Approximating functions in Clifford algebras: What to do with negative eigenvalues?

Paul Leopardi

Mathematical Sciences Institute, Australian National University.

For presentation at Australian Mathematical Society Meeting, Adelaide.

September 2009

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

Motivation

Functions in Clifford algebras are a special case of matrix functions, as can be seen via representation theory. The square root and logarithm functions, in particular, pose problems for the author of a general purpose library of Clifford algebra functions. This is partly because the *principal* square root and logarithm of a matrix do not exist for a matrix containing a negative eigenvalue.

(Higham 2008)

Problems

1. Define the square root and logarithm of a multivector in the case where the matrix representation has negative eigenvalues.
2. Predict or detect negative eigenvalues.

Topics

- Clifford algebras
- Functions in Clifford algebras
- Dealing with negative eigenvalues
- Predicting negative eigenvalues?
- Detecting negative eigenvalues

Clifford algebras

Each Clifford algebra $\mathbb{R}_{p,q}$ is a real associative algebra generated by $n = p + q$ anticommuting generators, p of which square to 1 and q of which square to -1.

(Braden 1985; Lam and Smith 1989; Porteous 1995; Lounesto 1997)

Start with a group of signed integer sets

Generators: $\{k\}$ where $k \in \mathbb{Z}^*$.

Relations: Element (-1) in the centre.

$$\begin{aligned}
(-1)^2 &= 1, \\
(-1)\{k\} &= \{k\}(-1) \quad \text{(for all } k\text{)}, \\
\{k\}^2 &= \begin{cases} -1 & (k < 0), \\ 1 & (k > 0), \end{cases} \\
\{j\}\{k\} &= (-1)\{k\}\{j\} \quad (j \neq k).
\end{aligned}$$

Canonical ordering:

$$\{j, k, \ell\} := \{j\}\{k\}\{\ell\} \quad (j < k < \ell), \text{ etc.}$$

Product of signed sets is signed XOR.

(Braden 1985; Lam and Smith 1989; Lounesto 1997; Dorst 2001)

Extend to a real linear algebra

Vector space: Real linear combination of $\mathbb{Z}^*$ sets.

$$v = \sum_{S \subset \mathbb{Z}^*} v_S S.$$

Multiplication: Extends group multiplication.

$$\begin{aligned} vw &= \sum_{S \in \mathbb{Z}^*} v_S S \sum_{T \subset \mathbb{Z}^*} w_T T \\ &= \sum_{S \in \mathbb{Z}^*} \sum_{T \subset \mathbb{Z}^*} v_S w_T ST. \end{aligned}$$

Clifford algebra $\mathbb{R}_{p,q}$ uses subsets of $\{-q, \ldots, p\}^*$.

(Braden 1985; Lam and Smith 1989; Wene 1992; Lounesto 1997; Dorst 2001; Ashdown)

Some examples of Clifford algebras

$$\mathbb{R}_{\mathbf{0,0}} \equiv \mathbb{R}.$$

$$\mathbb{R}_{\mathbf{0,1}} \equiv \mathbb{R} + \mathbb{R}\{-1\} \equiv \mathbb{C}.$$

$$\mathbb{R}_{\mathbf{1,0}} \equiv \mathbb{R} + \mathbb{R}\{1\} \equiv {}^{2}\mathbb{R}.$$

$$\mathbb{R}_{\mathbf{1,1}} \equiv \mathbb{R} + \mathbb{R}\{-1\} + \mathbb{R}\{1\} + \mathbb{R}\{-1, 1\} \equiv \mathbb{R}(2).$$

$$\mathbb{R}_{\mathbf{0,2}} \equiv \mathbb{R} + \mathbb{R}\{-2\} + \mathbb{R}\{-1\} + \mathbb{R}\{-2, -1\} \equiv \mathbb{H}.$$

