x) = x^{1/x} = e^{\frac{1}{x}\ln x}.$$

Then $f(n) = n^{1/n}$ (i.e. $\sqrt[n]{n}$). Moreover,

$$f'(x) = \left(-\frac{1}{x^2}\ln x + \frac{1}{x^2}\right) e^{\frac{1}{x}\ln x},$$

which is negative for $x > e$.

It follows that the sequence $\sqrt[n]{n}$ is decreasing if $n \geq 3$.

It is clear that the sequence is bounded below by 1, i.e. $1 \leq \sqrt[n]{n}$ for all n.

By the previous theorem, the sequence must converge. □

In one of the assignment problems we will see that $\sqrt[n]{n} \to 1$.

★ The analogue of Theorem 4.22 is not true for the rational numbers. In other words, a bounded increasing (or decreasing) sequence of rational numbers need not converge to a *rational* number. (Of course it *will* converge to a real number — this is just what the theorem says.)

The sequence in (4.18) is an example of a bounded increasing sequence of rational numbers which does not converge to a rational number; instead it converges to the irrational number $\sqrt{2}$.

4.6. Subsequences and the Bolzano-Weierstrass Theorem

A sequence need not of course converge, even if it is bounded. But by the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence.

Suppose (a_n) is a sequence of real numbers. A subsequence is just a sequence obtained by skipping terms. For example, the following are subsequences:

$$a_2, a_4, a_6, \ldots,$$

$$a_1, a_{27}, a_{31}, a_{44}, a_{101}, \ldots.$$

We usually write a subsequence of (a_n) as

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots,$$

---

[7] The length of the interval $[a, b]$ is defined to be $b - a$.

[8] ★ This actually requires the Archimedean Axiom. We have $(M - a)/2^{k-1} \leq (M - a)/k$. Now use the Archimedean Axiom to choose k so that $k \geq (M - a)/\varepsilon$.

[9] Recall that if $x > 0$ then $x^a = e^{\ln x^a} = e^{a \ln x}$.

or just as (a_{n_k}). Thus in the above two examples, we have

$$n_1 = 2, n_2 = 4, n_3 = 6, \ldots,$$
$$n_1 = 1, n_2 = 27, n_3 = 31, n_4 = 44, n_5 = 101, \ldots,$$

respectively.

A bounded sequence certainly need not be convergent. For example, the sequences

$$1, -1, 1, -1, 1, -1, 1, -1, \ldots,$$
$$1 + \frac{1}{2}, -1 + \frac{1}{3}, 1 + \frac{1}{4}, -1 + \frac{1}{5}, 1 + \frac{1}{6}, -1 + \frac{1}{7}, \ldots, \tag{4.19}$$

do not converge.

But there are subsequences of each sequence which converge to 1. For example,

$$1, 1, 1, 1, 1, 1, \ldots,$$
$$1, -1, 1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots,$$

are two different subsequences of the first sequence which converge to 1. In fact there are infinitely many different subsequences converging to 1.

Similarly, there are subsequences converging to -1. There are also subsequences of the second sequence converging to 1 and subsequences converging to -1.

The following theorem turns out to have many important applications.

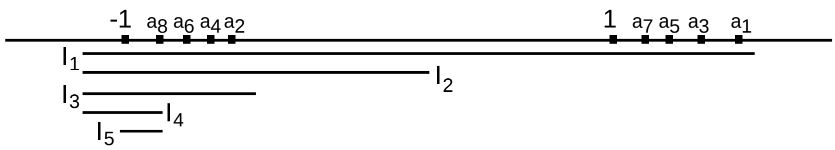
THEOREM 4.24 (Bolzano-Weierstrass Theorem). *If a sequence is bounded then it has a convergent subsequence.*

PROOF. ★ Let (a_n) be bounded. We will construct a convergent subsequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, a_{n_6}, \ldots.$$

Since the sequence (a_n) is bounded, there is a closed bounded interval I_1 which contains all terms from the sequence. Choose one such term and denote it by a_{n_1} ($n_1 = 1$ will do).

(The following diagram, not to scale, corresponds to the sequence (4.19).)



Divide I_1 into two closed intervals of equal length (and having only the midpoint of I_1 in common). At least one of these intervals contains an *infinite* number of different terms[10] from (a_n); call this interval I_2. Choose one such term from (a_n) and denote it by a_{n_2}, but with the condition $n_1 < n_2$. (In the diagram, we could take $n_2 = 2$.)

Divide I_2 into two closed intervals of equal length (and having only the midpoint of I_2 in common). At least one of these intervals contains an *infinite* number of different terms from (a_n); call this interval I_3. Choose one such term from (a_n) and denote it by a_{n_3}, but with the condition $n_2 < n_3$. (In the diagram, we could take $n_3 = 4$.)

