# SPHERICAL CODES WITH GOOD SEPARATION, DISCREPANCY AND ENERGY

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*Equal-area partitions of* $\mathbb{S}^d$ *.*

An *equal area partition* of $\mathbb{S}^d$ is a nonempty finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d \subset \mathbb{R}^d$ , such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$ ,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where $\sigma$ is the Lebesgue area measure on $\mathbb{S}^d$ .

*Diameter bounded sets of partitions.*

The *diameter* of a region $R \subset \mathbb{R}^{d+1}$ is defined by

$$\operatorname{diam} R := \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in R\}.$$

A set $\Xi$ of partitions of $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is *diameter-bounded* with *diameter bound* $K \in \mathbb{R}_+$ if for all $\mathcal{P} \in \Xi$, for each $R \in \mathcal{P}$ ,

$$\operatorname{diam} R \leqslant K \, |\mathcal{P}|^{-1/d} .$$

*Spherical polar coordinates on* $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ *.*

*Spherical polar coordinates* describe $\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ by one longitude, $\xi_1 \in \mathbb{R}$ (modulo $2\pi$ ), and $d-1$ colatitudes, $\xi_j \in [0, \pi]$ , for $j \in \{2, \ldots, d\}$ .

The spherical polar to Cartesian coordinate map $\odot : \mathbb{R} \times [0,\pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\odot(\xi_1, \xi_2, \ldots, \xi_d) = (x_1, x_2, \ldots, x_{d+1}),$$

$$\text{where} \quad x_1 := \cos \xi_1 \prod_{j=2}^{d} \sin \xi_j, \quad x_2 := \prod_{j=1}^{d} \sin \xi_j,$$

$$x_k := \cos \xi_{k-1} \prod_{j=k}^{d} \sin \xi_j, \quad k \in \{3, \ldots, d+1\}.$$

*Spherical caps, zones, and collars.*

The *spherical cap* $S(\mathbf{p}, \theta) \subset \mathbb{S}^d$ is

$$S(\mathbf{p}, \theta) := \left\{ \mathbf{q} \in \mathbb{S}^d \mid \mathbf{p} \cdot \mathbf{q} \geqslant \cos(\theta) \right\} .$$

For $d > 1$ , a *zone* can be described by

$$Z(\tau, \beta) := \left\{ \odot(\xi_1, \ldots, \xi_d) \in \mathbb{S}^d \mid \xi_d \in [\tau, \beta] \right\} ,$$

where $0 \leqslant \tau < \beta \leqslant \pi$ .

$Z(0, \beta)$ is a North polar cap and $Z(\tau, \pi)$ is a South polar cap.

If $0 < \tau < \beta < \pi$ , $Z(\tau, \beta)$ is a *collar.*

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*Mesh norm (covering radius).*

The *mesh norm* of $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^d$ is

$$\text{mesh norm } X := \sup_{\mathbf{y}\in\mathbb{S}^d} \min_{\mathbf{x}\in X} \cos^{-1}(\mathbf{x}\cdot\mathbf{y}).$$

*Minimum distance and packing radius.*

The *minimum distance* of $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^d$ is

$$\text{min dist } X := \min_{\mathbf{x}\neq\mathbf{y}\in X} \|\mathbf{x}-\mathbf{y}\|,$$

and the *packing radius* of $X$ is

$$\text{prad } X := \min_{\mathbf{x}\neq\mathbf{y}\in X} \cos^{-1}(\mathbf{x}\cdot\mathbf{y})/2.$$

*Mesh ratio and packing density.*

The *mesh ratio* of $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^d$ is

$$\text{mesh ratio } X := \text{mesh norm } X \;/\; \text{prad } X.$$

The *packing density* of $X$ is

$$\text{pdens } X := \mathcal{N}\sigma\big(S(\mathbf{x}, \text{prad } X)\big)/\ \sigma(\mathbb{S}^d).$$

*Normalized spherical cap discrepancy.*

We use the probability measure $\overset{*}{\sigma} := \sigma/\sigma(\mathbb{S}^d)$.

For $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^d$ the *normalized spherical cap discrepancy* is

$$\text{disc } X := \sup_{\mathbf{y}\in\mathbb{S}^d} \sup_{\theta\in[0,\pi]} \left| \frac{|X\cap S(\mathbf{y},\theta)|}{\mathcal{N}} - \overset{*}{\sigma}\big(S(\mathbf{y},\theta)\big) \right|.$$

*Normalized $s$-energy.*

For $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_\mathcal{N}\} \subset \mathbb{S}^d$ the *normalized $s$-energy* is

$$E_s(X) := \mathcal{N}^{-2} \sum_{i=1}^{\mathcal{N}} \sum_{\mathbf{x}_i\neq\mathbf{x}_j\in X} \|\mathbf{x}_i-\mathbf{x}_j\|^{-s},$$

and the *normalized energy double integral* is

$$I_s := \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \|\mathbf{x}-\mathbf{y}\|^{-s}\, d\overset{*}{\sigma}(\mathbf{x}) d\overset{*}{\sigma}(\mathbf{y}).$$

*Separation and discrepancy imply energy.*

**Theorem.**

*Let $(X_1, X_2, \ldots \in \mathbb{N})$ be a sequence of $\mathbb{S}^d$ codes for which there exist $c_1, c_2 > 0$ such that each $X_\mathcal{N} = \{\mathbf{x}_{\mathcal{N},1}, \ldots, \mathbf{x}_{\mathcal{N},\mathcal{N}}\}$ satisfies*

$$\|\mathbf{x}_{\mathcal{N},i} - \mathbf{x}_{\mathcal{N},j}\| > c_1\ \mathcal{N}^{-1/d}, \quad (i\neq j)$$
$$\textit{disc } X_\mathcal{N} \leqslant c_2\ \mathcal{N}^{-q}.$$

*Then for the normalized $s$ energy for $0 < s < d$, we have for some $c_3 \geqslant 0$,*

$$E_s(X_\mathcal{N}) \leqslant I_s + c_3\ \mathcal{N}^{(s/d-1)q}.$$

*For EQSP Matlab code.*

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

`http://eqsp.sourceforge.net`

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