Notes for MATH1115
Foundations, 2001

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CHAPTER 1

Introduction

The aim of the Foundations part of the course is to give an introduction to modern mathematics. In the process you will prove the major results used in the Calculus part of the course and thereby obtain a more fundamental understanding of that material.

Mathematics is the study of pattern and structure. It is studied both for its universal applicability and its internal beauty. In mathematics we make certain specific assumptions (or axioms) about the objects we study and then develop the consequences of these assumptions in a precise and careful manner. The axioms are chosen because they are "natural" in some sense; it usually happens that these axioms also describe phenomena in other subjects, in which case the mathematical conclusions we draw will also apply to these phenomena.

Areas of mathematics developed for "mathematical" reasons usually turn out to be applicable to a wide variety of subjects; a spectacular recent example being the applications of differential geometry to understanding the fundamental forces of nature studied in physics, and another being the application of partial differential equations and geometric measure theory to the study of visual perception in biology and robotics. There are countless other examples in engineering, economics, and the physical and biological sciences. On the other hand, the study of these disciplines can usually only be done by applying the techniques and language of mathematics. Mathematics is used as a tool in such investigations. But the study of these subjects can also lead to the development of new fields of mathematics and insights into old fields.

In this course we will study the real number system, the concepts of limit and continuity, differentiability and integrability, and differential equations. While most of these terms will be familiar from high school in a more or less informal setting, we will study them in a much more precise way. This is necessary both for applications and as a basis for generalising these concepts to other mathematical settings.

One important question we will investigate next semester is: when do certain types of differential equations have a solution, when is there exactly one solution, and when is there more than one solution? The solution of this problem uses almost all the material that is developed throughout the course. The study of differential equations is of tremendous importance in mathematics and for its applications. Any phenomena that changes with position and/or time is usually represented by one or more such equations.

The ideas we develop are basic to further developments in mathematics. The concepts we study generalise in many ways, such as to functions of more than one variable, and to functions whose variables are themselves functions (!); all these generalisations are fundamental to further applications.

At the end of the first semester you should have a much better understanding of all these ideas.

These Notes are intended so that you can concentrate on the lectures rather than trying to write everything down. Occasionally there may be lecture material not mentioned in the Notes, in which case I will indicate this precisely, but generally you should *not* take your own notes. So why come to the lectures? Because the Notes are frequently rather formal (this is a consequence of the precision of mathematics) and it is often very difficult to see the underlying concepts. In the lectures I explain the material in a less formal manner, single out and discuss the key and underlying ideas, and generally explain and discuss the subject in a manner which it is not possible to do efficiently in print. It would be a *very big mistake* to skip lectures. If you still do not believe me, ask students from my previous courses.

Do not think that you have covered any of this material in school; the topics may not appear new, but the material certainly will be. Do the assignments, read the lecture notes *before* class. Mathematics is not a body of isolated facts; each lecture will depend on certain previous material and you will understand the lectures much better if you keep up with the course. In the end this approach will be more efficient as you will gain more from the lectures.

Throughout the course I will make various digressions and additional remarks, marked clearly by a star *. This is *non-examinable* and generally more challenging material. But you should still read and think about it. It is included to put the subject in a broader perspective, to provide an overview, to indicate further directions, and to generally "round out" the subject. In addition, studying this more advanced material will help your understanding of the examinable material.

References are the text by Adams, the first few chapters of the book *Fundamental Ideas of Analysis* by Michael Reed, and the book *Calculus* by Michael Spivak. The book by Reed is also the text for the second year Analysis course MATH2320.

If you are having difficulty with some of the concepts, ask your tutor or come and see me during office hours. Do not let things slide!

CHAPTER 2

The Real Numbers

A reference for this chapter is Adams, Chapter P, mainly P1, P2 to page 14, P4 to page 29, and P5. However, the treatement here is considefably more sophisticated.

We will begin with a brief discussion of a few properties of the real numbers.

We then discuss the algebraic and order properties and indicate how they follow from 13 basic properties called the *Algebraic and Order Axioms.* Finally we discuss the *Completeness Axiom.* All properties of the real numbers follow from these 14 axioms.

2.1. Preliminary remarks

2.1.1. Decimal expansions. Real numbers have decimal expansions, for example:

$$
\begin{aligned}
&2 = 2.000\ldots \\
&1\frac{1}{2} = 1.5 = 1.5000\ldots \\
&\pi = 3.1459\ldots \\
&.4527\dot{1}4\dot{6}, \qquad \text{also written } .4527\overline{146}
\end{aligned}
$$

The "…" indicate the expansions go on forever, and the $\dot{1}4\dot{6}$ indicate that the pattern 146 is repeated forever. In the first two case the expansion continues with zeros and in the third case one can compute the expansion to any required degree of accuracy.

Instead of counting with base 10, we could count with any other integer base $b \geq 2$. In this case, for integers we write

$$b_1 b_2 \ldots b_n = b^{n-1} b_1 + \cdots + b^2 b_{n-2} + b b_{n-1} + b_n,$$

where each b_i takes values in the set $\{0, 1, \ldots, b-1\}$.(Some primitive societies did count to other than base 10.)

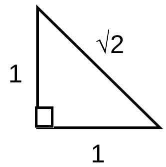
How is 123 *writen in base* 16 *and in base* 2*?*

We can also write nonintegers using base b. In particular,

$$\cdot b_1 b_2 \ldots b_n = \frac{b_1}{b} + \frac{b_2}{b^2} + \cdots + \frac{b_n}{b^n}.$$

2.1.2. Geometric representation. Real numbers can represented geometrically as points on an infinite line.

The ancient Greeks thought of real numbers as *lengths* of lines, and they knew that if x is the length of the hypotenuse of the following right angled triangle, then its square must satisfy $x^2 = 1^2 + 1^2 = 2$ (Pythagoras's theorem). We write $\sqrt{2}$ for this number x.



The Greeks also thought of real numbers as *ratios* of integers (or what we now call *rational numbers*).

So when they discovered the following theorem, they were very upset.

THEOREM 2.1.1. *$\sqrt{2}$ is not rational.*

PROOF. We argue by contradiction. That is, we *assume*

$$\sqrt{2} = m/n$$

where m and n are integers.

Multiplying numerator and denominator by -1 if necessary, we can take m and n to be positive. By canceling if necessary, we can reduce to the situation where m and n have no common factors. Squaring both sides of the equation, we have for these new m and n that

$$2 = m^2/n^2$$

and hence

$$m^2 = 2n^2.$$

It follows that m is even, since the square of an odd number is odd. (More precisely, if m were odd we could write $m = 2r + 1$ for some integer r; but then $m^2 = (2r+1)^2 = 4r^2 + 4r + 1 = 2(2r^2 + 2r) + 1$, which is odd, not even.) But since m is even, we can write

$$m = 2p$$

for some integer p, and hence

$$m^2 = 4p^2.$$

Substituting this into $m^2 = 2n^2$ gives

$$4p^2 = 2n^2,$$

and hence

$$2p^2 = n^2.$$

But now we can argue as we did before for m, and deduce that n is also even. Thus m and n both have the common factor 2, which contradicts the fact they have no common factors.

This contradiction implies that our original assumption was wrong, and so $\sqrt{2}$ is not rational. □

Use a similar argument to prove $\sqrt{3}$ is irrational. HINT: Instead of considering even and odd integers (i.e. remainder 0 or 1 after dividing by 2), you will need to consider integers with remainders 0, 1 or 2 after dividing by 3.

2.1.3. Different decimal expansions for the same number. There is one point that sometimes causes confusion. Is it the case that

$$1 = .\dot{9}\ ?,$$

or is it that $.\dot{9}$ is a "little" less than one? By $.\dot{9}$ we mean, as usual, $.999\ldots$, with the 9's repeated forever.

Any of the approximations to $.\dot{9}$,

$$.9 = \frac{9}{10},\ .99 = \frac{99}{100},\ .999 = \frac{999}{1000},\ .9999 = \frac{9999}{10000},\ \ldots$$

is certainly strictly less than one.

On the other hand, $.\dot{9}$ is defined to be the "limit" of the above infinite sequence (we discuss limits of sequences in a later chapter). Any *mathematically useful* way in which we define the limit of this sequence will in fact imply that $.\dot{9} = 1$. To see this, let

$$a = .\dot{9} = .999\ldots.$$

Then, for any reasonable definition of infinite sequence and limit, we would want that

$$10a = 9.999\ldots.$$

Subtracting, gives $9a = 9$, and hence $a = 1$.

The *only* way a real number can have two decimal expansions is for it to be of the form

$$.a_1a_2\ldots a_{n-1}a_n = .a_1a_2\ldots a_{n-1}(a_n - 1)\dot{9}\ .$$

For example,

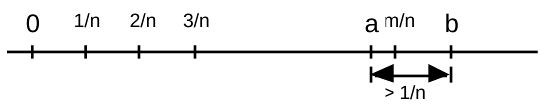
$$.2356 = .2355\dot{9}\ .$$

2.1.4. Density of the rationals and the irrationals. We claim that between any two real numners there is a rational number (in fact an infinite number of them) and an irrational number (in fact an infinite number).

To see this, first suppose $0 < a < b$. Choose an integer n such that $\frac{1}{n} < b - a$. Then at least one member $\frac{m}{n}$ of the sequence

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \frac{5}{n}, \ldots$$

will lie between a and b.



Namely, take the first integer m such that $\frac{m}{n} \geq a$.

Since we can then obtain another rational between $\frac{m}{n}$ and b, and yet another rational between *this* rational and b, etc., etc., there is in fact an infinite number of rationals between a and b.

If $a < 0$, a similar argument works with the sequence

$$-\frac{1}{n}, -\frac{2}{n}, -\frac{3}{n}, -\frac{4}{n}, -\frac{5}{n}, \ldots$$

Finally, choosing n so $\frac{\sqrt{2}}{n} < b - a$ and applying a similar argument to the sequence

$$\frac{\sqrt{2}}{n}, \frac{2\sqrt{2}}{n}, \frac{3\sqrt{2}}{n}, \frac{4\sqrt{2}}{n}, \frac{5\sqrt{2}}{n}, \ldots$$

gives the result for irrational[1] numbers.

[1]The number $\frac{m\sqrt{2}}{n}$ is irrational, because if it were rational then on multiplying by $\frac{n}{m}$ we would get that $\sqrt{2}$ is rational, which we know is not the case.

2.2. Algebraic and Order Properties

We introduce the algebraic and order axioms for the real number system and indicate how all the usual algebraic and order properties follow from these.

2.2.1. Algebraic and Order axioms. The *real number system* consists of the real numbers, together with the two operations *addition* (denoted by $+$) and *multiplication* (denoted by $\times$) and the *less than* relation (denoted by $<$). One also singles out two particular real numbers, *zero* or 0 and *one* or 1.

If a and b are real numbers, then so are $a+b$ and $a \times b$. We say that the real numbers are *closed* under addition and multiplication. We usually write

$$ab \quad \text{for} \quad a \times b.$$

For any two real numbers a and b, the statement $a < b$ is either true or false.

We will soon see that one can define subtraction and division in terms of $+$ and $\times$; and $\le$, $>$ etc. can be defined from $<$.

There are three categories of properties of the real number system: the *algebraic* properties, the *order* properties and the *completeness* property.

We will discuss the completeness property in a later section of this chapter. Here we begin with certain basic algebraic and order properties, usually called the *algebraic and order axioms*, from which we can prove all the other algebraic and order properties of the real numbers.