Divide I_3 into two closed intervals of equal length (and having only the midpoint of I_3 in common). At least one of these intervals contains an *infinite* number of terms from (a_n); call this interval I_4. Choose one such term from (a_n) and denote it by a_{n_4}, but with the condition $n_3 < n_4$. (In the diagram, we could take $n_4 = 8$.)

etc.

The sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots,$$

is a subsequence of (a_n).

Moreover, it is Cauchy.

[10]For example, a_3 and a_4 are always considered to be different terms, even if they have the same *value*.

To see this, first note that for each N,

$$a_{n_N}, a_{n_{N+1}}, a_{n_{N+2}}, a_{n_{N+3}}, \dots \in I_N.$$

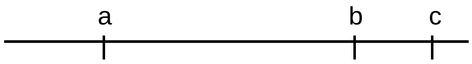
Now let $\varepsilon > 0$ be a given positive number. The length of I_N is $|I_N| = \frac{|I_1|}{2^{N-1}}$. Choose N so $\frac{|I_1|}{2^{N-1}} \leq \varepsilon$ [11]. It follows that

$$|a_{n_p} - a_{n_q}| \leq \varepsilon \quad \text{whenever} \quad p, q \geq N.$$

This means the sequence (a_{n_k}) is Cauchy. Hence it converges. This proves the theorem. $\square$

REMARK 4.25. *If all members of the original sequence (a_n) belong to a closed bounded interval $[a, b]$, then so does the limit of any subsequence.*

PROOF. To simplify notation, denote the subsequence by (x_k). Suppose in order to obtain a contradiction that $x_k \to c$ and $x_k \leq b$ for all k, but $c > b$ (see the next diagram).



Let $\varepsilon = (c - b)/2$, or any number less than the distance from b to c. Eventually all members of the sequence are within ε of c, and so in particular are $> b$. This contradicts the fact that $x_k \leq b$ for all k.

Similarly if $x_k \geq a$ for all k and $x_k \to c$, then $c \geq a$. $\square$

EXAMPLE 4.26. ★ We have seen examples of bounded sequences which have subsequences converging to 1 and subsequences converging to -1. Given m distinct real numbers $x_1, \dots, x_m$ it is easy to see that there is a sequence which has a subsequence converging to x_1, another converging to x_2, ... , and another converging to x_m.

What is more surprising is that there is a sequence, which for each real number $x \in [0, 1]$, has a subsequence converging to x.

The sequence can be constructed as follows. First enumerate all decimal expansions of length one, then all of length two, then all of length three, then all of length four, etc. Thus the sequences is

$$\cdot 0, \cdot 1, \cdot 2, \cdot 3, \dots, \cdot 9, \cdot 00, \cdot 01, \cdot 02, \cdot 03, \dots, \cdot 99, \cdot 000, \cdot 001, \cdot 002, \cdot 003,$$
$$\dots, \cdot 999, \cdot 0000, \cdot 0001, \cdot 0002, \cdot 0003, \dots, \cdot 9999, \cdot 0000, \cdot 0001, \dots.$$

Note that various values are repeated, e.g. $\cdot 2, \cdot 20, \cdot, 200, \dots$ etc., but this does not matter.

By using decimal expansions, we see that for any number x there is a subsequence converging to x. For example, the subsequence converging to $\sqrt{2}/2 = \cdot 7071067810\dots$ is

$$\cdot 7, \cdot 70, \cdot 707, \cdot 7071, \cdot 70710, \cdot 707106, \dots.$$

It is possible to change this construction a little and obtain a sequence which for *every* real number x has a subsequence converging to x. (The sequence cannot be bounded, since it must contain arbitrarily large real numbers.)

---

[11]★ This needs the Archimedean Axiom. It is equivalent to choosing N such that $2^{N-1} \geq |I_1|/\varepsilon$. But since $2^{N-1} \geq N$, for example by induction, it is sufficient to use the Archimedean Axiom to find a natural number $N \geq |I_1|/\varepsilon$.

CHAPTER 5

Continuous Functions

5.1. Definition and examples

The notion of continuity of a function is defined in terms of sequences. Some examples of continuous and discontinuous functions are given. (We also show that between any two numbers there are an infinite number of rationals and an infinite number of irrationals.)

Recall that the domain of a function f, denoted by $\mathcal{D}(f)$, is the set of numbers x such that $f(x)$ is defined. We will usually be interested in functions whoxe domains are intervals $[a, b]$, (a, b), (a, ∞)[1], etc. But it is possible for the domain to be a more complicated set of real numbers.

We will define the notion of continuity in terms of convergence of sequences. The informal idea of continuity of a function f at a point c is that "as x appraoches c then $f(x)$ approaches $f(c)$".

More accurately, we have the following natural definition.

Definition 5.1. A function f *is continuous at a point* $c \in \mathcal{D}(f)$ if for every sequence (x_n) such that $x_n \in \mathcal{D}(f)$ and $x_n \to c$, we have $f(x_n) \to f(c)$.

We say f is *continuous* (on its domain) if f is continuous at every point in its domain.

