Positive quadrature on the sphere and conjectures on monotonicities of Jacobi polynomials

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Outline of talk

- Some definitions,
- Property (R) and Reimer's proofs,
- Conjectures on Jacobi polynomials,
- Partial results in $[-1/2, 1/2]^2$,
- Weaker result for $\alpha \geqslant \beta > -1/2$,
- Application to Property (R).

Some definitions: 1 – notation

$$\mathbb{S}^d := \left\{ \mathrm{x} \in \mathbb{R}^{d+1} \mid \sum_{k=1}^{d+1} x_k^2 = 1 \right\},$$

$$\omega_d := \sigma(\mathbb{S}^d),$$

$$\widetilde{P}_n^{(\alpha,\beta)} := P_n^{(\alpha,\beta)} / P_n^{(\alpha,\beta)}(1),$$

$$\Theta_n^{(\alpha,\beta)} := \text{smallest zero in } \theta \text{ of } P_n^{(\alpha,\beta)}(\cos\theta),$$

$$Z_\alpha(z) := \Gamma(\alpha+1) \ \left(\tfrac{2}{z}\right)^\alpha \ J_\alpha(z).$$

Some definitions: 2 – polynomial spaces

We use $\mathbb{P}_n(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of maximum total degree n, restricted to $\mathbb{S}^d$, with dimension

$$\mathcal{D}(d,n) := \dim \mathbb{P}_n(\mathbb{S}^d) = \binom{n+d}{d} + \binom{n+d-1}{d}$$

and reproducing kernel $\Phi_n^{(d+1)}(\mathrm{x},\mathrm{y}) := \Phi_n^{(d+1)}(\mathrm{x}\cdot\mathrm{y})$, where

$$\Phi_n^{(d+1)} := \frac{2}{\omega_d}\,\frac{(d+1)_{n-1}}{(\frac{d}{2}+1)_{n-1}}\,P_n^{(\frac{d}{2},\frac{d}{2}-1)}$$

$$= \frac{\mathcal{D}(d,n)}{\omega_d}\,\widetilde{P}_n^{(\frac{d}{2},\frac{d}{2}-1)}.$$

Property (R)

Quadrature regularity: Le Gia and Sloan (1999), Sloan and Womersley (2000). Later refined into Property (R).

An admissible sequence of quadrature rules $(Q_1, \ldots)$ on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, has rule $Q_t = (X_t, W_t)$ with strength t and cardinality $|X_t| = \mathcal{N}_t$, with all weights $W_{t,k}$ positive.

An admissible sequence of quadrature rules has property (R) (Hesse and Sloan, 2003, 2004) if and only if, given $\phi \in]0, \frac{\pi}{2}]$, there exists positive constants γ and t_0 such that for all $\mathrm{y} \in \mathbb{S}^d$ and each rules Q_t in the sequence, if $t \geqslant t_0$ then

$$\sum_{\mathrm{x}_{t,k} \in \mathsf{S}\left(\mathrm{y}, \frac{\phi}{t}\right)} w_{t,k} \leqslant \gamma\, \sigma\left(\mathsf{S}\left(\mathrm{y}, \frac{\phi}{t}\right)\right).$$

Property (R) and Reimer's proofs: 1

Reimer (2000, 2003) proved that any admissible sequence of quadrature rules is quadrature regular and satisfies Property (R).

The (2000) proof uses $P_n^{(\frac{d}{2},\frac{d}{2}-1)}$, and the following limit theorem (Szegö 1939 – 1975).

Theorem 1. *For* $\alpha, \beta > -1$, $z \in \mathbb{C}$,

$$\lim_{n \to \infty} \widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = Z_\alpha(z).$$

The formula holds uniformly in every bounded region of the complex z *plane.*

Property (R) and Reimer's proofs: 2

From Reimer's proofs (2000, 2003) immediately follows:

Lemma 1. *Let* $Q := (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *of strength* $2n$.

Let $K := \Phi_n^{(d+1)}$.