For all real numbers a, b and c:

1. $a + b = b + a$ (commutative axiom for addition)
2. $(a+b)+c = a+(b+c)$ (associative axiom for addition)
3. $a + 0 = 0 + a = a$ (additive identity axiom)
4. there is a real number, denoted $-a$, such that
   $a + (-a) = (-a) + a = 0$ (additive inverse axiom)
5. $a \times b = b \times a$ (commutative axiom for multiplication)
6. $(a \times b) \times c = a \times (b \times c)$ (associative axiom for multiplication)
7. $a \times 1 = 1 \times a = a$, moreover $0 \neq 1$ (multiplicative identity axiom)
8. if $a \neq 0$ there is a real number, denoted a^{-1}, such that
   $a \times a^{-1} = a^{-1} \times a = 1$ (multiplicative inverse axiom)
9. $a \times (b+c) = a \times b + a \times c$ (distributive axiom)
10. exactly one of the following holds:
    $a < b$ or $a = b$ or $b < a$ (trichotomy axiom)
11. if $a < b$ and $b < c$, then $a < c$ (transitivity axiom)
12. if $a < b$ then $a + c < b + c$ (addition and order axiom)
13. if $a < b$ and $0 < c$, then $a \times c < b \times c$ (multiplication and order axiom)

There are a number of points that need to be made at this stage, before we proceed to discuss the consequences of these axioms

- By the symbol "=" for equality we mean "denotes the same thing as", or equivalently, "is the same real number as". We take "=" to be a *logical notion* and do not write down axioms for it.[2] Instead, we use any properties of "=" which follow from its logical meaning. For example: $a = a$; if $a = b$ then $b = a$; if $a = b$ and $b = c$ then $a = c$; if $a = b$ and something is true of a then it is also true of b (since a and b denote the same real number!).

  When we write $a \neq b$, we just mean that a is *not* the same real number as b.

[2]* One *can* write down basic properties, i.e. axioms, for "=" and the logic we use. See later courses on the foundations of mathematics.

- We are not really using subtraction in axiom 4; we are merely asserting that a real number, with a certain property, exists. It is convenient to denote this number by $-a$. A similar remark applies for axiom 8.
- The assertion $0 \neq 1$ in axiom 7 may seem silly. But it does not follow from the other axioms, since all the other axioms hold for the set containing just the number 0.
- Some of the axioms are redundant. For example, from axiom 1 and the property $a + 0 = a$ it follows that $0 + a = a$. Similar comments apply to axiom 4; and because of axiom 5 to axioms 7 and 8.

2.2.2. Algebraic consequences. All the usual algebraic properties of the real numbers follow from Axioms 1–9. In particular, one can solve simultaneous systems of linear equations. We will not spend much time on indicating how one deduces algebraic properties from these axioms, but will continue to use all the usual properties of addition, multiplication, subtraction and division that you have used in the past.

None-the-less, it is useful to have some idea of the methods involved in making deductions from axioms.

2.2.2.1. *Sum of three or more numbers.* The expression $a + b + c$ is at first ambiguous. Does it mean $(a+b)+c$ or $a+(b+c)$? The first expression means add a to b, then add c to the result; the second means add a to (the result of adding b to c). By the associative axiom, the result is the same in either case, and so we can define $a+b+c$ to be *either* $(a+b)+c$ or $a+(b+c)$.[3]

Using also the commutative axiom we also have

$$(a+b)+c = (b+a)+c = b+(a+c) = b+(c+a),$$

etc.

Similar remarks apply to the product of three or more numbers.

2.2.2.2. *Subtraction and Division.* We use the axioms to *define* these operations in terms of addition and multiplication by

$$\begin{aligned} a-b &= a+(-b) \\ a \div b &= a \times b^{-1} \quad \text{for } b \neq 0. \end{aligned}$$

This may look like a circular definition; it may appear that we are defining "subtraction" in terms of "subtraction". But this is not the case. Given b, from Axiom 4 there is a certain real number, which we denoted by $-b$, with certain properties. We then define $a-b$ to be the *sum* of a and this real number $-b$.

Similar comments apply to the definition of division. We also write a/b or $\frac{a}{b}$ for $a \div b$. Division by 0 is never defined.

2.2.2.3. *Other definitions.* We can also now define other numbers and operations. For example, we define $2 = 1+1$, $3 = 2+1$, etc.

Also, we define $x^2 = x \times x$, $x^3 = x \times x \times x$, $x^{-2} = \left(x^{-1}\right)^2$, etc. etc.

2.2.2.4. *Cancellation property of addition.* As an example of how to use Axioms 1–9 to derive other algebraic properties, we prove the cancellation property of addition, which says informally that if $a+c = b+c$ then we can "cancel" the number c.

Theorem 2.2.1 (Cancellation Theorem). *If a, b and c are real numbers and $a+c = b+c$, then $a = b$. Similarly, if $c+a = c+b$ then $a = b$.*

Proof. Assume

$$a+c = b+c.$$

[3] A similar remark is not true for subtraction, since $(a-b)-c$ and $a-(b-c)$ are in general not equal.

Since $a + c$ and $b + c$ denote the same real number, we obtain the same result if we add $-c$ to both; i.e.

$$(a + c) + (-c) = (b + c) + (-c).$$

(This used the existence of the number $-c$ from Axiom 4.) Hence

$$a + (c + (-c)) = b + (c + (-c))$$

from Axiom 2 applied twice, once to each side of the equation. Hence

$$a + 0 = b + 0$$

from Axiom 4 again applied twice. Finally,

$$a = b$$

from Axiom 3.

If $c + a = c + b$, then from the commutative axiom $a + c = b + c$, and we have just seen that this implies $a = b$. □

2.2.2.5. *Characterisation of* 0 *and of* $-a$*.* One of the axioms is that 0 has the property:

$$a + 0 = 0 + a = a \tag{1}$$

for every real number a. Does this property characterise 0? In other words, is there any other real number x with the property that

$$a + x = x + a = a \tag{2}$$

for every real number a?

Of course we know the answer, but the point here is that the answer follows from the axioms.

In fact from (1) and (2) we have $a + 0 = a + x$, and so from (the second part of) Theorem 2.2.1 with c, a, b there replaced by $a, 0, x$ respectively, it follows that $0 = x$.

One of the axioms asserts that for each number a there is a number (denoted by $-a$) which when added to a gives zero. But could there be another number which when added to a also gives zero? We know the answer is no, but this fact does not need to be asserted as a separate axiom, because it also follows from the existing axioms.

In fact, suppose $x + a = 0$. Because we already know $(-a) + a = 0$ for some specific number $-a$, it follows that $x + a = (-a) + a$. We can "cancel" the a by Theorem 2.2.1, giving $x = -a$.

2.2.2.6. *More algebraic consequences.* Certain not so obvious "rules", such as "the product of minus times minus is plus" and the rule for adding two fractions, follow from the axioms. If we want the properties given by Axioms 1–9 to be true for the real numbers (and we do), then there is no choice other than to have $(-a)(-b) = ab$ and $(a/c) + (b/d) = (ad + bc)/cd$ (see the following theorem).

We will not emphasise the idea of making deductions from the axioms, and for this reason I have marked the proofs of the assertions in the following theorem as * material. Nonetheless, you should have some appreciation of the ideas involved, and so you should work through a couple of the proofs.

THEOREM 2.2.2. *If* a, b, c, d *are real numbers and* $c \neq 0$, $d \neq 0$ *then*

1. $ac = bc$ *implies* $a = b$.
2. $a0 = 0$
3. $-(-a) = a$
4. $(c^{-1})^{-1} = c$
5. $(-1)a = -a$

6. $a(-b) = -(ab) = (-a)b$
7. $(-a) + (-b) = -(a + b)$
8. $(-a)(-b) = ab$
9. $(a/c)(b/d) = (ab)/(cd)$
10. $(a/c) + (b/d) = (ad + bc)/cd$

PROOF*. Each line in the following proofs will be

1. an example of one (or occasionally more) of axioms 1–9;
2. a previously proved result;
3. or follow from previously proved results by rules of logic[4] (which include the properties of equality).

Fill in any missing steps. Go through the proofs line by line and indicate what is used to justify each step.

1. Write out your own proof, following the ideas of the proof of the similar result for addition.

2. The trick here is to use the fact $0 + 0 = 0$ (from A3), together with the distributive axiom. The proof is as follows:
One has $a(0 + 0) = a0$
But the left side equals $a0 + a0$
and the right side equals $0 + a0$
Hence $a0 + a0 = 0 + a0$
Hence $a0 = 0$.

3. We want to show $-(-a) = a$.
By $-(-a)$ we mean the negative of $-a$, and hence by Axiom 4 we know that[5]

$$(-a) + (-(-a)) = 0.$$

But from Axiom 3

$$(-a) + a = 0$$

Hence

$$(-a) + (-(-a)) = (-a) + a$$

and so $-(-a) = a$ from the Cancellation Theorem.

4. Write out your own proof, along similar lines to the preceding proof. You should first prove a cancellation theorem for multiplication.

5. (As in the proof of **2.**) it is sufficient to show $(-1)a + a = 0$, because then

$$(-1)a + a = (-a) + a \quad \text{(additive inverse axiom)}$$

and so

$$(-1)a = -a$$

by the Cancellation Theorem.

[4]For example, if we prove that some statement P implies another statement Q, and if we also prove that P is true, then it follows from rules of logic that Q is true.

[5]Since a can represent *any* number in Axiom 4, we can replace a in Axiom 4 by $-a$. This might seem strange at first, but it is quite legitimate.

The proof is as follows:

$$\begin{aligned}(-1)a + a &= (-1)a + 1a \\ &= a((-1)+1) \quad (\textit{two} \text{ axioms were used for this step}) \\ &= a0 \\ &= 0\end{aligned}$$

This completes the proof.

6.

$$\begin{aligned}a(-b) &= a((-1)b) \\ &= (a(-1))b \\ &= ((-1)a)b \\ &= (-1)(ab) \\ &= -(ab)\end{aligned}$$

Prove the second equality yourself.

7. Prove this yourself using, in particular, use **4** and Axiom 9.

8.

$$\begin{aligned}(-a)(-b) &= ((-1)a)(-b) \\ &= (-1)(a(-b)) \\ &= -(a(-b)) \\ &= -(-(ab)) \\ &= ab\end{aligned}$$

9. First note that $(a/c)(b/d) = (ac^{-1})(bd^{-1})$
and $(ab)/(cd) = (ab)(cd)^{-1}$.
But $(ac^{-1})(bd^{-1}) = (ab)(c^{-1}d^{-1})$
(fill in the steps to prove this equality; which involve a number of applications of Axioms 5 and 6).

If we can show that $c^{-1}d^{-1} = (cd)^{-1}$ then we are done.
Since, by Axiom 8, $(cd)^{-1}$ is the *unique* real number such that $(cd)(cd)^{-1} = 1$, it is sufficient to show[6] that $(cd)(c^{-1}d^{-1}) = 1$.
Do this; use A5–A8

This completes the proof.

Important Remark: There is a tricky point in the preceding that is easy to overlook; but will introduce some important ideas about logical reasoning.
We used the number $(cd)^{-1}$.
To do this we need to know that $cd \neq 0$.
We know that $c \neq 0$ and $d \neq 0$ and we want to prove that $cd \neq 0$.
This is *equivalent* to proving that if $cd \neq 0$ is false, i.e. if $cd = 0$, then at least one of $c \neq 0$ and $d \neq 0$ is false, i.e. at least one of $c = 0$ or $d = 0$ is true.
In other words, *we want to show that if $cd = 0$ then either $c = 0$ or $d = 0$ (possibly both).*

The argument is written out as follows:

Claim: If $c \neq 0$ and $d \neq 0$ then $cd \neq 0$

[6] When we say "it is sufficient to show ... " we mean that if we can show ... then the result we want will follow.

We will establish the *claim* by proving that if $cd = 0$ then $c = 0$ or $d = 0$.[7] There are two possibilities concerning c;

either $c = 0$, in which case we are done
or $c \neq 0$. But in this case, since $cd = 0$, it follows
$c^{-1}(cd) = c^{-1}0$ and so
$d = 0$
why?; fill in the steps

Thus we have shown that if $cd = 0$ then $c = 0$ or $d = 0$. Equivalently, if $c \neq 0$ and $d \neq 0$, then $cd \neq 0$. This completes the proof of the claim.