In other words,

$$x_n \in \mathcal{D}(f) \text{ and } x_n \to c \quad \Rightarrow \quad f(x_n) \to f(c).$$

We will often not write $x_n \in \mathcal{D}(f)$, although this is always understood in order that $f(x_n)$ be defined.

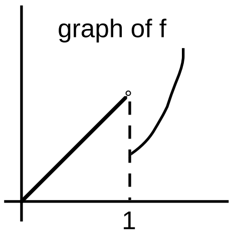
Thus in order to show f *is* continuous at c, we have to show that for *every* sequence $x_n \to c$ one has $f(x_n) \to f(c)$.

In order to show f *is not* continuous at c, we only have to show there is *one* ("bad") sequence $x_n \to c$ with $f(x_n) \not\to f(c)$.[2]

Example 5.2. Suppose

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ \frac{1}{2}x^2 & 1 \le x \le \frac{3}{2} \end{cases}$$

The domain of f is $[0, \frac{3}{2}]$. The following is an attempt to sketch the graph of f.



[1] I emphasise that ∞ is *not* a numbr, and that for us the symbol ∞ has no neaning by itself. The interval (a, ∞) is just the set of real numbers strictly greater than a.

[2] If there is one, there will in fact be many such "bad" sequences — we can always change the first million or so terms — but the point is that to show continuity fails it is sufficient to just prove there is one "bad" sequence.

It is clear that f is not continuous at 1. For example, take the sequence $x_n = 1 - \frac{1}{n}$. Then $x_n \to 1$ but $f(x_n)$ $(= 1 - \frac{1}{n}) \not\to f(1)$ since $f(1) = \frac{1}{2}$.

On the other hand, if $c \neq 1$ and $c \in \mathcal{D}(f)$ then

$$x_n \to c \quad \Rightarrow \quad f(x_n) \to f(c).$$

To see this, first suppose $x_n \to c$ and $1 < c \leq \frac{3}{2}$. Then $x_n \geq 1$ for all sufficiently large n, and so $f(x_n) = \frac{1}{2}x_n^2$ for all sufficiently large n. From properties of sequences if $x_n \to c$ then $x_n^2 \to c^2$ and so $\frac{1}{2}x_n^2 \to \frac{1}{2}c^2$. But $f(x_n) = \frac{1}{2}x_n^2$ for all sufficiently large n, and so $\lim f(x_n) = \lim \frac{1}{2}x_n^2 = \frac{1}{2}c^2$.

In particular, f is not continuous on its domain.

The case $0 \leq c < 1$ is similar, and easier.

If we vary this example a little, and define

$$g(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2}x^2 & 1 < x \leq \frac{3}{2}, \end{cases}$$

then the domain of g is $[0, 1) \cup (1, \frac{3}{2}]$. The function g is continuous at each $c \in \mathcal{D}(g)$, and so *is* continuous on its domain.

Example 5.3. *The absolute value function f (given by $f(x) = |x|$) is continuous.*

We first show continuity at 0. For this, suppose $x_n \to 0$. Then $|x_n| \to 0$ (this is immediate from the definition of convergence, since $|x_n - 0| \leq \varepsilon$ iff $|\,|x_n| - 0\,| \leq \varepsilon$), i.e. $f(x_n) \to f(0)$.

To prove continuity at $c \neq 0$ is similar to the previous example.

The following result is established directly from the properties of convergent sequences.

Proposition 5.4. *Every polynomial function is continuous.*

Proof. Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k.$$

To show that f is continuous at some point c, suppose $x_n \to c$.

Then $x_n^2 \to c^2$, $x_n^3 \to c^3$, etc., by the theorem about products of convergent sequences. It follows that $a_1x_n \to a_1c$, $a_2x_n^2 \to a_2c^2$, $a_3x_n^3 \to a_3c^3$, etc., by the theorem about multiplying a convergent sequence by a constant. Finally,

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \to a_0 + a_1c + a_2c^2 + \cdots + a_kc^k$$

by repeated applications of the theorem about sums of convergent sequences (a_0 is here regarded as a constant sequence). □

Example 5.5. Here is a surprising example.

Let

$$f(x) = \begin{cases} x & x \text{ rational} \\ x & -x \text{ irrational.} \end{cases}$$

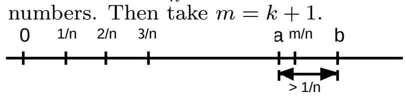
The following diagram is misleading, since *between any two real numbers there is both a rational and an irrational number* (in fact an infinite number of each).[3]

---

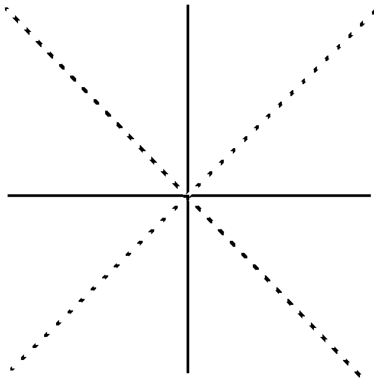
[3]To see this, first suppose $0 < a < b$. Choose an integer n such that $\frac{1}{n} < b - a$ (from the Archimedean axiom!). Then at least one member $\frac{m}{n}$ of the sequence

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \frac{5}{n}, \ldots$$

will lie between a and b. This is very plausible from the diagram. To show it rigorously requires showing there is a largest k such that $\frac{k}{n} < a$, which in turn requires a slightly more careful definition of the natural numbers. Then take $m = k + 1$.



