Positive quadrature on the sphere and conjectures on monotonicities of Jacobi polynomials

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Outline of talk

- Some definitions,
- Property (R) and Reimer's proofs,
- Conjectures on Jacobi polynomials,
- Partial results in $[-1/2, 1/2]^2$,
- Weaker result for $\alpha \geqslant \beta > -1/2$,
- Application to Property (R).

Some definitions: 1 – notation

$$\mathbb{S}^d := \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \mid \sum_{k=1}^{d+1} x_k^2 = 1 \right\},$$

$$\omega_d := \sigma(\mathbb{S}^d),$$

$$\widetilde{P}_n^{(\alpha,\beta)} := P_n^{(\alpha,\beta)} / P_n^{(\alpha,\beta)}(1),$$

$$\Theta_n^{(\alpha,\beta)} := \text{ smallest zero in } \theta \text{ of } P_n^{(\alpha,\beta)}(\cos\theta),$$

$$Z_\alpha(z) := \Gamma(\alpha+1)\ \left(\tfrac{2}{z}\right)^\alpha J_\alpha(z).$$

Some definitions: 2 – polynomial spaces

We use $\mathbb{P}_n(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of maximum total degree n, restricted to $\mathbb{S}^d$, with dimension

$$\mathcal{D}(d,n) := \dim \mathbb{P}_n(\mathbb{S}^d) = \binom{n+d}{d} + \binom{n+d-1}{d}$$

and reproducing kernel $\Phi_n^{(d+1)}(\mathrm{x},\mathrm{y}) := \Phi_n^{(d+1)}(\mathrm{x}\cdot\mathrm{y})$, where

$$\Phi_n^{(d+1)} := \frac{2}{\omega_d} \frac{(d+1)_{n-1}}{(\frac{d}{2}+1)_{n-1}} P_n^{(\frac{d}{2},\frac{d}{2}-1)}$$
$$= \frac{\mathcal{D}(d,n)}{\omega_d} \widetilde{P}_n^{(\frac{d}{2},\frac{d}{2}-1)}.$$

Property (R)

Quadrature regularity: Le Gia and Sloan (1999), Sloan and Womersley (2000). Later refined into Property (R).

An admissible sequence of quadrature rules $(Q_1, \ldots)$ on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, has rule $Q_t = (X_t, W_t)$ with strength t and cardinality $|X_t| = \mathcal{N}_t$, with all weights $W_{t,k}$ positive.

An admissible sequence of quadrature rules has property (R) (Hesse and Sloan, 2003, 2004) if and only if, given $\phi \in (0, \frac{\pi}{2}]$, there exists positive constants γ and t_0 such that for all $\mathrm{y} \in \mathbb{S}^d$ and each rules Q_t in the sequence, if $t \geqslant t_0$ then

$$\sum_{\mathrm{x}_{t,k} \in \mathsf{S}\left(\mathrm{y}, \frac{\phi}{t}\right)} w_{t,k} \leqslant \gamma\, \sigma\left(\mathsf{S}\left(\mathrm{y}, \frac{\phi}{t}\right)\right).$$

Property (R) and Reimer's proofs: 1

Reimer (2000, 2003) proved that any admissible sequence of quadrature rules is quadrature regular and satisfies Property (R).

The (2000) proof uses $P_n^{(\frac{d}{2},\frac{d}{2}-1)}$, and the following limit theorem (Szegö 1939 – 1975).

Theorem 1. *For* $\alpha, \beta > -1$, $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = Z_\alpha(z).$$

The formula holds uniformly in every bounded region of the complex z *plane.*

Property (R) and Reimer's proofs: 2

From Reimer's proofs (2000, 2003) immediately follows:

Lemma 1. *Let* $Q := (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *of strength* $2n$.

Let $K := \Phi_n^{(d+1)}$.