10. *Exercise*
HINT: We want to prove

$$ac^{-1} + bd^{-1} = (ad + bc)(cd)^{-1}.$$

First prove that

$$(ac^{-1} + bd^{-1})(cd) = ad + bc.$$

Then deduce the required result.

□

2.2.3. Order consequences. All the standard properties of inequalities for the real numbers follow from Axioms 1–13.

2.2.3.1. *More definitions.* One defines "$>$", "$\leq$" and "$\geq$" in terms of $<$ as follows:

$$\begin{aligned} &a > b \text{ if } b < a, \\ &a \leq b \text{ if } (a < b \text{ or } a = b), \\ &a \geq b \text{ if } (a > b \text{ or } a = b). \end{aligned}$$

(Note that the statement $1 \leq 2$, although it is not one we are likely to make, is indeed true, *why*?)

We define $\sqrt{b}$, for $b \geq 0$, to be that number $a \geq 0$ such that $a^2 = b$. Similarly, if n is a natural number, then $\sqrt[n]{b}$ is that number $a \geq 0$ such that $a^n = b$. To prove there *is* always such a number a requires the "completeness axiom" (see later). (To prove that there is a *unique* such number a requires the order axioms.)

If $0 < a$ we say a is *positive* and if $a < 0$ we say a is *negative*.

2.2.3.2. *Some properties of inequalities.* The following are consequences of the axioms, although we will not stop to prove them.

THEOREM 2.2.3. *If a, b and c are real numbers then*

1. $a < b$ *and* $c < 0$ *implies* $ac > bc$
2. $0 < 1$ *and* $-1 < 0$
3. $a > 0$ *implies* $1/a > 0$
4. $0 < a < b$ *implies* $0 < 1/b < 1/a$
5. $|a + b| \leq |a| + |b|$ *(triangle inequality)*
6. $||a| - |b|| \leq |a - b|$ *(a consequence of the triangle inequality)*

2.2.4. Our approach henceforth. From now on, unless specifically noted or asked otherwise, we will use all the standard algebraic and order properties of the real numbers that you have used before. In particular, we will use any of the definitions and results in Adams Section P1

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[7] *Note*; in mathematics, if we say P or Q (is true) then we *always* include the possibility that *both* P and Q are true.

2.2.5. Natural and Rational numbers. The set $\mathbb{N}$ of natural numbers is defined by

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

Here $2 = 1 + 1$, $3 = 2 + 1$, $\dots$. (Thus $\mathbb{N}$ is described by listing its members.)

The set $\mathbb{Z}$ of integers is defined by

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The set $\mathbb{Q}$ of rational numbers is defined by

$$\mathbb{Q} = \{\, m/n \mid m, n \in \mathbb{Z},\ n \neq 0 \,\}.$$

(We read the right side after the equality as "the set of m/n such that m, n are members of $\mathbb{Z}$ and $n \neq 0$.)

A real number is *irrational* if it is not rational. It can be proved that π and e are irrational, see *Calculus* by M. Spivak.

The set $\mathbb{N}$ is not a model of Axiom 3, as 0 is not a member. The set $\mathbb{Z}$ is a model of all of Axioms 1–13, except for Axiom 8, since the multiplicative inverse a^{-1} of an integer is not usually an integer.

The set $\mathbb{Q}$ is a model of all of Axioms 1–13 but not of the Completeness Axiom (see later).

2.2.6. Fields*.

The real numbers and the rationals, as well as the integers modulo a fixed prime number, form a field.

Any set S, together with two operations $\oplus$ and $\otimes$ and two members $0_\oplus$ and $1_\otimes$ of S, which satisfies the corresponding versions of Axioms 1–9, is called a *field*.

Thus $\mathbb{R}$ (together with the operations of addition and multiplication and the two real numbers 0 and 1) is a field. The same is true for $\mathbb{Q}$, but not for $\mathbb{Z}$ since Axiom 8 does not hold, *why?*.

An interesting example of a field is the set

$$F_p = \{0, 1, \dots, p-1\}$$

for any fixed *prime* p, together with addition and multiplication defined "modulo p"; i.e. one performs the usual operations of addition and multiplication, but then takes the "remainder" after dividing by p.

Thus for $p = 5$ one has:

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

It is not too hard to convince yourself that the analogues of Axioms 1–9 hold for any prime p. The axiom which fails if p is not prime is Axiom 8, *why?*. Note that since F_p is a field, we can solve simultaneous linear equations in F_p.

The fields F_p are very important in coding theory and cryptography.

2.2.7. Groups*. Any set S, together with an operation $\otimes$ and a particular member $e \in S$, which satisfies:

for all $a, b, c \in S$:

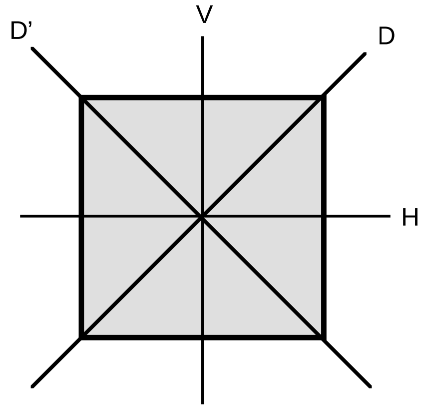
1. $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (associative axiom)
2. $a \otimes e = e \times a = a$ (identity axiom)

3. there is a member of S, denoted a^{-1}, such that $a \otimes a^{-1} = a^{-1} \otimes a = e$ (inverse axiom)

Examples are the reals or rationals, with $\otimes$ replaced by $\times$ and e replaced by 1, or with $\otimes$ replaced by $+$ and e replaced by 0. Another example is $\mathbb{Z}$ with $\otimes$ and e replaced by $+$ and 0.

The notion of a group pervades much of mathematics and its applications. Important examples are groups of transformations and applications to classification of crystals.

As a simple case, consider a square.



If the square is reflected in any of the four axes V (vertical axis), H (horizontal axis), D or D' (two diagonals), then the image coincides with the original square. We denote these four transformations by V, H, D, D' respectively.

We could also rotate the square through 90^0, 180^0 or 270^0; these operations are denoted by R, R' and R'' respectively. Finally, we could rotate through 360^0, which has the same effect as doing nothing; this operation is called the *identity* operation and is denoted by I.

We say the square is *invariant* under the 8 transformations I, V, H, D, D', R, R' and R''.

If we first apply R and then apply H, the result is written as $H \otimes R$ (read from right to left, just as for the composition of functions in general). This is not a new operation — it has the same effect as applying D'. One way to see this is to check what happens to each of the four vertices of the square. For example, if we apply $H \otimes R$ the top right vertex is first mapped to the top left vertex and then to the bottom right vertex. This is the same as applying D'. Similarly for the other three vertices.

However, if we apply H and R in the reverse order, i.e. $R \otimes H$, then the result is different. This time it is the same as D. In particular, the operation $\otimes$ is not commutative.

If we draw up a table we obtain

$\otimes$	I	V	H	D	D'	R	R'	R"
I	I	V	H	D	D'	R	R'	R"
V	V	I	R'	R	R"	D	H	D'
H	H	R'	I	R"	R	D'	V	D
D	D	R"	R	I	R	H	D'	V
D'	D'	R	R"	R'	I	V	D	H
R	R	D'	D	V	H	R'	R"	I
R'	R'	H	V	D'	D	R"	I	R
R"	R"	D	D'	H	V	I	R	R'

A crystal is classified by classifying the group of transformations which leaves it invariant. These ideas are treated in MATH2302 and MATH2322.

Another important example of a group is the set of 2×2 matrices, where now the operation is matrix multiplication and the identity is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In fact this is related to the previous example, the 8 operations correspond to certain 2×2 matrices (Can you find them? See Lay Chapter 1.8). More generally, very important examples are the groups of $n \times n$ matrices, and various "subgroups".

See "A Survey of Modern Algebra" by Birkhofff and MacLane for an introduction to fields and groups. (An old classic.)

2.3. Completeness Axiom

The Completeness Axiom is introduced. It is true for the real numbers, but the analogous axiom is not true for the rationals. We define the notion of upper bound (lower bound) and least upper bound (greatest lower bound) of a set of real numbers.

See Adams pages 4,5 and a few brief remarks after Example 1 on page A24 (in one of the Appendices)

The *Completeness Axiom* is the final axiom for the real number system, and is probably not one you have met before. It is more difficult to understand than the other properties, but it is essential in proving many of the important results in calculus.

2.3.1. Statement of the Axiom.

Axiom 14 (Completeness Axiom): *If A is any non-empty set of real numbers with the property that there is some real number x such that $a \leq x$ for every $a \in A$, then there is a* smallest *(or* least*) real number x with this same property.*

A is *non-empty* means that A contains at least one number.

Note that the number x in the axiom need not belong to A. For example, if A is the interval $[0, 1)$ then the smallest (or "least") number x as above is 1, but $1 \notin A$. On the other hand, if $A = [0, 1]$ then the smallest number x as above is again 1, but now $1 \in A$.

There is some useful notation associated with the Completeness Axiom.

Definition 2.3.1. If A is a set of real numbers and x is a real number such that $a \leq x$ for every $a \in A$, then x is called an *upper bound* for A. If also $x \leq b$ for every upper bound b then x is called the *least upper bound* or *supremum* of A. In this case one write

$$x = \operatorname{lub} A \quad \text{or} \quad x = \sup A.$$

If $x \leq a$ for every $a \in A$, then x is called a *lower bound* for A. If also $x \geq c$ for every lower bound c, then x is called the *greatest lower bound* or *infimum* of A. In this case one write

$$x = \operatorname{glb} A \quad \text{or} \quad x = \inf A.$$

Thus we have:

Axiom 14 (Completeness Axiom): If a non-empty set A has an upper bound then it has a least upper bound.

(Remember that when we say A "has" an upper bound or a least upper bound x we do *not* require that $x \in A$.)

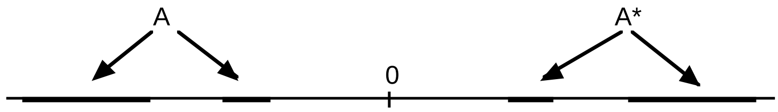
See Adams page A24, Example 1.

2.3.2. An equivalent formulation. There is an equivalent form of the axiom, which says: *If A is any non-empty set of real numbers with the property that there is some real number x such that $x \leq a$ for every $a \in A$, then there is a* largest *real number x with this same property.* In other words if a non-empty set A has a lower bound then it has a greatest lower bound.

It is not too hard to see that this form does indeed follow from the Completeness Axiom. The trick is to consider, instead of A, the set

$$A^* := \{ -x : x \in A \},$$

which is obtained by "reflecting" A about 0.



Lower bounds for A correspond under reflection to upper bounds for A^*, and a glb corresponds to a lub. If A is bounded below then A^* is bounded above, and so by the Completeness Axiom has a lub. After reflection, this lub for A^* gives a glb for A. (To actually write this out carefully needs some care—you need to check from the relevant definitions and the properties of inequalities that the first three sentences in this paragraph are indeed correct.)

(Similarly, the Completeness Axiom follows from the above version.)

Unlike in the case of Axioms 1–13, we will always indicate when we use the Completeness Axiom.

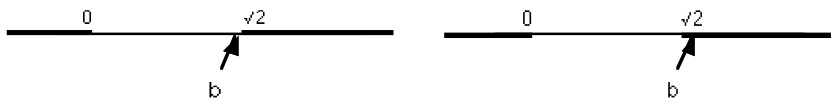
2.3.3. Interpretation of the Completeness Axiom. The Completeness Axiom implies there are no "gaps" in the real numbers.