Matrix representations of Clifford algebras

Each Clifford algebra $\mathbb{R}_{p,q}$ is isomorphic to a matrix algebra over $\mathbb{R}$, ${}^2\mathbb{R} := \mathbb{R} + \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or ${}^2\mathbb{H}$ per the following table, with periodicity of 8. The $\mathbb{R}$ and ${}^2\mathbb{R}$ matrix algebras are highlighted in red.

p \ q	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$
1	${}^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
2	$\mathbb{R}(2)$	${}^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	${}^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$
5	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
6	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(64)$	$\mathbb{R}(128)$

(Hile and Lounesto 1990; Porteous 1995; Lounesto 1997; Leopardi 2004)

Real representations

A real matrix representation is obtained by representing each complex or quaternion value as a real matrix.

$$\mathbb{R}_{0,1} \equiv \mathbb{C}: \quad \rho(x + y\{-1\}) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

$$\mathbb{R}_{1,0} \equiv {}^2\mathbb{R}: \quad \rho(x + y\{1\}) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

$$\mathbb{R}_{0,2} \equiv \mathbb{H}:$$

$$\rho(\, w + x\{-2\} + y\{-1\} + z\{-2,-1\}\,) = \begin{bmatrix} w & -y & -x & z \\ y & w & -z & -x \\ x & -z & w & -y \\ z & x & y & w \end{bmatrix}$$

$2^n \times 2^n$ real matrices for some n.

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Real chessboard

p ↓ \ q →	0	1	2	3	4	5	6	7
0	1	2	4	8	8	8	8	16
1	2	2	4	8	16	16	16	16
2	2	4	4	8	16	32	32	32
3	4	4	8	8	16	32	64	64
4	8	8	8	16	16	32	64	128
5	16	16	16	16	32	32	64	128
6	16	32	32	32	32	64	64	128
7	16	32	64	64	64	64	128	128

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Definition of matrix functions

For a function f analytic in $\Omega \subset \mathbb{C}$,

$$f(X) := \frac{1}{2\pi i} \int_{\partial\Omega} f(z)\,(zI - X)^{-1}\,dz,$$

where the spectrum $\Lambda(X) \subset \Omega$.

For f analytic on an open disk $D \supset \Lambda(X)$ with $0 \in D$,

$$f(X) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} X^k.$$

For invertible Y, $\quad f(YXY^{-1}) = Y f(X) Y^{-1}$.

(Rinehart 1955; Golub and van Loan 1983, 1996; Horn and Johnson 1994)

Functions in Clifford algebras

For f analytic in $\Omega \subset \mathbb{C}$, $\mathbf{x}$ in a Clifford algebra,

$$f(\mathrm{x}) := \frac{1}{2\pi i} \int_{\partial\Omega} f(z)\,(z - \mathrm{x})^{-1}\,dz,$$

where the spectrum $\Lambda(\rho\mathrm{x}) \subset \Omega$, with $\rho\mathbf{x}$ the matrix representing $\mathbf{x}$ (adapted from Higham 2008).

What about negative eigenvalues?

Use an algebra with $\mathbf{i}$: $\mathbf{i}^2 = -1$, $\mathbf{i}\mathbf{x} = \mathbf{x}\mathbf{i}$ for all $\mathbf{x}$:
Full $\mathbb{C}$ matrix algebra.

Embeddings:

$$\mathbb{R} \equiv \mathbb{R}_{0,0} \subset \mathbb{R}_{0,1} \equiv \mathbb{C}.$$

$${}^2\mathbb{R} \equiv \mathbb{R}_{1,0} \subset \mathbb{R}_{1,2} \equiv \mathbb{C}(2).$$

$$\mathbb{R}(2) \equiv \mathbb{R}_{1,1} \subset \mathbb{R}_{1,2} \equiv \mathbb{C}(2).$$

$$\mathbb{H} \equiv \mathbb{R}_{0,2} \subset \mathbb{R}_{1,2} \equiv \mathbb{C}(2).$$