Since we can then obtain another rational between $\frac{m}{n}$ and b, etc., etc., there is in fact an infinite number of rationals between a and b.



The function f is continuous at 0. To see this, suppose $x_n \to 0$. Then $|x_n| \to 0$ (this follows from the definition of a limit). Since $-|x_n| \leq f(x_n) \leq |x_n|$, it follows from the Squeeze theorem that $f(x_n) \to 0$, i.e. $f(x_n) \to f(0)$.

On the other hand, *f is not continuous at c if $c \neq 0$.* For example if c is irrational then we can choose a sequence of rationals x_n such that $x_n \to c$ (by repeated applications of the remark above in italics). It follows that $f(x_n) = x_n \to c \neq f(c)$. Similarly if c is irrational.

We will later define the exponential, logarithm, and trigonometric functions, and show they are continuous. Meanwhile, we will use them in examples (but not in the development of the theory).

5.2. Properties of continuous functions

The basic properties of continuous functions follow easily from the analogous properties of sequences.

Theorem 5.6. *Let f and g be continuous functions and let $D = \mathcal{D}(f) \cap \mathcal{D}(g)$.*[4] *Then*

1. *$f + g$ is continuous on D,*
2. *fg is continuous on D,*
3. *αf is continuous on $\mathcal{D}(f)$ (α any real number)*
4. *f/g is continuous at any point $c \in D$ such that $g(c) \neq 0$.*

Proof. Suppose $c \in D$. Let (x_n) be any sequence such that $x_n \to c$ (and as usual, $x_n \in D$).

Then $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$, since f and g are continuous at c. It follows

$$f(x_n) + g(x_n) \to f(c) + g(c)$$

by Theorem 4.12 about sums of convergent sequences. That is,

$$(f + g)(x_n) \to (f + g)(c).$$

It follows that $f + g$ is continuous at c.

The proof in the other cases is similar. Just note for the case f/g that if $x_n \to c$ and $g(c) \neq 0$, then $g(x_n) \neq 0$ for all sufficiently large n[5]. $\square$

If $a < 0$, a similar argument works with the sequence

$$-\frac{1}{n}, -\frac{2}{n}, -\frac{3}{n}, -\frac{4}{n}, -\frac{5}{n}, \dots$$

Finally, choosing n so $\frac{\sqrt{2}}{n} < b - a$ and applying a similar argument to the sequence

$$\frac{\sqrt{2}}{n}, \frac{2\sqrt{2}}{n}, \frac{3\sqrt{2}}{n}, \frac{4\sqrt{2}}{n}, \frac{5\sqrt{2}}{n}, \dots$$

gives the result for irrational numbers.

[4]If A and B are sets, then their intersection $A \cap B$ is the set of numbers in *both* A and B. Their *union* $A \cup B$ is the set of numbers in *at least one* of A and B.

[5]If $g(c) > 0$, by continuity of g at c and the definition of convergence of a sequence, $g(x_n) \in [\frac{1}{2}g(c), \frac{3}{2}g(c)]$ for all sufficiently large n and so it is positive. The argument in case $g(c) < 0$ is similar.

The composition of two continuous functions is continuous. (See Adams page 35 for a discussion about the composition of functions, or the 1998 Calculus Notes page 16.)

THEOREM 5.7. *Suppose f and g are continuous. Then $f \circ g$ is continuous.*

PROOF. The domain D of $f \circ g$ is the set of numbers x such that both $x \in \mathcal{D}(g)$ and $g(x) \in \mathcal{D}(f)$.

Suppose $c \in D$. Let $x_n \to c$ and $x_n \in D$. It follows that $g(x_n) \to g(c)$ since g is continuous at c. It then follows that $f(g(x_n)) \to f(g(c))$ since f is continuous at $g(c)$ (note that $g(x_n) \in \mathcal{D}(f)$). In other words, $(f \circ g)(x_n) \to (f \circ g)(c)$, and so $f \circ g$ is continuous at c. $\square$

It follows from our results so far that rational functions (quotients of polynomials) and in general functions defined from other continuous functions by means of algebraic operations and composition, will be continuous on their domain.