For example, the rational numbers are *not* a model of the corresponding version of Axiom 14. In other words, Axiom 14 is not true if, in the first statement of the axiom, the three occurence of the word "real" are replaced by "rational".

For example, let

$$A = \{\, a \in \mathbb{Q} \mid 0 \le a \text{ and } a^2 < 2 \,\} = \{\, a \in \mathbb{Q} \mid 0 \le a < \sqrt{2} \,\}.$$

(The first definition for A has the advantage that A is defined without actually referring to the existence of the irrational number $\sqrt{2}$.) There are certainly rational numbers x which are upper bounds for A, i.e. such that $a \le x$ for every $a \in A$, just take $x = 23$. But we claim *there is no rational number b which is a* least *upper bound for A.*



This b is no good as there is a member of A (in fact infinitely many) between b and √2

This b is no good as there are smaller upper bounds (in fact infinitely many) between b and √2

Proof of claim. Since $\sqrt{2}$ is not rational it cannot be the required rational number b.

On the other hand, if $b < \sqrt{2}$, since there is always a rational number between b and $\sqrt{2}$ this gives a member of A between b and $\sqrt{2}$, and so b cannot be an upper bound.

Finally, if $b > \sqrt{2}$, there is always a smaller rational number between $\sqrt{2}$ and b, and so b cannot be the *least* rational number which is an upper bound for A.

We have ruled out the three possibilities $b = \sqrt{2}$, $b < \sqrt{2}$ and $b > \sqrt{2}$. This completes the proof of the claim. Hence there is no rational number which is a least upper bound for A. □

2.3.4. Archimedean Property. This property of the real numbers is not surprising, but it does not follow from the algebraic and order axioms alone. It says, informally, that there are no real numbers beyond all the natural numbers.

Theorem 2.3.2 (Archimedean Property). *For every real number x there is a natural number n such that $x < n$. Equivalently, the set $\mathbb{N}$ is not bounded above.*

Proof*. Suppose that the theorem were false. Then there would be a real number x with the property that $n < x$ for all $n \in \mathbb{N}$. This implies $\mathbb{N}$ is bounded above and so there must be a *least* upper bound b (say) by the Completeness Axiom.

In other words,

$$n \leq b \text{ for } \textit{every } n \in \mathbb{N}.$$

It follows that

$$n + 1 \leq b \text{ for } \textit{every } n \in \mathbb{N},$$

since $n + 1 \in \mathbb{N}$ if $n \in \mathbb{N}$. But this implies

$$n \leq b - 1 \text{ for } \textit{every } n \in \mathbb{N}.$$

In other words, $b - 1$ is also an upper bound for $\mathbb{N}$, which contradicts the fact that b is the *least* upper bound.

Since we have obtained a contradiction by assuming the statment of the theorem is false, the statement must in fact be true. □

The only surprising thing about the Archimedean property is that it needs the Completeness Axiom to prove it. But there are in fact models of the algebraic and order axioms in which the Archimedean property is false. They are sometimes called the *Hyperreals*!. See the next starred section.

The following corollary says that between zero and any positive number (no matter how small) there is always a number of the form $1/n$, where $n \in \mathbb{N}$. This is the same type of result as in Section 2.1.4, that between any two different real numbers there is always a rational number. But in Section 2.1.4 we did not go back to the axioms to prove the result.

The symbol ε in the following is called "epsilon" and is a letter of the Greek alphabet. You could replace ε by any other symbol such as x or a, and the Corollary would have *exactly* the same meaning. Hoever, it is traditional in mathematics to use ε when we are thinking of a very small positive number. Sometimes we use the symbol δ or "delta", another letter of the Greek alphabet, in the same way.

Corollary 2.3.3. *For any real number $\varepsilon > 0$ there is a natural number n such that $\frac{1}{n} < \varepsilon$.*

Proof. By the Archimedean Property there is a natural number n such that $n > \frac{1}{\varepsilon}$. But then $\frac{1}{n} < \varepsilon$ (by standard properties for manipulating inequalities). □

Note that this corollary was actually used in the proof of the results in Section 2.1.4, *where*?

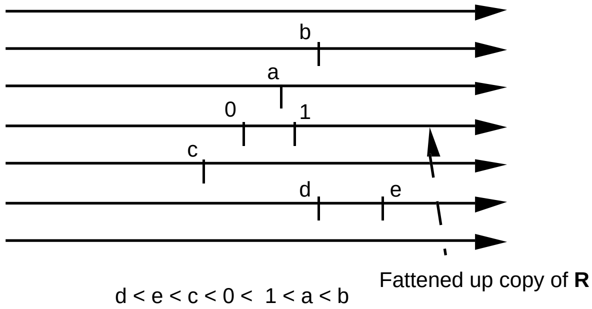
It is probably confusing as to why the Archimedean property and its Corollary should rely on the Completeness Axiom for their proofs. What is the difference between them and the other standard properties of inequalities such as in Theorem 2.2.3?

Well, the main difference is that the usual properties of inequalities, such as in Theorem 2.2.3, essentially tell us how to *manipulate* inequalities. The Archimedean Property is essentially a *non-existence* property of the infinite set $\mathbb{N}$ of real numbers (i.e. $\mathbb{N}$ has no upper bound).

Don't worry! There will not be any more surprises like this. There will be important situations where we rely on the Completeness Axiom, such as in proving certain properties of continuous functions, but these applications will not be so surprising.

2.3.5. Hyperreals*. *Part* of any model of the hyperreals looks like a "fattened up" copy of $\mathbb{R}$, in the sense that it contains a copy of $\mathbb{R}$ together with "infinitesimals" squeezed between each real a and all reals greater than a. (In particular there are hyperreals bigger than 0 and less than any positive real number!) This part is followed and preceded by infinitely many "copies" of itself, and between any two copies there are infinitely many other copies. See the following crude diagram.

In particular there are hyperreals bigger than any natural number, as the natural numbers will all lie on the fattened up copy of $\mathbb{R}$.



A "number" on any line is less than any number to the right, and less than any any number on any higher line. Between any two lines there is an infinite number of other lines.

The hyperreals were discovered by the U.S. mathematician Abraham Robinson in the 1960's. They turn out to be quite useful in proving results about the usual real numbers and have been used in many areas of mathematics (for example in probability theory and stochastic processes). They are one way of giving a rigorous meaning to the notion of an infinitesimal — i.e. it is possible to interpret the expression $\frac{dy}{dx}$ as the ratio of two hyperreal numbers dy and dx.

There are even courses on Calculus that are based on using the hyperreals (also called "nonstandard" numbers). See "Elementary Calculus" by H. Jerome Keisler, and the corresponding Instructors Manual "Foundations of Infinitesimal Calculus". However, this approach to teaching Calculus has not been particularly popular!

We will not use the hyperreals. You certainly should not refer to them in any of your proofs.

2.4. Sets

The notion of a set is basic in mathematics. We will not need to study the theory of sets, but we will need to know some notation and a few basic properties.

2.4.1. Notation for sets. By a *set* (sometimes called a *class* or *family*) we mean a collection, often infinite, of objects of some type.[8] Members of a set are often called *elements* of the set. If a is a member (i.e. element) of the set S, we write

$$a \in S.$$

If a is not a member of S we write

$$a \notin S.$$

If a set is finite, we may describe it by listing its members. For example,

$$A = \{ 1, 2, 3 \}.$$

Note that $\{1, 2, 3\}$, $\{2, 3, 1\}$, $\{1, 1, 1, 2, 3\}$ are different descriptions of *exactly* the same set. Some infinite sets can also be described by listing their members, provided the pattern is clear. For example, the set of even positive integers is

$$E = \{ 2, 4, 6, 8, \dots \}.$$

We often use the notation

$$S = \{ x \mid P(x) \},$$

where $P(x)$ is some statement involving x. We read this as "S is the set of all (real numbers) x such that $P(x)$ is true". It is usually understood from the context of the discussion that x is restricted to be a real number. But if there is any possible ambiguity, then we write

$$S = \{ x \in \mathbb{R} \mid P(x) \},$$

which we read as "S is the set of elements x in $\mathbb{R}$ such that $P(x)$ is true". Note that this is *exactly* the same set as

$$\{ y \mid P(y) \} \quad \text{or equivalently} \quad \{ y \in \mathbb{R} \mid P(y) \}.$$

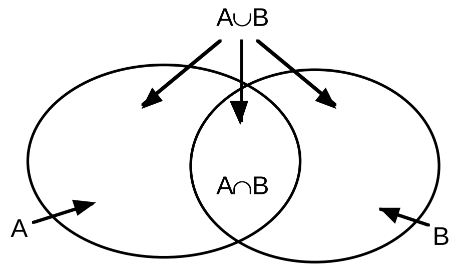
The variables x and y are sometimes called "dummy" variables, they are meant to represent any real number with the specified properties.

One also sometimes uses "$:$" instead of "$\mid$" when describing sets.

The *union* of two or more sets is the set of numbers belonging to at least one of the sets. The *intersection* of two or more sets is the set of numbers belonging to all of the sets. We use $\cup$ for union and $\cap$ for intersection. Thus if A and B are sets, then

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \},$$
$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}.$$

[8]* There is a mathematical theory of sets, and in fact all of mathematics can be formulated within the theory of sets. However, this is normally only useful or practical when considering fundamental questions about the foundations of mathematics.



The set A is a *subset* of B, and we write $A \subseteq B$, if every element of A is also an element of B. In symbols

$$A \subseteq B \quad \text{iff} \quad x \in A \Rightarrow x \in B.$$

(Where $\Rightarrow$ is shorthand for "implies".)

For example

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

It is also true that $\mathbb{N} \subseteq \mathbb{N}$, etc., although we would not normally make this statement.

Two sets are *equal* iff they have the same elements. It follows that

$$A = B \quad \text{iff} \quad A \subseteq B \text{ and } B \subseteq A.$$

In particular, we frequently prove two sets A and B are equal by first proving that every member of A is a member of B (i.e. $A \subseteq B$) and then proving that every member of B is a member of A (i.e. $B \subseteq A$).

For example, using the standard notation for intervals of real numbers,

$$\{\, x \mid 0 < x < 2 \text{ and } 1 \le x \le 3 \,\} = (0,2) \cap [1,3] = [1,2),$$
$$\{\, x \mid 0 < x < 1 \text{ or } 2 < x \le 3 \,\} = (0,1) \cup (2,3].$$

Also

$$(0,2) = (0,1) \cup [1,2) = (0,1] \cup [1,2) = (0,1) \cup (\frac{1}{2}, 2),$$

etc.

2.4.2. Ordered pairs of real numbers. In the Linear Algebra course we used both the notation $\begin{bmatrix} a \\ b \end{bmatrix}$ and (a,b) to represent vectors in $\mathbb{R}^2$, which we also regard as ordered pairs, or 2-tuples, of real numbers. Of course, (a,b) and (b,a) are distinct, unless $a = b$. This is different from the situation for the *set* containing a and b; i.e. $\{a,b\}$ and $\{b,a\}$ are just different ways of describing the same set.

We also have ordered triples (a,b,c), and more generally ordered n-tuples $(a_1, \ldots, a_n)$, of real numbers.

Remark* It is sometimes useful to know that we can define ordered pairs in terms of sets. The only property we require of ordered pairs is that

$$(a,b) = (c,d) \quad \text{iff} \quad (a = c \text{ and } b = d). \tag{3}$$

We could not define $(a,b) = \{a,b\}$, because we would not be able to distinguish between (a,b) and (b,a). But there are a number of ways that we can define ordered pairs in terms of sets. The standard definition is

$$(a,b) := \{\{a\}, \{a,b\}\}.$$

To show this is a good definition, we need to prove (3).

Proof. It is immediate from the definition that if $a = c$ and $b = d$ then $(a, b) = (c, d)$.