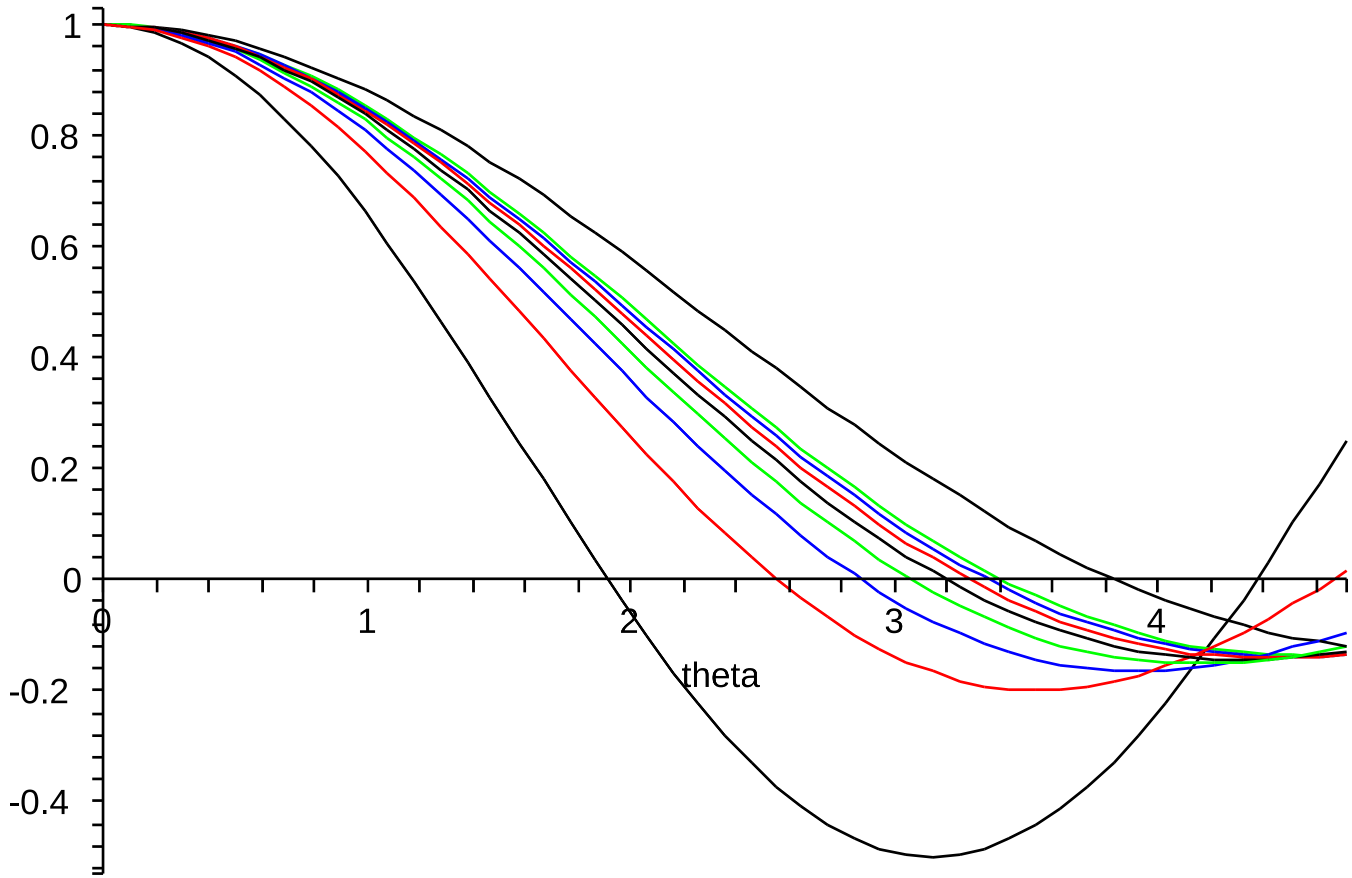
Then for $\theta \in \left]0, n\Theta_n^{(\frac{d}{2},\frac{d}{2}-1)}\right[$, *for any* $\mathrm{y} \in \mathbb{S}^d$,

$$\sum_{\mathrm{x}_k \in \mathsf{S}\left(\mathrm{y}, \frac{\theta}{n}\right)} w_k \leqslant \frac{K(1)}{K^2\left(\cos\frac{\theta}{n}\right)}$$

$$= \frac{\omega_d}{\mathcal{D}(d,n)} \left(\widetilde{P}_n^{(\frac{d}{2},\frac{d}{2}-1)}\left(\cos\frac{\theta}{n}\right)\right)^{-2}.$$

Monotonicity of $\widetilde{P}_n^{(1,0)}(\cos\theta/n)$?