EXAMPLE 5.8. The function

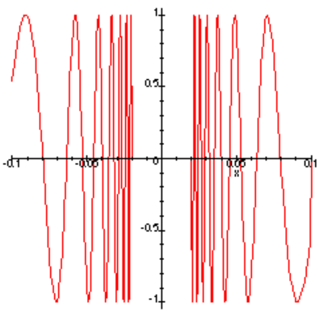
$$f_1(x) = \sin \frac{1}{x}$$

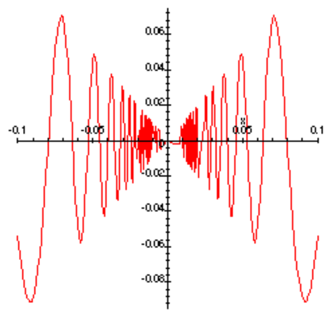
is the composition of the two continuous functions $\sin(x)$ and $1/x$ [6] and so is continuous. The domain of f_1 is the set of real numbers x such that $x \neq 0$. That is, $\mathcal{D}(f_1) = \{\, x \mid x \neq 0 \,\}$.

Similarly, the function

$$f_2(x) = x \sin \frac{1}{x}$$

is continuous on its domain, which is the *same* domain as for f_1.





However, there is an interesting difference between f_1 and f_2. In the case of the latter we can define a new function g_2 by

$$g_2(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\mathcal{D}(g_2) = \mathbb{R}$ and $g_2(x) = f_2(x)$ if $x \neq 0$, i.e. if $x \in \mathcal{D}(f_2)$. Moreover, g_2 is continuous on its domain $\mathbb{R}$.

To show continuity of g_2 at $x \neq 0$, take any sequence $x_n \to x$. For all sufficiently large n, $x_n \in \mathcal{D}(f_2)$, and so $g_2(x_n) = f_2(x_n)$. It follows that $g_2(x_n) \to g_2(x)$ since $f_2(x_n) \to f(x)$ by the continuity of f. This means g_2 is continuous at x if $x \neq 0$.

To show continuity of g_2 at $x = 0$, take any sequence $x_n \to 0$. Then

$$-|x_n| \leq g_2(x_n) \leq |x_n|,$$

---

[6] The notation may seem a bit confusing. You may ask "is it the same x in both cases"? But this is not the right way to look at it. By the function $\sin x$, is meant the function which assigns to each real number x (say) the real number $\sin x$. If we said the function $\sin y$, or just sin, we would mean the same thing.

Similarly, the function $1/x$, or $1/y$, or "the reciprocal function", all mean the same thing.

and so $g_2(x_n) \to 0$ $(= g_2(0))$ by the Squeeze Theorem. (We need to be a bit careful since some of the x_n may equal zero.) This means g_2 is continuous at 0.

In the case of f_1 there is *no* way of extending the function to a continuous function g_1 defined on all of $\mathbb{R}$. This is essentially because there is no number y such that $f_1(x_n) \to y$ for *every* sequence $x_n \to 0$ (with $x_n \neq 0$.)

We sometimes say that f_2 has a *removable singularity* at 0, and that the singularity of f_1 at 0 is *not removable*.

5.3. Supremum of a set

A set which is bounded above need not have a maximum member, but there will always be a supremum. If the supremum belongs to the set then it is just the maximum, if the supremum does not belong to the set then there is no maximum. There is always a sequence of elements from the set which converges to the supremum.

For the important results in the next section, and elsewhere, we need the idea of the supremum of a set.

If we consider sets such as $[1, 2]$ or $(1, 2]$ or the set $A = \{2, 1, 0, -1, \dots\}$ of integers less than or equal to 2, we see that each of these sets contains a largest (or maximum) member, namely 2.

On the other hand, the set $(1, 2)$, or the set B of irrrationals less than 2, does *not* contain a largest member. But we say that 2 is the *supremum* of the set. In the first three examples we also say that 2 is the supremum of the set.

The precise definitions are a follows:

DEFINITION 5.9. Let S be a set of real numbers.

We say a is the *maximum* member of S if

1. $x \le a$ for every $x \in S$,
2. $a \in S$.

We write $a = \max S$.

We say a is the *supremum* (or *sup*) of S if

1. $x \le a$ for every $x \in S$,
2. for every $\varepsilon > 0$ there is at least one member of S in the interval $[a - \varepsilon, a]$.

We write $a = \sup S$.

First note that by the previous definition, 2 is the maximum element of $[1, 2]$, $(1, 2]$ and A. Next note by the definition that 2 is the supremum of these three sets as well as of the sets $(1, 2)$ and B.

Note that if S has a maximum member a, then a is also the supremum of S. This is immediate, since property 1 is the same in each case, and if $a \in S$ then we can just take a as the member of S in $[a - \varepsilon, a]$.

Note that if the supremum a of S exists and is also a member of S, then it is the maximum member of S (this is immediate from the definition of "maximum member").

Note that there can be at most one maximum member of S. For if a_1 and a_2 were both maximum elements[7], then we would have from property 1 that $a_2 \le a_1$ and $a_1 \le a_2$, which implies $a_1 = a_2$. Similarly, there can be at most one supremum of S, again by property 1.