Next suppose $(a, b) = (c, d)$, i.e. $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. We consider the two cases $a = b$ and $a \neq b$ separately.

If $a = b$ then $\{\{a\}, \{a, b\}\}$ contains exactly one member, namely $\{a\}$, and so $\{\{c\}, \{c, d\}\}$ also contains exactly the one member $\{a\}$. This means $\{a\} = \{c\} = \{c, d\}$. Hence $a = c$ and $c = d$. In conclusion, $a = b = c = d$.

If $a \neq b$ then $\{\{a\}, \{a, b\}\}$ contains exactly two (distinct) members, namely $\{a\}$ and $\{a, b\}$. Since $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ it follows $\{c\} \in \{\{a\}, \{a, b\}\}$ and so $\{c\} = \{a\}$ or $\{c\} = \{a, b\}$. The second equality cannot be true since $\{a, b\}$ contains two members whereas $\{c\}$ contains one member, and so $\{c\} = \{a\}$, and so $c = a$.

Since also $\{c, d\} \in \{\{a\}, \{a, b\}\}$ it now follows that $\{c, d\} = \{a, b\}$ (otherwise $\{c, d\} = \{a\}$, but since also $\{c\} = \{a\}$ this would imply $\{\{c\}, \{c, d\}\}$ and hence $\{\{a\}, \{a, b\}\}$ has only one member, and we have seen this is not so). Since a and b are distinct and $\{c, d\} = \{a, b\}$, it follows c and d are distinct; since $a = c$ it then follows $b = d$. In conclusion, $a = c$ and $b = d$.

This completes the proof. $\square$

CHAPTER 3

Sequences

The reference here is Adams Section 9.1 and most of page A 25, but we do considerably more. Another reference is *Fundamentals of Analysis* by M. Reed, Chapter 2. This is also the text for the second year course MATH2320.

3.1. Examples and Notation

A *sequence* is an infinite list of numbers with a first, but no last, element. Simple examples are

$$1, 2, 1, 3, 1, 4, \ldots$$
$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$
$$1, -1,\ 1, -1,\ 1, \ldots$$

A sequence can be written in the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

More precisely, a sequence is a function f whose domain is the set of natural numbers, where in the above example $f(n) = a_n$. We often just write (a_n) or $(a_n)_{n\geq 1}$ to represent the sequence.

See Adams, page 518, Example 1.

See Adams pages 519, 520 for the definitions of the following terms:

1. bounded below, lower bound; bounded above, upper bound; bounded;
2. positive, negative;
3. increasing, decreasing, monotonic;
4. alternating;
5. ultimately (or "eventually").

See Examples 2, 3 in Adams page 520.

3.2. Convergence of Sequences

The most fundamental concept in the study of sequences is the notion of *convergence* of a sequence.

The informal idea is that a sequence (a_n) converges to a, and we write

$$\lim a_n = a,$$

if no matter how small a positive number is chosen, the distance between a_n and a, i.e. $|a_n - a|$, will ultimately be less than this positive number. (The smaller the positive numner, the further out in the sequence we will need to go.)

It is important to note that this is a condition that must be satisfied by *any* positive number. For example, we may have

$$|a_n - a| < .1 \text{ if } n > 50, \quad \text{(here the positive number is .1)}$$
$$|a_n - a| < .01 \text{ if } n > 300, \quad \text{(here the positive number is .01)}$$
$$|a_n - a| < .001 \text{ if } n > 780, \quad \text{(here the positive number is .001)}$$

etc.

DEFINITION 3.2.1. We say that *the sequence* (a_n) *converges to a limit* a, and write

$$\lim_n a_n = a, \quad \lim a_n = a, \quad a_n \to a, \quad \text{or } \lim_{n\to\infty} a_n = a,$$

if for every positive number ε there exists an integer N such that

$$n > N \quad \text{implies} \quad |a_n - a| < \varepsilon. \tag{4}$$

(You should note that there is no such number as ∞. The symbol ∞ in the above has no meaning by iteslf, just as the word “lim” has no meaning by itself.)

See Adams page 521, Figure 9.1, for a graphical illustration of limit.

EXAMPLE 3.2.2. Show that the sequence given by $a_n = 1 + \frac{1}{n^2}$ converges to 1 according to the definition.

SOLUTION. Let $\varepsilon > 0$ be given.

We want to find N such that (4) is true with $a = 1$. But

$$|a_n - 1| = \frac{1}{n^2}.$$

Since

$$\frac{1}{n^2} < \varepsilon \quad \text{if} \quad n^2 > \frac{1}{\varepsilon},$$

i.e.

$$\text{if} \quad n > \frac{1}{\sqrt{\varepsilon}},$$

we can take

$$N = \left[\frac{1}{\sqrt{\varepsilon}}\right],$$

or any larger integer, where [ ] denotes “the integer part of”. □

Thus if $\varepsilon = .1$ we can take any integer $N > 1/\sqrt{.1}$, for example $N = 4$ (or anything larger). If $\varepsilon = .01$ we can take $N = 10$ (or anything larger). If $\varepsilon = .001$ we can take $N = 32$ (or anything larger). But the above proof works of course for *any* $\varepsilon > 0$.

*In the above example we took $N = \left[\frac{1}{\sqrt{\varepsilon}}\right]$, the integer part of $\left[\frac{1}{\sqrt{\varepsilon}}\right]$, or equivalently the smallest integer greater than $\left[\frac{1}{\sqrt{\varepsilon}}\right] - 1$. In order to prove that such an integer always exists, strictly speaking we need the Archimedean Property. A similar remark applies to the following examples, but we will not usually explicitly state this fact.

Example 3.2.3. Consider the sequence defined by $a_1 = 1$, and $a_{n+1} = \frac{1}{2}a_n + 2$ for $n \geq 1$.

It is easy to calculate that the first few terms are

$$1, 2.5, 3.25, 3.625, 3.8125, 3.90625, 3.953125, 3.9765625, \ldots .$$

It seems reasonable that the sequence is converging to 4. One way to prove this is as follows.

Proof. Let $\varepsilon > 0$ be given.

We want to find N such that[1]

$$n > N \quad \Rightarrow \quad |a_n - 4| < \varepsilon. \tag{5}$$

We have a formula for a_{n+1} in terms of a_n, and we first use this to get a formula for $|a_{n+1} - 4|$ in terms of $|a_n - 4|$. Thus

$$|a_{n+1} - 4| = \left|\frac{1}{2}a_n + 2 - 4\right| = \left|\frac{1}{2}a_n - 2\right| = \left|\frac{1}{2}(a_n - 4)\right| = \frac{1}{2}|a_n - 4|.$$

Thus $|a_1 - 4| = 3$, $|a_2 - 4| = 3/2$, $|a_3 - 4| = 3/2^2$, $|a_4 - 4| = 3/2^3$, $\ldots$. In general[2] $|a_n - 4| = 3/2^{n-1}$.

It follows that

$$|a_n - 4| < \varepsilon \quad \text{for those } n \text{ such that} \quad \frac{3}{2^{n-1}} < \varepsilon.$$

This last inequality is equivalent to $2^{n-1}/3 < 1/\varepsilon$, i.e. $2^{n-1} > 3/\varepsilon$, i.e. $(n-1)\ln 2 > \ln(3/\varepsilon)$, i.e. $n > 1 + \ln(3/\varepsilon)/\ln 2$.[3]

Hence (5) is true for

$$N = 1 + \left[\frac{\ln \frac{3}{\varepsilon}}{\ln 2}\right].$$

□

You may object that we used ln, the natural logarithm, in the previous example, but we have not yet shown how to define logarithms and establish their properties from the axioms. This is a valid criticism. But in order to have interesting examples, we will often do this sort of thing.

However, we will not do it when we are establishing the underlying theory. In particular, the development of the theory will not depend on the examples.

See Adams page 521 Example 4.

Definition 3.2.4. If a sequence (a_n) does not converge, then we say that it *diverges*.

We say that the sequence (a_n) *diverges to* $+\infty$, and write

$$\lim_n a_n = \infty, \quad \lim a_n = \infty, \quad a_n \to \infty, \quad \text{or } \lim_{n\to\infty} a_n = \infty,$$

---

[1]We will often write "⇒" for "implies".

[2]This could easily be proved by induction, but it is not necessary to do so.

[3]* As in the previous example, to prove the existence of such an n requires, strictly speaking, the Archimedean Property.

if for each real number M there exists an integer N such that

$$n > N \quad \text{implies} \quad a_n > M.$$

(The interesting situation is M large and positive.)

Similarly, we say (a_n) to *diverges to* $-\infty$ if for each real number M there exists an integer N such that

$$n > N \quad \text{implies} \quad a_n < M.$$

(The interesting situation is M large and negative.)

A sequence may diverge, without diverging to $\pm\infty$. See Adams page 521 Example 5.

3.3. Properties of limits

It is normally not very efficient to use the definition of a limit in order to prove that a sequence converges. Instead, we prove a number of theorems which will make things much easier.

The first theorem shows that if two sequences converge, then so does their sum, and moreover the limit of the new sequence is just the sum of the limits of the original sequences. Similar results are true for products and quotients, and if we multiply all terms in a sequence by the same real number.

In (9) we need to assume $b \neq 0$. This will imply that ultimately $b_n \neq 0$ (i.e. $b_n \neq 0$ for all sufficiently large n), and hence that the sequence (a_n/b_n) is defined for all sufficiently large n.

THEOREM 3.3.1. *Suppose*

$$\lim a_n = a, \quad \lim b_n = b,$$

and c is a real number. Then the following limits exist and have the given values.

$$\lim(a_n \pm b_n) = a \pm b, \tag{6}$$

$$\lim ca_n = ca, \tag{7}$$

$$\lim a_n b_n = ab, \tag{8}$$

$$\lim \frac{a_n}{b_n} = \frac{a}{b}, \qquad \textit{assuming } b \neq 0. \tag{9}$$

We will give the proofs in the next section. The theorem is a partial justification that Definition 3.2.1 does indeed capture the informal notion of a limit.

The results are not very surprising. For example, if a_n is getting close to a and b_n is getting close to b then we expect that $a_n + b_n$ is getting close to $a + b$.

EXAMPLE 3.3.2. Let $a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^2 - (1 + 2^{-n})$.

We can prove directly from the definition of convergence that $\frac{1}{\sqrt{n}} \to 0$ and $2^{-n} \to 0$. It then follows from the previous theorem that $1 + \frac{1}{\sqrt{n}} \to 1$ (since we can think of $1 + \frac{1}{\sqrt{n}}$ as obtained by adding the term 1 from the constant sequence (1) to the term $\frac{1}{\sqrt{n}}$). Applying the theorem again, $\left(1 + \frac{1}{\sqrt{n}}\right)^2 \to 1$. Similarly, $1 + 2^{-n} \to 1$.

Hence (again from the theorem) $a_n \to 1 - 1 = 0$.

EXAMPLE 3.3.3. Let $a_n = \frac{2n^2-1}{3n^2-7n+1}$.

Write

$$\frac{2n^2 - 1}{3n^2 - 7n + 1} = \frac{2 - \frac{1}{n^2}}{3 - \frac{7}{n} + \frac{1}{n^2}}.$$

Since the numerator and denominator converge to 2 and 3 respectively, it follows $a_n \to 2/3$.

See also Adams, pages 523, Example 6.

Before we prove Theorem 3.3.1 there is a technical point. We should prove that a convergent sequence cannot have two different limits. This is an easy consequence of the definition of convergence.

THEOREM 3.3.4. *If (a_n) is a convergent sequence such that $a_n \to a$ and $a_n \to b$ then $a = b$.*

The next easy result is useful in a number of situations.