Sequence of $\widetilde{P}_n^{(1,0)}(\cos\theta/n)$ seems monotonic to the first zero:



Conjectures on Jacobi polynomials

Conjecture 1. *For* $\alpha > -1$, $\beta > -1$,
if for $\theta \in]0, \Theta_1^{(\alpha,\beta)}]$ *we have*

$$\widetilde{P}_1^{(\alpha,\beta)}(\cos\theta) < \widetilde{P}_2^{(\alpha,\beta)}\left(\cos\frac{\theta}{2}\right) \tag{1}$$

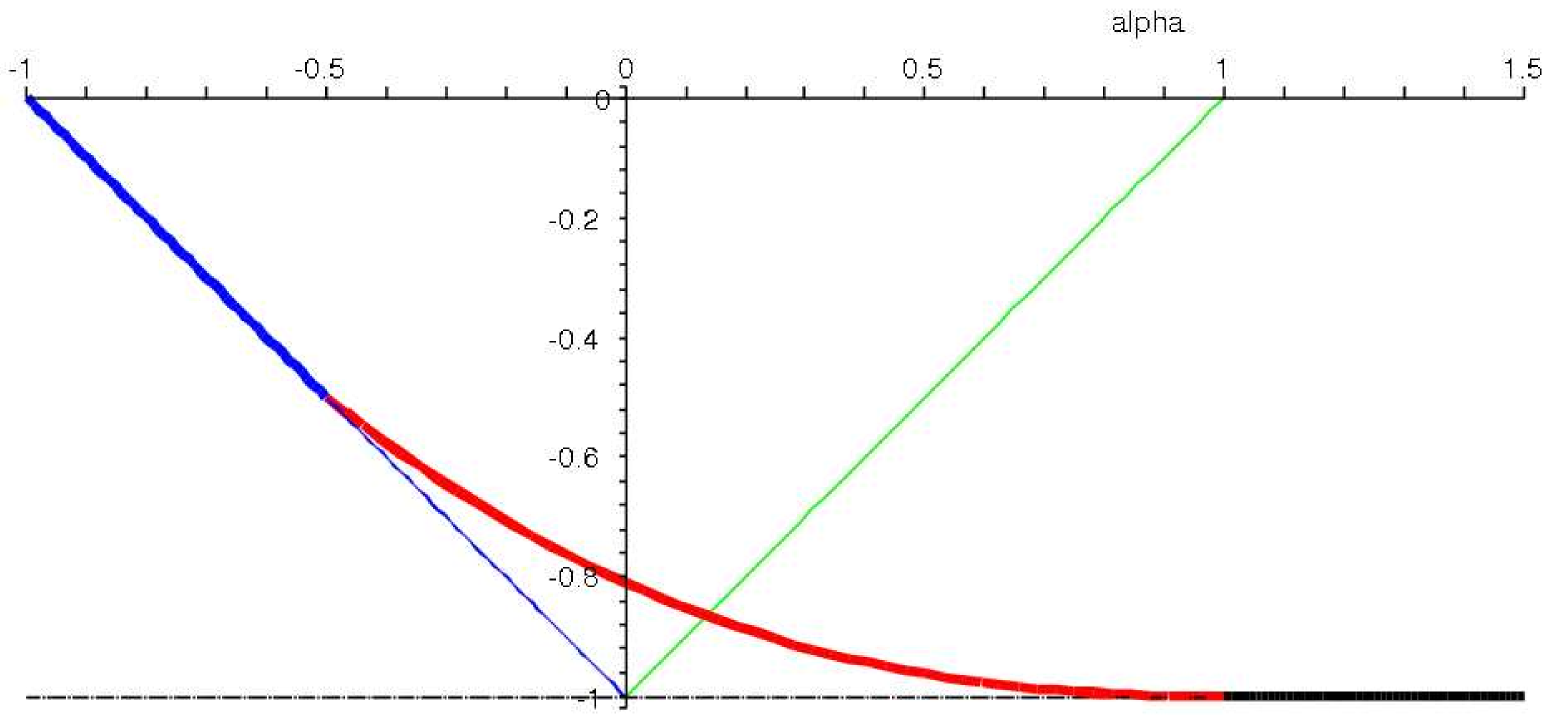
then for $n \geqslant 1$, $\theta \in]0, n\Theta_n^{(\alpha,\beta)}]$, *we have*

$$\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) < \widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right) \tag{2}$$

and therefore

$$n\Theta_n^{(\alpha,\beta)} < (n+1)\Theta_{n+1}^{(\alpha,\beta)}. \tag{3}$$

Where does premise (1) hold?



$$\left(3\alpha^2+2\alpha\beta-\beta^2+9\alpha+\beta+4\right)\sqrt{\frac{\beta+1}{\alpha+\beta+2}}+(\alpha+\beta)^2+3\alpha+7\beta+4=0.$$

Partial results in $[-1/2, 1/2]^2$

Previously known (Gegenbauer polymonials):

$$\left(n + \tfrac{1}{2} + \alpha\right) \Theta_n^{(\alpha,\alpha)} < \left(n + \tfrac{3}{2} + \alpha\right) \Theta_{n+1}^{(\alpha,\alpha)}$$

for $n \geqslant 1$, $\alpha \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ (Szegö (1939)).