THEOREM 5.10. *Suppose S is a set of real numbers and $a = \sup S$. Then there exists a sequence (x_n) of elements of S such that $x_n \to a$.*

PROOF. From property 2 of the definition of supremum, for each $\varepsilon > 0$ there is a member of S in the interval $[a - \varepsilon, a]$.

For each natural number n there is thus a member of S in $[a - \frac{1}{n}, a]$. For each n, choose one such member and denote it by x_n. Then $x_n \to a$, and we are finished. □

[7]A *member* of an *element* of a set means the same thing.

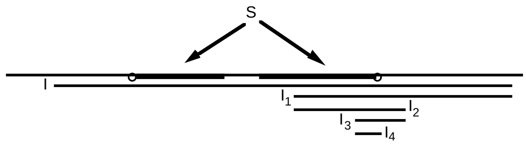
EXAMPLE 5.11. Going back to the five examples at the beginning of this section, we can take $x_n = 2 - \frac{1}{n}$ for each of the sets $[1, 2]$, $(1, 2]$ and $(1, 2)$. For the sets $[1, 2]$ and $(1, 2]$ we could also take the constant sequence $x_n = 2$, and this sequence also works for the set A. For the set B we could take $x_n = 2 - \frac{\sqrt{2}}{n}$.

Although a set need not have a maximum member, even if it is bounded above[8], the following theorem shows that every set which is bounded above does have a supremum. (although it may or may not be a member of S).

THEOREM 5.12. *Let S be set of real numbers which is bounded above. Then S has a supremum.*

PROOF. ★ We use a bisection argument.

Since S is bounded above we can find an interval $I = [a, b]$ such that $x \le b$ for every $x \in S$ and such that there is at least one member of S in I.



Divide I into two closed bounded intervals of equal length, meeting only at the midpoint of I. If the right interval contains at least one member of S, then we take I_1 to be this interval. Otherwise (if the right interval contains no members of S) we take I_1 to be the left interval. Let $I_1 = [a_1, b_1]$. In either case it follows that $x \le b_1$ for every $x \in S$ and there is at least one member of S in I_1.

Divide I_1 into two closed bounded intervals of equal length, meeting only at the midpoint of I_1. If the right interval contains at least one member of S, then we take I_2 to be this interval. Otherwise, (if the right interval contains no members of S) we take I_2 to be the left interval. Let $I_2 = [a_2, b_2]$. In either case it follows that $x \le b_2$ for every $x \in S$ and there is at least one member of S in I_2.

Divide I_2 into two closed bounded intervals of equal length, meeting only at the midpoint of I_2. If the right interval contains at least one member of S, then we take I_3 to be this interval. Otherwise, (if the right interval contains no members of S) we take I_3 to be the left interval. Let $I_3 = [a_3, b_3]$. In either case it follows that $x \le b_3$ for every $x \in S$ and there is at least one member of S in I_3.

Etc.

In this way we construct a sequence of intervals (I_n), each containing the next. That is

$$I_1 \supset I_2 \supset I_3 \supset \cdots .$$

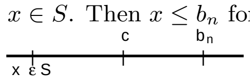
Moreover,

$$a_1 \le a_2 \le a_3 \le \cdots \ldots \cdots \ge b_3 \ge b_2 \ge b_1.$$

Since (a_n) is increasing and bounded it follows that $a_n \to a^*$, say. Similarly $b_n \to b^*$, say.

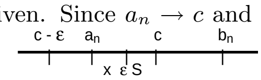
It follows that $a^* = b^*$. To see this, note that for every n we have $a_n \le a^* \le b_n$ and $a_n \le b^* \le b_n$, and so $|a^* - b^*| \le b_n - a_n$. But $b_n - a_n = (b - a)/2^n$ and so $a^* = b^*$, as otherwise we would have a contradiction for sufficiently large n (by the Archimedean Axiom).

We now define $c = a^* = b^*$ and show it is the supremum of S.

To check property 1, suppose $x \in S$. Then $x \le b_n$ for *every* n, and so $x \le \lim b_n$ (see the following proposition). 

. That is, $x \le c$ for all $x \in S$.

[8]Recall that a set S (of real numbers) is *bounded above* if there is a number K such that $x \le K$ for all $x \in S$.

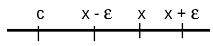
To check property 2, let $\varepsilon > 0$ be given. Since $a_n \to c$ and a_n is increasing, we can choose n so $a_n \in [c-\varepsilon, c]$. 

. From before we have $a_n \le c \le b_n$. There is at least one member $x \in S$ in $I_n = [a_n, b_n]$ by the way we constructed the I_n, and every member of S is $\le c$ by property 1. Hence this member of S is in fact in $[a_n, c]$ and hence in $[c-\varepsilon, c]$. This proves property 2. □

In the previous theorem we used the following easy result, which is also useful elsewhere.

