The Riesz energy of point sets on the unit sphere under weak-star convergence

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The players: the sphere, point sets, potentials

The unit sphere is $\mathbb{S}^d := \{ x \in \mathbb{R}^{d+1} \mid \|x\| = 1 \}$.

We treat sequences of point sets of the form $(X_1, X_2, \ldots)$, with

$$X_n := \{x_{n,1}, \ldots, x_{n,n}\} \subset \mathbb{S}^d, \text{ so that } |X_n| = n,$$

the Riesz potential

$$U_s(r) := r^{-s}, \quad 0 < s < d,$$

and Riesz kernel

$$\Phi_s(x, y) := U_s(\|x - y\|) = \|x - y\|^{-s}.$$

The story: separation, convergence, energy

For a sequence of point sets X on $\mathbb{S}^d$, define

$$\mathrm{E}_n(X)\Phi_s := \frac{1}{n^2}\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\Phi_s(x_{n,k}, x_{n,j}).$$

Let μ be the normalized Lebesgue area measure on $\mathbb{S}^d$. Define

$$\mathrm{I}\,\Phi_s := \int\int \Phi_s(x, y)\, d\mu(x)\, d\mu(y).$$

Theorem 1. *For a well-separated and weak-star convergent sequence of point sets* X *on* $\mathbb{S}^d$*, for* $0 < s < d$,

$$\mathrm{E}_n(X)\Phi_s \to \mathrm{I}\,\Phi_s \quad \textit{as} \quad n \to \infty.$$

Separation

We say our sequence X is well separated

if there is a constant γ such that

$$\|x - y\| > \gamma n^{-1/d}$$

for all $x, y \in X_n$.

Weak convergence

A sequence of measures $(\nu_1, \nu_2, \ldots)$

on a compact metric space S

converges weakly to the measure ν if and only if

$$\int_S f(x)\, d\nu_n(x) \to \int_S f(x)\, d\nu(x)$$

for all continuous f.

(Billingsley 1968, 1999)

Weak-star convergence

Our point set $X_n \subset \mathbb{S}^d$ defines a normalized counting measure via equal weight quadrature,

$$\mu_n(A) := \frac{|X_n \cap A|}{|X_n|} = \frac{|X_n \cap A|}{|n|},$$

$$\int_{\mathbb{S}^d} f(x)\, d\mu_n(x) = \frac{1}{n} \sum_{k=1}^{n} f(x_{n,k}).$$

We say that X is weak-star convergent if μ_n converges weakly to μ, the normalized Lebesgue area measure on $\mathbb{S}^d$.

Precedents: minimum energy points

Let Ω_s be a sequence of minimum $\mathbf{E}_n(.)\Phi_s$ energy point sets on $\mathbb{S}^d$, for $0 < s < d$. Then

$\mathbf{E}_n(\Omega_s)\Phi_s = \mathbf{I}\,\Phi_s + O(n^{s/d-1})$ and
$\mathbf{E}_n(\Omega_s)\Phi_s \leqslant \mathbf{I}\,\Phi_s - Cn^{s/d-1}$.

(Wagner 1990, 1992, Rakhmanov, Saff and Zhou 1994, Brauchart 2004)

For $d - 1 \leqslant s < d$, Ω_s is well separated.
(Dahlberg 1978, Kuijlaars, Saff and Sun 2004)

It is well known that for $0 < s < d$, Ω_s is weak-star convergent.
(Götz 2000, 2003, Götz and Saff 2000, Damelin and Grabner 2003)

Examples

1. Minimum energy sequences.

   For $d-1 \leqslant s' < d$ and $0 < s < d$,

   $$\mathbf{E}_n(\Omega_{s'})\Phi_s \longrightarrow \mathbf{I}\,\Phi_s.$$

2. Well-separated, diameter-bounded equal area sequences.
   (Alexander 1972, Stolarsky 1973,
   Rakhmanov, Saff and Zhou 1994, Zhou 1995,
   Kuijlaars and Saff 1998)
3. Well-separated spherical designs.
   (Grabner and Tichy 1993, Hesse and Leopardi (submitted))
   - For strength t, spherical cap discrepancy is $O(t^{-1})$.

Punctured measures

For a measure ν and a point x,

define the punctured measure $\nu[x]$ by

$$\nu[x](A) := \nu\left(A \setminus \{x\}\right).$$

Then, eg.

$$\int_{\mathbb{S}^d} f(y)\, d\mu_n[x_{n,k}](y) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} f(x_{n,j}).$$

Energy functionals

For a measure ν and a kernel ϕ defined on a space S define

$$\mathcal{I}(\nu)\phi := \int_S \int_S \phi(x,y)\, d\nu(y)\, d\nu(x),$$

$$\mathcal{E}(\nu)\phi := \int_S \int_S \phi(x,y)\, d\nu[x](y)\, d\nu(x).$$

For $\mathbb{S}^d$ and the measures μ, and μ_n for a fixed X, define

$$\mathrm{I}_n := \mathcal{I}(\mu_n), \qquad \mathrm{I} := \mathcal{I}(\mu),$$

$$\mathrm{E}_n := \mathcal{E}(\mu_n), \qquad \mathrm{E} := \mathcal{E}(\mu).$$

These definitions agree with those used in Theorem 1.