So far proved:

$$n\Theta_n^{(\alpha,\beta)} < (n+1)\Theta_{n+1}^{(\alpha,\beta)} \quad \text{for } n \geqslant 1,\ (\alpha,\beta) \in \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2$$

(Sturm comparison or Gatteschi (1987)),

$$\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) < \widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right)$$

for $n \geqslant 1$, $\theta \in]0,\pi[$, $(\alpha,\beta) \in \left\{\left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ (Koumandos 2005).

Weaker result for $\alpha \geqslant \beta > -1/2$

Theorem 2. *For* $n \geqslant 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in]0, \frac{\pi}{2}]$, *we have*

$$\left(2n\,\sin\frac{\theta}{2n}\right)^{\alpha-\beta}\left(n\,\sin\frac{\theta}{n}\right)^{\beta+\frac{1}{2}}\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) <$$
$$\left((2n+2)\,\sin\frac{\theta}{2n+2}\right)^{\alpha-\beta}\left((n+1)\,\sin\frac{\theta}{n+1}\right)^{\beta+\frac{1}{2}}\widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right).$$

Proved by Sturm comparison using

$$F_n^{(\alpha,\beta)}(\theta) := \frac{1}{n^2}\left(\frac{\frac{1}{4}-\alpha^2}{4\sin^2\frac{\theta}{2n}} + \frac{\frac{1}{4}-\beta^2}{4\cos^2\frac{\theta}{2n}}\right) + \left(1+\frac{\alpha+\beta+1}{2n}\right)^2,$$

$$V_n^{(\alpha,\beta)}(\theta) := \left(2n\,\sin\frac{\theta}{2n}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2n}\right)^{\beta+\frac{1}{2}}\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right),$$

$$\frac{\partial^2}{\partial\theta^2}V_n^{(\alpha,\beta)}(\theta) + F_n^{(\alpha,\beta)}(\theta)\,V_n^{(\alpha,\beta)}(\theta) = 0.$$

Application to Property (R): 1

From Lemma 1 and Conjecture 1 immediately follows:

Conjecture 2. *Let* $Q := (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *of strength* $2n$.

Then for $\theta \in \left]0, \Theta_1^{(\frac{d}{2},\frac{d}{2}-1)}\right[$*, for any* $\mathbf{y} \in \mathbb{S}^d$*,*

$$\sum_{\mathbf{x}_k \in \mathsf{S}\left(\mathbf{y}, \frac{\theta}{n}\right)} w_k \leqslant \frac{\omega_d}{\mathcal{D}(d,n)} \left(\widetilde{P}_1^{(\frac{d}{2},\frac{d}{2}-1)}(\cos\theta)\right)^{-2}.$$

Application to Property (R): 2

Conjecture 3. *For* $t \geqslant t_0 \geqslant 2$, *let* $Q = (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *which is exact on* $\mathbb{P}_t(\mathbb{S}^d)$.
Then for $\phi \in \,]0, \pi[$, *for any* $\mathbf{y} \in \mathbb{S}^d$, *we have*

$$\sum_{\mathbf{x}_k \in S\left(\mathbf{y}, \frac{\phi}{t}\right)} w_k \leqslant c_1 \, t^{-d} \leqslant c_1 \, c_2 \, \sigma \left(S \left(\mathbf{y}, \frac{\phi}{t} \right) \right),$$

where

$$c_1 := 2^{d-1} \, \omega_d \, d! \, \left(\widetilde{P}_1^{\left(\frac{d}{2}, \frac{d}{2}-1\right)} \left(\cos \frac{\phi}{2} \right) \right)^{-2},$$

$$c_2 := \frac{d}{\omega_{d-1}} \left(\operatorname{sinc} \frac{\phi}{t_0} \right)^{-d+1} \phi^{-d}.$$

Application to Property (R): 3

Lemma 1 and our weaker result, Theorem 2, give us only:

Theorem 3. *With the same conditions and notation as Conjecture 3, for* $\phi \in \,]0, \pi[$, *for any* $\mathrm{y} \in \mathbb{S}^d$, *we have*

$$\sum_{\mathrm{x}_k \in S\left(\mathrm{y}, \frac{\phi}{t}\right)} w_k \leqslant c_3 \, t^{-d} \leqslant c_3 \, c_2 \, \sigma \left(S \left(\mathrm{y}, \frac{\phi}{t} \right) \right),$$

where

$$c_3 := c_1 \left(\operatorname{sinc} \frac{\phi}{2} \right)^{-d-1},$$

$$c_1, c_2 \quad \text{as per Conjecture 3.}$$