PROPOSITION 5.13. *Suppose* (x_n) *is a convergent sequence and* $x_n \le c$ $(x_n \ge c)$ *for all* n*. Then* $\lim x_n \le c$ $(\lim x_n \ge c$ *).*

PROOF. Let $x = \lim x_n$ and suppose $x_n \le c$ for all n. Suppose, in order to gain a contradiction, that $x > c$.

Let ε be any positive number less than $x - c$ 

. Then for all suficiently large n, $x_n \in [x-\varepsilon, x+\varepsilon]$. This contradicts $x_n \le c$.

Hence we must have $x \le c$.

A similar proof applies for the "$\le$" result. □

Note that if $x_n < c$ for all n, then it does not necessarily follow $\lim x_n < c$; we can still only deduce in general that $\lim x_n \le c$. For example, $-\frac{1}{n} < 0$ for all n, but $\lim \frac{1}{n} = 0$.

REMARK 5.14. We can also define the *minimum member* of a set and the *infimum* of a set, and prove that if a set of real numbers is bounded below, then it has an infimum.

5.4. Three big theorems

A continuous function defined on a closed bounded interval is bounded above and below, and takes a maximum and a minimum value. If a continuous function defined on an interval takes two values, then it takes all values in between.

The theorems in this section are *global*, in that they refer to properties of continuous functions over their entire domain, or assert the *existence* of a point in the domain with a particular property. The require the Bolzano Weierstrass Theorem in their proof, which in turn uses the Cauchy Completeness Axiom.

The properties of continuous functions in Section 5.2 followed from the definitions in a relatively straightforward way. They are *local* properties in that they essentially refer to properties of continuous function near a prescribed point (although the point may be an arbitrary one in the domain).

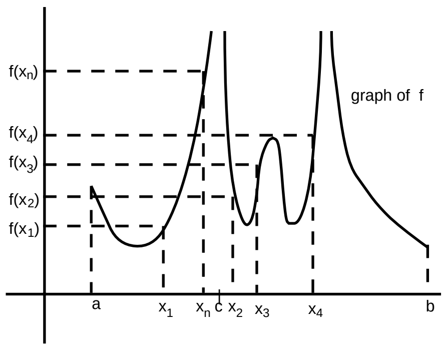
We say a function f defined on a set E is *bounded* on E iff there exists a real number M such that

$$|f(x)| \le M \quad \text{for all} \quad x \in S.$$

THEOREM 5.15. *Suppose f is a continuous function defined on a closed bounded interval. Then f is bounded.*

PROOF. Suppose (in order to obtain a contradiction) that f is a continuous function defined on an interval $[a,b]$ but f is *not* bounded. This means that for each real number M there must be some $x \in [a,b]$ such that $f(x) > M$.

(In the following diagram, f is of course not continuous. The diagram is just to give an indication of the argument used to obtain a contradiction.)



In particular, for each n choose $x_n \in [a, b]$ such that $|f(x_n)| > n$. (Note that $|f(x_n)|$ diverges to infinity.)

By the Bolzano Weierstrass theorem (and the remark which follows it) there is a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $c \in [a, b]$. By continuity of f, $f(x_{n_k}) \to f(c)$. But this contradicts the fact that for each n, $|f(x_n)| > n$.

Hence f *is* bounded. □

REMARK 5.16.

1. The corresponding result is not true for open intervals or unbounded intervals. For example, if $f(x) = 1/x$ for $x \in (0., 1]$, then f is continuous on $(0, 1]$ but is not bounded on $(0, 1]$. Also, if $f(x) = x$ for $x \in [0, \infty)$ then f is continuous on $[0, \infty)$ but is not bounded on $[0, \infty)$.
2. The result is not as obvious as might first appear; it is possible to construct some pretty wild continuous functions.

   ★ For example, there are continuous functions which are *nowhere* differentiable. See Section 5.3 p 55 of the 1998 Calculus Notes
3. ★ The analogous result is not true for the rationals. For example, let $f(x) = 1/(1 - x^2)$ for $1 \leq x \leq 2$. This is continuous at every point other than $x = \sqrt{2}$ (and in particular is continuous at every rational point) and it takes rational values at rational points. However, it is not bounded.

   This explains why we need the Cauchy Completeness Theorem in the proof of the previous theorem (it was used to prove the Bolzano Weierstrass Theorem) — if we only required the other axioms in the proof then because these axioms are true for the rationals it would follow that the analogous result would also be true for the rationals.

We say a function f defined on a set E *takes its maximum value at* $c \in E$ and c is a *maximum point* iff $f(c) \geq f(x)$ for all $x \in [a, b]$. We say f *takes its minimum value at* $d \in E$ and d is a *minimum point* iff $f(d) \leq f(x)$ for all $x \in [a, b]$.

A function can take its maximum or minimum value at more than one point (a constant function is a simple example).

A function can be bounded and not take a maximum or a minimum value. For example, if

$$f(x) = \begin{cases} |x| & -1 \leq x \leq 1,\ x \neq 0 \\ 1 & x = 0, \end{cases}$$

then f is bounded on $[-1, 1]$ but does not take a minimum value.