THEOREM 3.3.5. *Suppose $a_n \to a$. Then the sequence is bounded; i.e. there is a real number M such that $|a_n| \leq M$ for all n.*

The next theorem is not true if we replace both occurrences of "$\leq$" by "'$<$". For example $-1/n < 1/n$ for all n, but the sequences $(1/n)$ and $(-1/n)$ have the same limit 0.

THEOREM 3.3.6. *Suppose $a_n \to a$, $b_n \to b$, and $a_n \leq b_n$ ultimately. Then $a \leq b$.*

The following theorem says that if a sequence is "squeezed" between two sequences which both converge to the same limit, then the original sequence also converges, and it converges to the same limit.

THEOREM 3.3.7. *Suppose $a_n \leq b_n \leq c_n$ ultimately. Suppose $a_n \to L$ and $c_n \to L$. Then $b_n \to L$.*

EXAMPLE 3.3.8. Consider the sequence $3+(\sin\cos n)/n$. Since $-1 \leq \sin x \leq 1$, it follows that $3-1/n \leq 3+(\sin\cos n)/n \leq 3+1/n$. But $3-1/n \to 3$ and $3+1/n \to 3$. Hence $3+(\sin\cos n)/n \to 3$.

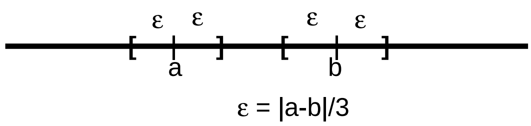
3.4. Proofs of limit properties

I have starred some of these proofs, as they are a bit technical. But you should aim to have some understanding of the ideas involved.

*PROOF OF THEOREM 3.3.4. Suppose $a_n \to a$ and $a_n \to b$.

Assume (in order to obtain a contradiction) that $a \neq b$.

Take $\varepsilon = |a-b|/3$ in the definition of a limit, Definition 3.2.1. (For motivation, look at the following diagram).



Since $a_n \to a$, it follows that

(10) $$a_n \in (a-\varepsilon, a+\varepsilon)$$

for all sufficiently large n, say for $n > N_1$.

Since $a_n \to b$, it follows that

(11) $$a_n \in (b-\varepsilon, b+\varepsilon)$$

for all sufficiently large n, say for $n > N_2$.

But this implies

$$a_n \in (a-\varepsilon, a+\varepsilon) \quad \text{and} \quad a_n \in (b-\varepsilon, b+\varepsilon)$$

for all $n > \max\{N_1, N_2\}$, which is impossible as $\varepsilon = |a-b|/3$.

Thus the assumption $a \neq b$ led to a contradiction and so $a = b$. □

*PROOF OF THEOREM 3.3.5. Assume $a_n \to a$.

From the definition of convergence, taking $\varepsilon = 1$, there is an integer N such that

(12) $$a - 1 < a_n < a + 1 \quad \text{for all } n > N.$$

Fix this N. Since the set of terms

$$a_1, a_2, \ldots, a_N$$

is *finite*, it follows that there exist real numbers M_1 and M_2 such that

(13) $$M_1 \leq a_n \leq M_2 \quad \text{for all } n \leq N.$$

(Just take $M_1 = \min\{a_1, a_2, \ldots, a_N\}$ and $M_2 = \max\{a_1, a_2, \ldots, a_N\}$.)

From (12) and (13),

$$M_1^* \leq a_n \leq M_2^* \quad \text{for } \textit{all } n,$$

where $M_1^* = \min\{a-1, M_1\}$, $M_2^* = \max\{a+1, M_2\}$.

Hence $|a_n| \leq M$ for all n where $M = \max\{|M_1^*|, |M_2^*|\}$. □

PROOF OF THEOREM 3.3.1 (6) ("+" CASE). Suppose $a_n \to a$ and $b_n \to b$.

Let $\varepsilon > 0$ be given.

Since $a_n \to a$ there exists an integer N_1 (by Definition 3.2.1) such that

(14) $$n > N_1 \quad \text{implies} \quad |a_n - a| < \varepsilon/2.$$

Since $b_n \to b$ there exists an integer N_2 (again by Definition 3.2.1) such that

(15) $$n > N_2 \quad \text{implies} \quad |b_n - b| < \varepsilon/2.$$

It follows that if $n > \max\{N_1, N_2\}$ then

$$\begin{aligned}|(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &< |a_n - a| + |b_n - b| \quad \text{by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by (14) and (15)} \\ &= \varepsilon.\end{aligned}$$

It follows from Definition 3.2.1, with $N = \max\{N_1, N_2\}$, that $(a_n + b_n)$ converges and the limit is $a + b$. □

Notice in the proof how the definition of a limit is used three times; once to get information from the fact $a_n \to a$, once to get information from the fact $b_n \to b$, and finally to deduce that $a_n + b_n \to a + b$.

By the way, why do we use $\varepsilon/2$ in (14) and (15), and why is this justifiable by Definition 3.2.1?

Proof of Theorem 3.3.1 (7). Suppose $a_n \to a$ and c is a real number.

Let $\varepsilon > 0$ be any positive number. We want to show

$$|ca_n - ca| < \varepsilon$$

for all sufficiently large n.

Since $a_n \to a$ there exists an integer N such that

$$|a_n - a| < \varepsilon/|c| \quad \text{for all} \quad n > N.$$

(This assumes $c \neq 0$. But if $c = 0$, then the sequence (ca_n) is the sequence all of whose terms are 0, and this sequence certainly converges to $ca = 0$.) Multiplying both sides of the inequality by $|c|$ we see

$$|c|\,|a_n - a| < \varepsilon \quad \text{for all} \quad n > N,$$

i.e.

$$|ca_n - ca| < \varepsilon \quad \text{for all} \quad n > N,$$

and so $ca_n \to ca$ by the definition of convergence. □

Proof of Theorem 3.3.1 (6) (“$-$” case). Suppose $a_n \to a$ and $b_n \to b$.

We can write

$$a_n - b_n = a_n + (-1)b_n.$$

But $(-1)b_n \to (-1)b$ by the previous result with $c = -1$, and so the result now follows from (6) for the *sum* of two sequences. □

*Proof of Theorem 3.3.1 (8). Suppose $a_n \to a$ and $b_n \to b$.

As usual, let $\varepsilon > 0$ be an arbitrary positive number.

We want to show there is an integer N such that

$$|a_nb_n - ab| < \varepsilon$$

for all $n > N$.

To see how to choose N, write

$$\begin{aligned}|a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &= |a_n|\,|b_n - b| + |b|\,|a_n - a|.\end{aligned} \tag{16}$$

(This trick of adding and subtracting the same term, here it is $a_n b$, is often very useful.) We will show that both terms are $< \varepsilon/2$ for all sufficiently large n.

For the second term $|b|\,|a_n - a|$, the result is certainly true if $b = 0$, since the term is then 0. If $b \neq 0$, since $a_n \to a$, we can choose N_1 such that

$$|a_n - a| < \frac{\varepsilon}{2|b|} \quad \text{for all} \quad n > N_1,$$

and so

$$|b|\,|a_n - a| < \frac{\varepsilon}{2} \quad \text{for all} \quad n > N_1. \tag{17}$$

For the first term $|a_n|\,|b_n - b|$, we use Theorem 3.3.5 to deduce for some M that $|a_n| \le M$ for all n. By increasing M if necessary take $M \neq 0$. By the same argument as for the second term, we can choose N_2 such that

$$M\,|b_n - b| < \frac{\varepsilon}{2} \quad \text{for all} \quad n > N_2,$$

and so

$$|a_n|\,|b_n - b| < \frac{\varepsilon}{2} \quad \text{for all} \quad n > N_2. \tag{18}$$

Putting (16), (17) and (18) together, it follows that if $n > N$, where $N = \max\{N_1, N_2\}$, then

$$|a_n b_n - ab| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves $a_n b_n \to ab$. □

*Proof of Theorem 3.3.1 (9). Suppose $a_n \to a$ and $b_n \to b$ where $b \neq 0$.

We will prove that $a_n/b_n \to a/b$ by first showing $1/b_n \to 1/b$ and then using the previous result about products of sequences.

We first prove

$$|b_n| > |b|/2 \text{ ultimately.} \tag{19}$$

The proof is similar to that in Theorem 3.3.5, and goes as follows:

First assume $b > 0$. Choose $\varepsilon = |b|/2$ (> 0) in the definition of convergence and deduce that for some integer N,

$$n > N \Rightarrow |b_n - b| < b/2,$$

and so in particular

$$n > N \Rightarrow |b_n| > b/2.$$

This proves (19) in case $b > 0$.

In case $b < 0$ we similarly prove that ultimately $b_n < b/2$, and so ultimately $|b_n| > |b|/2$. Ths completes the proof of (19).

We now proceed with the proof that $1/b_n \to 1/b$. For this let $\varepsilon > 0$ be any positive number.

In order to see how to choose N in the definition of convergence, we compute

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b_n|\,|b|} \le \frac{2|b - b_n|}{|b|^2}, \tag{20}$$

ultimately (this uses (19)). (The only reason for "$\le$" instead of "$<$" is that perhaps $|b - b_n| = 0$.)

Since $b_n \to b$ we can find an integer N such that for all $n > N$,

$$|b - b_n| < \frac{|b|^2}{2}\varepsilon.$$

It follows from (20) that if $n > N$ then

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $1/b_n \to 1/b$.

Since $a_n \to a$, it now follows from the result for products that $a_n/b_n \to a/b$. □

Proof of Theorem 3.3.6. (The proof is similar to that for Theorem 3.3.4.)
Suppose $a_n \to a$, $b_n \to b$, and ultimately $a_n \leq b_n$.
Assume (in order to obtain a contradiction) that $a > b$. Let $\varepsilon = \frac{1}{3}(a - b)$.
Then

$$a_n \in (a - \varepsilon, a + \varepsilon) \text{ ultimately,}$$

and in particular

$$a_n > a - \varepsilon \text{ ultimately.}$$

Similarly

$$b_n < b + \varepsilon \text{ ultimately.}$$

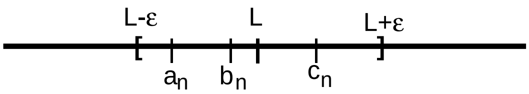
(Draw a diagram.) Since $\varepsilon = \frac{1}{3}(a - b)$, this implies

$$a_n > b_n \text{ ultimately.}$$

But this contradicts $a_n \leq b_n$, and so the assumption is false. Thus $a \leq b$. □

Proof of Theorem 3.3.7. Suppose $a_n \leq b_b \leq c_n$ ultimately. Suppose $a_n \to L$ and $c_n \to L$.

Let $\varepsilon > 0$ be given. (For motivation, look at the following diagram).



Since $a_n \to L$ there is some integer N_1 such that

$$n > N_1 \Rightarrow a_n \in (L - \varepsilon, L + \varepsilon). \tag{21}$$

Since $c_n \to L$ there is some integer N_2 such that

$$n > N_2 \Rightarrow c_n \in (L - \varepsilon, L + \varepsilon). \tag{22}$$

Let$N = \max\{N_1, N_2\}$. Then since $a_n \leq b_n \leq c_n$ it follows from (21) and (22) that

$$n > N \Rightarrow b_n \in (L - \varepsilon, L + \varepsilon).$$

But ε was an arbitrary positive number, and so it follows that $b_n \to L$. □

3.5. More results on sequences

We have seen that a convergent sequence is bounded.

The converse is false. For example, the sequence

$$1, -1, 1, -1, \ldots$$

is bounded but does not converge.

However, a bounded monotone sequence *does* converge. The proof needs the Completeness Axiom in order to give a "candidate" for the limit.

THEOREM 3.5.1. *If a sequence is bounded and ultimately monotone (i.e. either ultimately increasing or ultimately decreasing), then it converges.*

PROOF. (See Adams, Appendix III, Theorem 2) □

EXAMPLE 3.5.2. Prove the sequence (a_n) defined by

$$a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n}$$

is convergent, and find the limit.