Complex chessboard

$q \to$ / $p \downarrow$	0	1	2	3	4	5	6	7
0	1	1	2	4	4	4	8	16
1	2	2	2	4	8	8	8	16
2	2	4	4	4	8	16	16	16
3	2	4	8	8	8	16	32	32
4	4	4	8	16	16	16	32	64
5	8	8	8	16	32	32	32	64
6	8	16	16	16	32	64	64	64
7	8	16	32	32	32	64	128	128

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Real–complex chessboard

p ↓ \ q →	0	1	2	3	4	5	6	7
0	1	2	4	8	8	8	8	16
1	2	2	4	8	16	16	16	16
2	2	4	4	8	16	32	32	32
3	4	4	8	8	16	32	64	64
4	8	8	8	16	16	32	64	128
5	16	16	16	16	32	32	64	128
6	16	32	32	32	32	64	64	128
7	16	32	64	64	64	64	128	128

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Example: ${}^2\mathbb{R} \equiv \mathbb{R}_{1,0}$

$$ {}^2\mathbb{R} \equiv \mathbb{R}_{1,0} \subset \mathbb{R}_{1,2} \subset \mathbb{R}_{2,2} \equiv \mathbb{R}(4). $$

$$ \rho(x + y\{1\}) = \begin{bmatrix} x & y & & \\ y & x & & \\ & & x & y \\ & & y & x \end{bmatrix} $$

$$ \mathbf{i} = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} $$

(Cartan and Study 1908; Porteous 1969; Lounesto 1997)

Definitions of sqrt and log

When the matrix representing $\mathbf{x}$ has a negative eigenvalue and no imaginary eigenvalues, define

$$\begin{aligned}\mathtt{sqrt}(\mathbf{x}) &:= \frac{\mathbf{1}+\mathbf{i}}{\sqrt{\mathbf{2}}}\mathtt{sqrt}(-\mathbf{i}\mathbf{x}),\\ \mathtt{log}(\mathbf{x}) &:= \mathtt{log}(-\mathbf{i}\mathbf{x}) + \mathbf{i}\frac{\pi}{\mathbf{2}},\end{aligned}$$

where $\mathbf{i}^2 = -\mathbf{1}$ and $\mathbf{i}\mathbf{x} = \mathbf{x}\mathbf{i}$.

When $\mathbf{x}$ also has imaginary eigenvalues, the real matrix representing $-\mathbf{i}\mathbf{x}$ has negative eigenvalues. Find some ϕ such that $\mathtt{exp}(\mathbf{i}\phi)\mathbf{x}$ does not have negative eigenvalues, and define

$$\begin{aligned}\mathtt{sqrt}(\mathbf{x}) &:= \mathtt{exp}\Big(-\mathbf{i}\frac{\phi}{\mathbf{2}}\Big)\mathtt{sqrt}\big(\mathtt{exp}(\mathbf{i}\phi)\mathbf{x}\big),\\ \mathtt{log}(\mathbf{x}) &:= \mathtt{log}\big(\mathtt{exp}(\mathbf{i}\phi)\mathbf{x}\big) - \mathbf{i}\phi.\end{aligned}$$

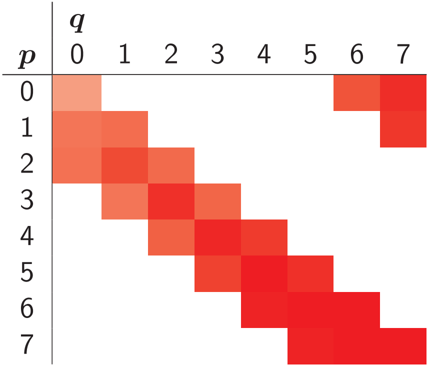
Predicting negative eigenvalues?

In Clifford algebras with a faithful irreducible *complex* or *quaternion* representation, a multivector with independent $N(0,1)$ random coefficients is *unlikely* to have a negative eigenvalue. In large Clifford algebras with an irreducible real representation, such a random multivector is very likely to have a negative eigenvalue.