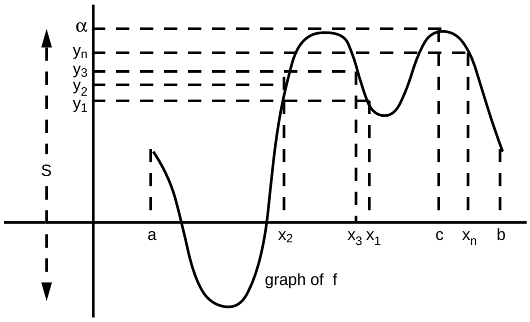
THEOREM 5.17. *Suppose f is a continuous function defined on a closed bounded interval. Then f takes a maximum value and a minimum value.*

PROOF. ★ Suppose f is a continuous function defined on the closed bounded interval $[a, b]$.

Let S be the set of all values taken by f. This means $S = \{\, f(x) \mid x \in [a, b] \,\}$.

The function f is bounded by the previous theorem, and another way of expressing this is to say that the set S is bounded. From Theorem 5.12 it follows that S has a supremum α, say.

We *claim* that α is a member of S, in other words that $\alpha = f(c)$ for some $c \in [a, b]$. This will imply that $f(x) \leq f(c)$ for all $x \in [a, b]$ and so c is a maximum point.



Proof of claim. There exists a sequence (y_n) of elements of S such that $y_n \to \alpha$ (by Theorem 5.10).

For each y_n there exists (at least one) $x_n \in [a, b]$ such that $f(x_n) = y_n$. By the Bolzano Weierstrass Theorem there exists a subsequence (x_{n_k}) of (x_n) which converges to some $c \in [a, b]$. By continuity of f, $f(x_{n_k}) \to f(c)$, i.e. $y_{n_k} \to f(c)$. On the other hand, (y_{n_k}) is a subsequence of (y_n) and so $y_{n_k} \to \alpha$.[9]

Because a sequence can have only one limit (Theorem 4.15) it follows that $f(c) = \alpha$. This proves the claim, and hence that c is a maximum point.

The proof that there is a minimum point is similar. □

Remark 5.18. ★ While it may seem clear that there is a maximum point c, particularly in the case of the diagram, there are some rather wild continuous functions as noted before. Just from the definition of $\sup S$ we can find for each $\varepsilon > 0$ a number $x \in [a, b]$ such that $f(x)$ is within ε of α, even if f is not continuous but is merely bounded. However, to show there is some c such that one actually has $f(c) = \alpha$ requires the continuity of f and uses the Cauchy Completeness Axiom.

The last result implies that if a continuous function defined on an interval I (not necessarily closed or bounded) takes two particular values, then it must take all values between. In other words, for any two points $a, b \in I$ and any γ between $f(a)$ and $f(b)$ then $f(c) = \gamma$ for some $c \in [a, b]$.

Theorem 5.19. *Suppose f is continuous on $[a, b]$. Then for any γ between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that then $f(c) = \gamma$.*

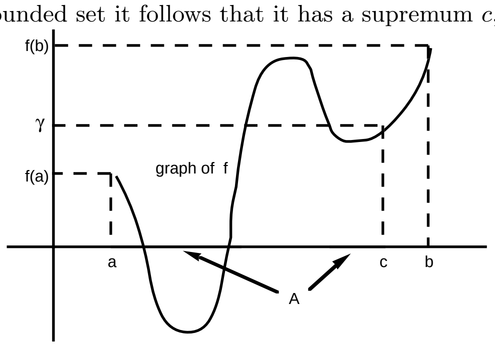
Proof. Suppose $f(a) < \gamma < f(b)$ (the case $f(a) > \gamma > f(b)$ is similar).

In order to prove there is some $c \in [a, b]$ such that $f(c) = \gamma$, let

$$A = \{x \in [a, b] \mid f(x) \leq \gamma\}.$$

(That is, A is the set of all $x \in [a, b]$ such that $f(x) \leq \gamma$.)

Since A is a bounded set it follows that it has a supremum c, say.



We want to show that $f(c) = \gamma$.

[9] It is clear, and not hard to prove, that if a sequence converges then any subsequence converges to the same limit.

There is a sequence $x_n \in A$ such that $x_n \to c$ (Theorem 5.10). By continuity, $f(x_n) \to f(c)$. Since $f(x_n) \leq \gamma$ for all n, it follows $f(c) \leq \gamma$ (Proposition 5.13). (So, in particular, $c \in A$.)

Since $c \neq b$ (because we now know that $f(c) \leq \gamma$, while $f(b) > \gamma$), there is a sequence $x'_n \to c$ and $c < x'_n < b$. But $f(x'_n) > \gamma$ (since otherwise $x'_n \in A$, which contradicts $c = \sup A$) and so $f(c) \geq \gamma$ (Proposition 5.13 again).

Because $f(c) \leq \gamma$ and $f(c) \geq \gamma$ it follows $f(c) = \gamma$. □

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