SOLUTION. (See Adams Example 8, Section 9.1, or Section 10.1 of 3rd edition, for details).

The idea is to show by induction that

1. (a_n) is monotone increasing,
2. $a_n \leq 3$.

It follows from the previous theorem that $a_n \to a$, say.

In order to find a, we use the facts that if $a_n \to a$ then $a_{n+1} \to a$ [4] and $\sqrt{6 + a_n} \to \sqrt{6 + a}$ [5]. By uniqueness of the limit of a sequence, $a = \sqrt{6 + a}$. Solving gives $a = -2$ or 3. The former is impossible, as it it easy to see $a_n \geq 1$ for every n. □

The following limits are often useful.

THEOREM 3.5.3.

1. *If* $|x| < 1$ *then* $\lim x^n = 0$.
2. *If* x *is any real number, then* $\lim \frac{x^n}{n!} = 0$.

PROOF. (Adams gives a proof (see Theorem 3 section 9.1, or 10.1) which uses continuity and properties of logs for the first part — here is another proof that does not use this.)

Since $|x| < 1$ the sequence $|x|^n$ is decreasing[6] and all terms are ≥ 0. Hence $|x|^n \to a$ (say) by Theorem 3.5.1.

Since $|x|^n \to a$, also $|x|^{n+1} \to a$ (see footnote 4). But $|x|^{n+1} = |x|\,|x|^n \to |x|\,a$. Hence $a = |x|a$ by uniqueness of limits, and so $a = 0$ as $|x| \neq 1$.

Because $-|x|^n \leq x^n \leq |x|^n$ and since both $|x|^n \to 0$ and $-|x|^n \to 0$, it follows from the Squeeze Theorem that $x^n \to 0$.

The second result follows from the first, see Adams. □

EXAMPLE 3.5.4. Find $\lim \frac{3^n + 4^n + 5^n}{5^n}$.

SOLUTION. Example 10 of Adams. □

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[4] It follows easily from the definition of convergence that if $a_n \to a$, then also $a_{n+1} \to a$ (*Exercise*). This is frequently a useful fact.

[5] This can either be proved from the definition of a limit (*Exercise*). Later it will follow easily from the fact that the function f given by $f(x) = \sqrt{6 + x}$ is continuous.

[6] We could prove this by induction, but that is not really required.

CHAPTER 4

Continuous Functions

The intuitive idea of a continuous function is one whose graph can be drawn without lifting pen from paper. But this is too vague to develop a useful theory.

The Definition we give here for a continuous function uses uses the ideas we have developed for convergent sequences. The approach in Adams [Definition 9 Section 1.5] is different but equivalent.

In Appendix III, Theorem 4, Adams shows that if a function is continuous according to the Definition he uses, then it is continuous according to the Definition here. One can also show the converse, that if a function is continuous according to the Definition here then it is continuous according to the Definition in Adams. We will see this later.

4.1. Definition and examples

Recall that the domain of a function f, denoted by $\mathcal{D}(f)$, is the set of numbers x such that $f(x)$ is defined. We will usually be interested in functions whose domains are intervals $[a, b]$, (a, b), (a, ∞)[1], etc. But it is possible for the domain to be a more complicated set of real numbers.

We will define the notion of continuity in terms of convergence of sequences. The informal idea of continuity of a function f at a point c is that "as x approaches c then $f(x)$ approaches $f(c)$".

More accurately, we have the following natural definition.

Definition 4.1.1. A function f *is continuous at a point* $c \in \mathcal{D}(f)$ if for every sequence (x_n) such that $x_n \in \mathcal{D}(f)$ and $x_n \to c$, we have $f(x_n) \to f(c)$.

We say f is *continuous* (on its domain) if f is continuous at every point in its domain.

In other words,

$$x_n \in \mathcal{D}(f) \text{ and } x_n \to c \quad \Rightarrow \quad f(x_n) \to f(c).$$

We will often not write $x_n \in \mathcal{D}(f)$, although this is always understood in order that $f(x_n)$ be defined.

Thus in order to show f *is* continuous at c, we have to show that for *every* sequence $x_n \to c$ one has $f(x_n) \to f(c)$.

In order to show f *is not* continuous at c, we only have to show there is *one* ("bad") sequence $x_n \to c$ with $f(x_n) \not\to f(c)$.[2]

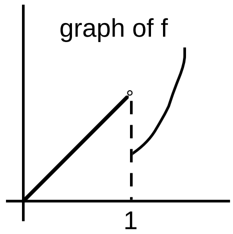
Example 4.1.2. Suppose

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ \frac{1}{2}x^2 & 1 \le x \le \frac{3}{2} \end{cases}$$

[1]Note that ∞ is *not* a number, and that for us the symbol ∞ has no neaning by itself. The interval (a, ∞) is just the set of real numbers strictly greater than a.

[2]If there is one, there will in fact be many such "bad" sequences — we can always change the first million or so terms — but the point is that to show continuity fails it is sufficient to just prove there is one "bad" sequence.

The domain of f is $[0, \frac{3}{2}]$. The following is an attempt to sketch the graph of f.



It is clear that f is not continuous at 1. For example, take the sequence $x_n = 1 - \frac{1}{n}$. Then $x_n \to 1$ but $f(x_n)\ (= 1 - \frac{1}{n}) \not\to f(1)$ since $f(1) = \frac{1}{2}$.

On the other hand, if $c \neq 1$ and $c \in \mathcal{D}(f)$ then

$$x_n \to c \quad \Rightarrow \quad f(x_n) \to f(c).$$

To see this, first suppose $x_n \to c$ and $1 < c \leq \frac{3}{2}$. Then $x_n \geq 1$ for all sufficiently large n, and so $f(x_n) = \frac{1}{2}x_n^2$ for all sufficiently large n. From properties of sequences if $x_n \to c$ then $x_n^2 \to c^2$ and so $\frac{1}{2}x_n^2 \to \frac{1}{2}c^2$. But $f(x_n) = \frac{1}{2}x_n^2$ for all sufficiently large n, and so $\lim f(x_n) = \lim \frac{1}{2}x_n^2 = \frac{1}{2}c^2$.

In particular, f is not continuous on its domain.

The case $0 \leq c < 1$ is similar, and easier.

If we vary this example a little, and define

$$g(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2}x^2 & 1 < x \leq \frac{3}{2}, \end{cases}$$

then the domain of g is $[0, 1) \cup (1, \frac{3}{2}]$. The function g is continuous at each $c \in \mathcal{D}(g)$, and so *is* continuous on its domain.

However, there is no *extension* of g to a continuous function defined on all of $[0, 3/2]$.

Example 4.1.3. *The absolute value function f (given by $f(x) = |x|$) is continuous.*

We first show continuity at 0. For this, suppose $x_n \to 0$. Then $|x_n| \to 0$ (this is immediate from the definition of convergence, since $|x_n - 0| \leq \epsilon$ iff $|\,|x_n| - 0\,| \leq \epsilon$), i.e. $f(x_n) \to f(0)$.

To prove continuity at $c \neq 0$ is similar to the previous example.

The following result is established directly from the properties of convergent sequences.

Proposition 4.1.4. *Every polynomial function is continuous.*

Proof. Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k.$$

To show that f is continuous at some point c, suppose $x_n \to c$.

Then $x_n^2 \to c^2$, $x_n^3 \to c^3$, etc., by the theorem about products of convergent sequences. It follows that $a_1x_n \to a_1c$, $a_2x_n^2 \to a_2c^2$, $a_3x_n^3 \to a_3c^3$, etc., by the theorem about multiplying a convergent sequence by a constant. Finally,

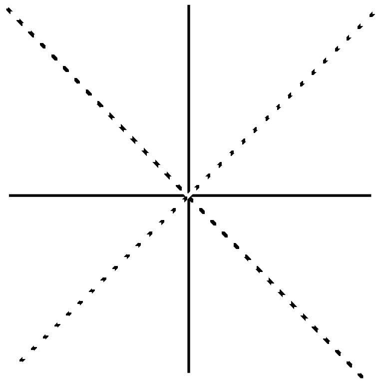
$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \to a_0 + a_1c + a_2c^2 + \cdots + a_kc^k$$

by repeated applications of the theorem about sums of convergent sequences (a_0 is here regarded as a constant sequence). □

EXAMPLE 4.1.5.* Here is a surprising example.
Let

$$f(x) = \begin{cases} x & x \text{ rational} \\ x & -x \text{ irrational.} \end{cases}$$

The following diagram is misleading, since between any two real numbers there is both a rational and an irrational number.



The function f is continuous at 0. To see this, suppose $x_n \to 0$. Then $|x_n| \to 0$ (this follows from the definition of a limit). Since $-|x_n| \le f(x_n) \le |x_n|$, it follows from the Squeeze theorem that $f(x_n) \to 0$, i.e. $f(x_n) \to f(0)$.

On the other hand, *f is not continuous at c if $c \neq 0$.* For example if c is irrational then we can choose a sequence of rationals x_n such that $x_n \to c$ (by repeated applications of the remark above in italics). It follows that $f(x_n) = x_n \to c \neq f(c)$. Similarly if c is irrational.

We will later define the exponential, logarithm, and trigonometric functions, and show they are continuous. Meanwhile, we will use them in examples (but not in the development of the theory).

4.2. Properties of continuous functions

The basic properties of continuous functions follow easily from the analogous properties of sequences.

THEOREM 4.2.1. *Let f and g be continuous functions and let $D = \mathcal{D}(f) \cap \mathcal{D}(g)$.*[3] *Then*

1. *$f + g$ is continuous on D,*
2. *fg is continuous on D,*
3. *αf is continuous on $\mathcal{D}(f)$ (α any real number)*
4. *f/g is continuous at any point $c \in D$ such that $g(c) \neq 0$.*

PROOF. Suppose $c \in D$. Let (x_n) be any sequence such that $x_n \to c$ (and as usual, $x_n \in D$).

Then $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$, since f and g are continuous at c. It follows

$$f(x_n) + g(x_n) \to f(c) + g(c)$$

by Theorem 3.3.1 about sums of convergent sequences. That is,

$$(f + g)(x_n) \to (f + g)(c).$$

It follows that $f + g$ is continuous at c.

The proof in the other cases is similar. Just note for the case f/g that if $x_n \to c$ and $g(c) \neq 0$, then $g(x_n) \neq 0$ for all sufficiently large n[4]. □

The composition of two continuous functions is continuous. (See Adams page 35 for a discussion about the composition of functions.)

THEOREM 4.2.2. *Suppose f and g are continuous. Then $f \circ g$ is continuous.*

PROOF. The domain D of $f \circ g$ is the set of numbers x such that both $x \in \mathcal{D}(g)$ and $g(x) \in \mathcal{D}(f)$.

Suppose $c \in D$. Let $x_n \to c$ and $x_n \in D$. It follows that $g(x_n) \to g(c)$ since g is continuous at c. It then follows that $f(g(x_n)) \to f(g(c))$ since f is continuous at $g(c)$ (note that $g(x_n) \in \mathcal{D}(f)$). In other words, $(f \circ g)(x_n) \to (f \circ g)(c)$, and so $f \circ g$ is continuous at c. □

It follows from our results so far that rational functions (quotients of polynomials) and in general functions defined from other continuous functions by means of algebraic operations and composition, will be continuous on their domain.

EXAMPLE 4.2.3. The function

$$f_1(x) = \sin \frac{1}{x}$$

is the composition of the two continuous functions $\sin(x)$ and $1/x$ [5] and so is continuous. The domain of f_1 is the set of real numbers x such that $x \neq 0$. That is, $\mathcal{D}(f_1) = \{\, x \mid x \neq 0 \,\}$.

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[3] If A and B are sets, then their intersection $A \cap B$ is the set of numbers in *both* A and B. Their *union* $A \cup B$ is the set of numbers in *at least one* of A and B.