Proof of Theorem 1: Cutoff potential

Fix $0 < s < d$.

Fix a weak-star convergent and well-separated sequence X,

with separation constant γ.

Define $\Delta_m := \gamma m^{-1/d}$.

Now define the cutoff potential U_m by

$$U_m(r) := \begin{cases} \Delta_m^{-s} & (0 < r \leqslant \Delta_m) \\ U(r) = r^{-s} & (r > \Delta_m). \end{cases}$$

Proof of Theorem 1: Split

By the triangle inequality, for any $m, n \in \mathbb{N}$,

$$|\mathrm{E}_n \Phi - \mathrm{I}\,\Phi| \leqslant a_n + b_{m,n} + c_{m,n} + d_{m,n} + e_m,$$

where

$$\begin{aligned} a_n &:= |\mathrm{E}_n \Phi - \mathrm{E}_n \Phi_n|, \\ b_{m,n} &:= |\mathrm{E}_n \Phi_n - \mathrm{E}_n \Phi_m|, \\ c_{m,n} &:= |\mathrm{E}_n \Phi_m - \mathrm{I}_n \Phi_m|, \\ d_{m,n} &:= |\mathrm{I}_n \Phi_m - \mathrm{I}\,\Phi_m|, \\ e_m &:= |\mathrm{I}\,\Phi_m - \mathrm{I}\,\Phi|. \end{aligned}$$

Convergence of $a_n = |\mathrm{E}_n(\Phi - \Phi_n)|$

Since $(U - U_n)(r) = 0$ for $r \geqslant \Delta_n$,

the separation condition guarantees that

$$a_n = 0 \quad \text{for any } n.$$

Convergence of $b_{m,n} = |\mathrm{E}_n(\Phi_n - \Phi_m)|$

For $m \geqslant n$, $b_{m,n} = 0$ by the same argument as for a_n.

Define $\Theta_n := 2\sin^{-1}(\Delta_n/2)$.

For $m < n$, for any k, separation gives a bound on

$$\mu_n\big(S(x_{n,k}, \theta) \setminus S(x_{n,k}, \Theta_n)\big)$$

in terms of the packing of small spherical caps $S(y, \Theta_n/2)$ within the larger cap $S(x_{n,k}, \theta + \Theta_n/2)$.

We use this bound with a Riemann-Stieltjes integral to show that
$b_{m,n} < \frac{\epsilon}{4}$ if both m and $n \geqslant M_b$,
where M_b is defined using ϵ, d, s and γ.

Convergence of $c_{m,n} = |(\mathrm{E}_n - \mathrm{I}_n)\Phi_m|$

$$c_{m,n} = \frac{1}{n^2}\sum_{k=1}^{n}\Phi_m(x_{n,k}, x_{n,k})$$
$$= \frac{U_m(0)}{n} = \frac{\Delta_m^{-s}}{n} = \gamma^{-s} m^{s/d} n^{-1}$$
$$< \frac{\epsilon}{4} \quad \text{if} \quad n \geqslant N_c(m),$$

where

$$N_c(m) := 4\gamma^{-s} m^{s/d} \epsilon^{-1} + 1.$$

Convergence of $d_{m,n} = |(\mathrm{I}_n - \mathrm{I})\Phi_m|$

For any m, the weak-star convergence of X ensures that

$$d_{m,n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

since weak convergence of $\mu_n \longrightarrow \mu$
implies weak convergence of $\mu_n \times \mu_n \longrightarrow \mu \times \mu$.

(Billingsley 1968, 1999)

So we have $d_{m,n} < \frac{\epsilon}{4}$ if $n \geqslant N_d(m)$,
where $N_d(m)$ depends on ϵ, m and X.

Convergence of $e_m = |\mathrm{I}(\Phi_m - \Phi)|$

We have

$$e_m = \frac{\omega_d}{\omega_{d+1}} \int_0^{\Theta_m} (r^{-s} - \Delta_m^{-s})(\sin\theta)^{d-1} d\theta$$

where $\omega_d := \sigma(\mathbb{S}^{d-1})$ and $r = 2\sin(\theta/2)$.

It can be shown that $e_m < \frac{\epsilon}{4}$ if $m \geqslant M_e$,
where M_e is defined using ϵ, d, s and γ.

Proof of Theorem 1: Reassembly

So we have $|\mathbf{E}_n \, \Phi - \mathbf{I} \, \Phi| < \epsilon$ if

$$n \geqslant \max \left(M, N_c(M), N_d(M) \right),$$

where

$$M := \max(M_b, M_e).$$

In other words, $|\mathbf{E}_n \, \Phi - \mathbf{I} \, \Phi| \to 0$ as $n \to \infty$.