(Ginibre 1965; Edelman, Kostlan and Shub 1994; Edelman 1997; Forrester and Nagao 2007)

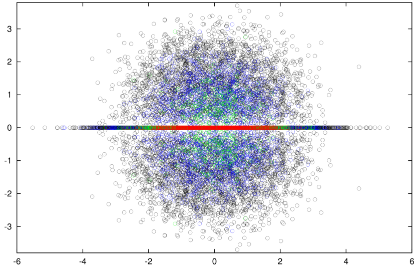
Predicting negative eigenvalues?

Probability of a negative eigenvalue is denoted by shades of red.



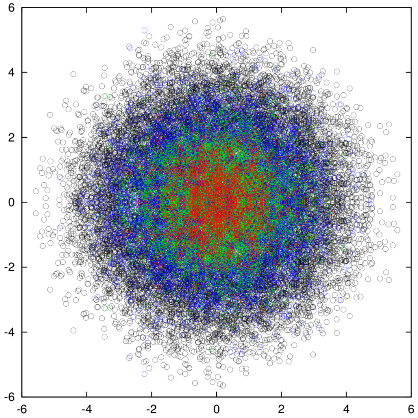
This phenomenon is a direct consequence of the eigenvalue density of the Ginibre ensembles.

Real Ginibre ensemble



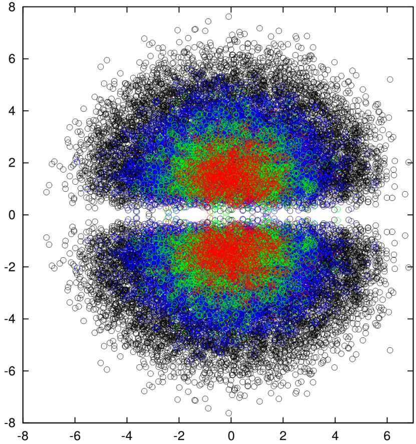
Eigenvalue density of real representations of Real Ginibre ensemble.

Complex Ginibre ensemble



Eigenvalue density of real representations of Complex Ginibre ensemble.

Quaternion Ginibre ensemble



Eigenvalue density of real representations of Quaternion Ginibre ensemble.

Detecting negative eigenvalues

Trying to predict negative eigenvalues using the p and q of $\mathbb{R}_{p,q}$ is futile. Negative eigenvalues are always possible, since $\mathbb{R}_{p,q}$ contains $\mathbb{R}_{p',q'}$ for all $p' \leqslant p$ and $q' \leqslant q$.

The eigenvalue densities of the Ginibre ensembles simply make testing more complicated.

In the absence of an efficient algorithm to detect negative eigenvalues only, it is safest to use a standard algorithm to find all eigenvalues. If the real Schur factorization is used this allows the `sqrt` and `log` algorithms to operate on triangular matrices only.

(Higham 2008).

Schur form and QR algorithm

Schur form:

Block triangular, eigenvalues on diagonal. Eg.

$$T = \begin{bmatrix} -2 & 7 & 19 & 3i \\ & -2 & -5 & 0 \\ & & i & 1 \\ & & & 9 \end{bmatrix}$$

QR algorithm:

Iterative algorithm for Schur decomposition
$X = QTQ^*$: originally iterated QR decomposition.

Schur form is more numerically stable than Jordan form.

(Golub and van Loan 1983, 1996; Davies and Higham 2002; Higham 2008)

Further problem

Devise an algorithm which detects negative eigenvalues only, and is more efficient than standard eigenvalue algorithms.

GluCat — Clifford algebra library

- Generic library of universal Clifford algebra templates.
- C++ template library for use with other libraries.
- Implements algorithms for matrix functions.

For details, see **http://glucat.sf.net**

(Lounesto et al. 1987; Lounesto 1992; Raja 1996; Leopardi 2001-2008)