[4] If $g(c) > 0$, by continuity of g at c and the definition of convergence of a sequence, $g(x_n) \in [\frac{1}{2}g(c), \frac{3}{2}g(c)]$ for all sufficiently large n and so it is positive. The argument in case $g(c) < 0$ is similar.

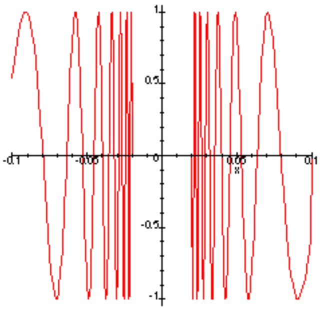
[5] The notation may seem a bit confusing. You may ask "is it the same x in both cases"? But this is not the right way to look at it. By the function $\sin x$, is meant the function which assigns to each real number x (say) the real number $\sin x$. If we said the function $\sin y$, or just sin, we would mean the same thing.

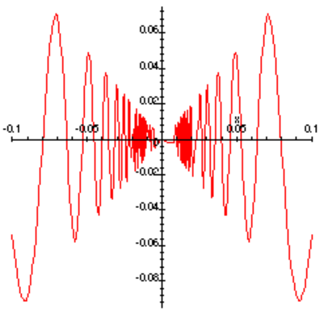
Similarly, the function $1/x$, or $1/y$, or "the reciprocal function", all mean the same thing.

Similarly, the function

$$f_2(x) = x \sin \frac{1}{x}$$

is continuous on its domain, which is the *same* domain as for f_1.





However, there is an interesting difference between f_1 and f_2. In the case of the latter we can define a new function g_2 by

$$g_2(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\mathcal{D}(g_2) = \mathbb{R}$ and $g_2(x) = f_2(x)$ if $x \neq 0$, i.e. if $x \in \mathcal{D}(f_2)$. Moreover, g_2 is continuous on its domain $\mathbb{R}$.

To show continuity of g_2 at $x \neq 0$, take any sequence $x_n \to x$. For all sufficiently large n, $x_n \in \mathcal{D}(f_2)$, and so $g_2(x_n) = f_2(x_n)$. It follows that $g_2(x_n) \to g_2(x)$ since $f_2(x_n) \to f(x)$ by the continuity of f. This means g_2 is continuous at x if $x \neq 0$.

To show continuity of g_2 at $x = 0$, take any sequence $x_n \to 0$. Then

$$-|x_n| \leq g_2(x_n) \leq |x_n|,$$

and so $g_2(x_n) \to 0 \; (= g_2(0))$ by the Squeeze Theorem. (We need to be a bit careful since some of the x_n may equal zero.) This means g_2 is continuous at 0.

In the case of f_1 there is *no* way of extending the function to a continuous function g_1 defined on all of $\mathbb{R}$. This is essentially because there is no number y such that $f_1(x_n) \to y$ for *every* sequence $x_n \to 0$ (with $x_n \neq 0$.)

We sometimes say that f_2 has a *removable singularity* at 0, and that the singularity of f_1 at 0 is *not removable*.

4.3.* Another Definition of Continuity

(Note that this entire section is * material. But you will use the definition of continuity, given in this section, next semester, and the ideas here should help your understanding of continuity.)

In Adams, a different definition is used for continuity of a function at a point. The two definitions are equivalent, as we will see.

The advantages of the definition via sequences which we use are that:

1. It is a little easier to understand.
2. It make the proofs of the theorems in the previous section easier, once we have the corresponding results for sequences.

Adams first defines what is meant by the limit of a finction f at a point c, written as $\lim_{x\to c} f(x)$ (see Definition 9 in Section 1.5 of Adams). A function f is then defined to be continuous at c if this limit exists and if $f(c) = \lim_{x\to c} f(x)$.

In the case of continuity, this becomes (Definition 1, Appendix III, Adams):

Definition 4.3.1. A function f *is continuous at a point* $c \in \mathcal{D}(f)$ if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that

$$x \in \mathcal{D}(f) \text{ and } |x-c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

We say f is *continuous* (on its domain) if f is continuous at every point in its domain.

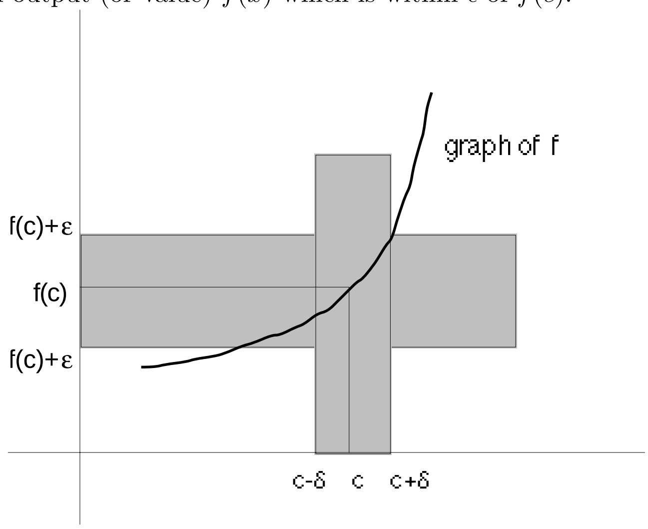
Notice that

$$|x-c| < \delta \quad \text{is the same as} \quad x \in (c-\delta, c+\delta).$$

and

$$|f(x) - f(c)| < \epsilon \quad \text{is the same as} \quad f(x) \in (f(c)-\epsilon, f(c)+\epsilon).$$

The idea of the definition is that for any given "tolerance" $\epsilon > 0$, there is a corresponding "tolerance" $\delta > 0$, such that any x in the domain of f within distance δ of c gives an output (or value) $f(x)$ which is within ϵ of $f(c)$.



$x \in (c-\delta, c+\delta) \to f(x) \in (f(c)-\varepsilon, f(c)+\varepsilon)$

i.e., $|x-c|<\delta \to |f(x)-f(c)|<\varepsilon$

For ε as shown, we have drawn the largest δ which will work.
But any smaller δ will also work.

The definition of continuity is given in Adams for functions defined on an interval. But the same definition works for functions defined on more general sets, and so to be consistent with Definition 4.1.1 we have stated the definition for the more general case. But to fix your ideas, you should think of the case where the domain of f, i.e. $\mathcal{D}(f)$, is an interval.

THEOREM 4.3.2. *A function f is continuous at c according to Definition 4.3.1 iff it is continuous at c according to Definition 4.1.1.*

PROOF. First suppose f is continuous at c according to Definition 4.3.1. We want to show that

$$x_n \to c \quad \Rightarrow \quad f(x_n) \to f(c). \tag{23}$$

In order to do this, suppose $\epsilon > 0$ is given (i.e. ϵ is some given 'tolerance"). We need to show there is a corresponding N such that

$$n > N \quad \Rightarrow \quad |f(x_n) - f(c)| < \epsilon. \tag{24}$$

First note from Definition 4.3.1 that given $\epsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

But if $x_n \to c$, then there is an N corresponding to δ such that

$$n > N \quad \Rightarrow \quad |x_n - c| < \delta.$$

Combining the last two implications gives (24). Since $\epsilon > 0$ was arbitrary, this gives (23). In other words, f is continuous at c according to Definition 4.1.1.

Converesely, suppose f is continuous at c according to Definition 4.1.1. Suppose $\epsilon > 0$ is given. We want to show there is some corresponding $\delta > 0$ such that

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon. \tag{25}$$

Assume there is no such δ (in order to obtain a contradiction). This means that for each $\delta > 0$ there is a real number x (with $x \in \mathcal{D}(f)$) such that

$$|x - c| < \delta \quad \text{but} \quad |f(x) - f(c)| \geq \epsilon.$$

Let $\delta = \frac{1}{n}$ and denote some x as above by x_n. Thus we have for each natural number n a real number x_n such that

$$|x_n - c| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - f(c)| \geq \epsilon.$$

It follows both that $x_n \to c$ and that it is *not* true that $f(x_n) \to f(c)$.

This contradicts the fact that f is continuous at c according to Definition 4.1.1, and so the assumption is false. In other words, there *is* a $\delta > 0$ corresponding to ϵ such that (25) is true. Since $\epsilon > 0$ was arbitrary, it follows that f is continuous at c according to Definition 4.3.1. □

4.4. Continuous functions on a closed bounded interval

See Adams pp 80–85 (pp 78–82 of the 3rd edn) and Appendix III. Although Adams uses a different definition of continuity (as discussed in the previous section) he uses the fact that his definition implies the definition here (first direction of Theorem 4.3.2, also Theorem 4 in Adams Appendix III), and then argues by using the definition here.

The following two theorems are quite "deep" and require the Completeness Axiom for their proof.

THEOREM 4.4.1 (Boundedness and Max-Min Theorems). *If f is continuous on $[a, b]$, then it is bounded there (i.e. there exists a constant K such that $|f(x)| \le K$ if $a \le x \le b$).*

Moreover, there exist points $u, v \in [a, b]$ such that for any $x \in [a, b]$ we have

$$f(v) \le f(x) \le f(u).$$

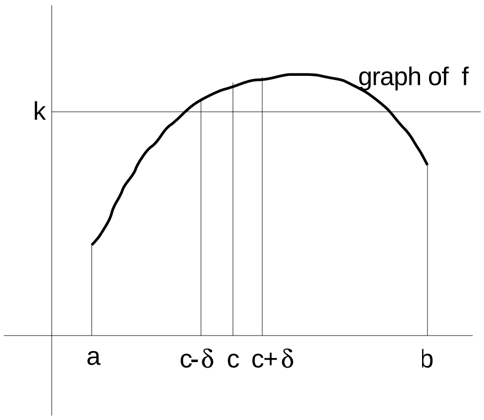
That is, f assumes maximum and minimum values on $[a, b]$.

PROOF. See Adams Appendix III, theorems 5 and 6. □

THEOREM 4.4.2 (Intermediate Value Theorem). *If f is continuous on $[a, b]$ and s is a number between $f(a)$ and $f(b)$ then there is a point $c \in [a, b]$ such that $f(c) = s$.*

PROOF. See Adams Appendix III, Theorem 7. □

REMARK 4.4.1. In the proofs, Adams uses the result that for any point $c \in [a, b]$, *if $f(c) > k$ then $f(x) > k$ on some interval* $(c - \delta, c + \delta)$ (if $c = a$ then we need to take an interval of the form $[a, a + \delta)$, if $c = b$ we need to take an interval of the form $(b - \delta, b]$).



f continuous and f(c)>k implies:
f(x)>k on some interval (c- δ,c+δ)

We prove this result by *assuming the result is not true* (in order to obtain a contradiction). This means that for each $\delta = \frac{1}{n}$ we can find an x_n such that

$$|x_n - c| < \frac{1}{n} \quad \text{but} \quad f(x_n) \le k.$$

Since $x_n \to c$, it follows by continuity of f that $f(x_n) \to f(c)$ Since $f(x_n) \le k$ for all n, it follows that $f(c) \le k$ (by Theorem 3.3.6 — taking $a_n = f(x_n)$ and $b_n = f(c)$).

This contradicts $f(c) > k$ and so the assumption is false and the required result is true.

EXAMPLE 4.4.3. See Example 11 in Section 1.4 of Adams. Here the Intermediate Value Theorem is used to justify the existence of a solution of the equation $x^3 - x - 1 = 0$.

One can also prove the existence of a number x such that $x^2 = 2$ in this manner. Just note that if $f(x) = x^2$ then $f(1) = 1$, $f(2) = 4$, and since f is continuous it follows by the Intermediate Value Theorem that $f(x) = 2$ for some x between 1 and 2.

Thus we have justified the existence of $\sqrt{2}$, i.e. a positive number whose square is 2.