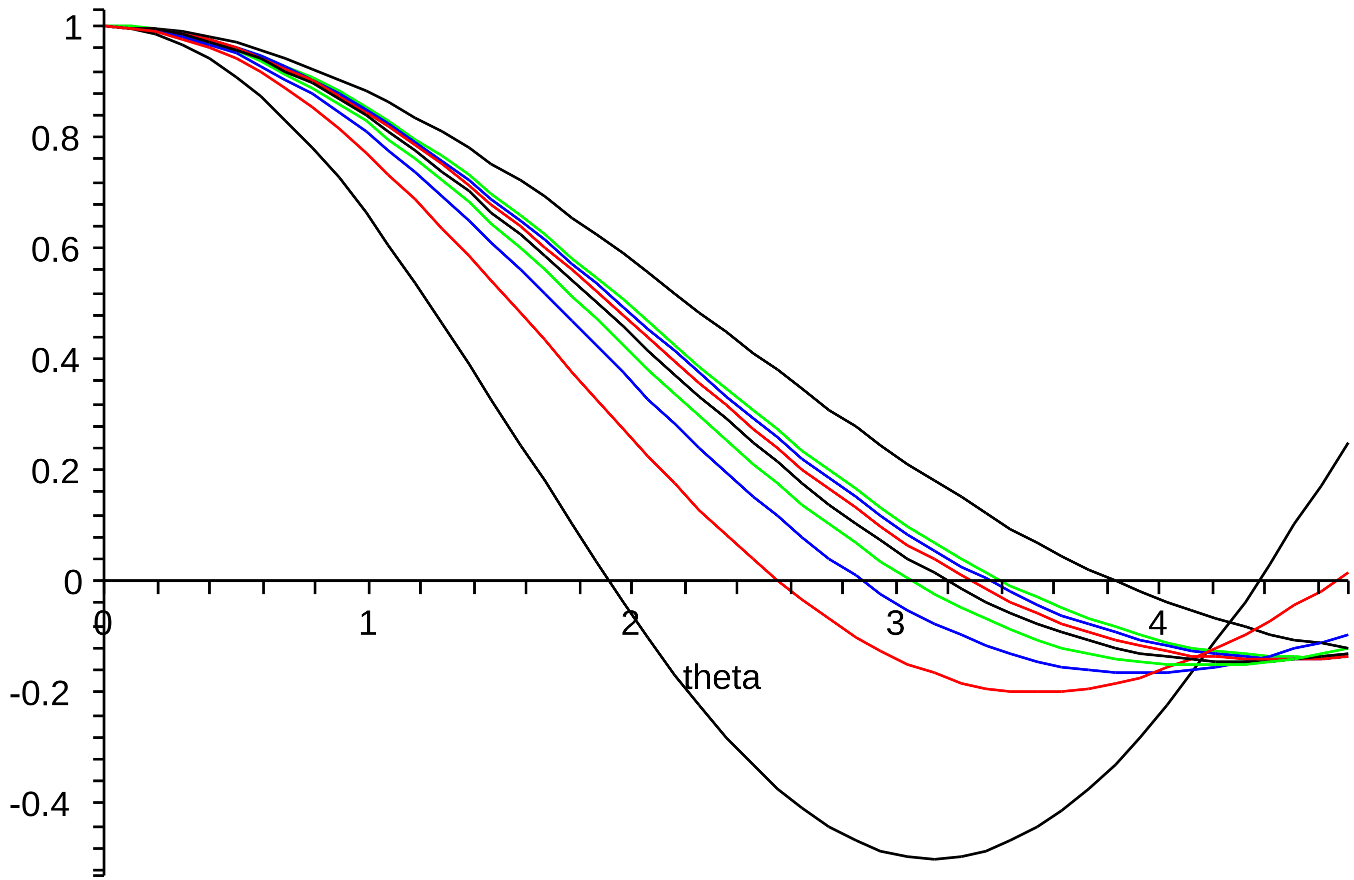
Then for $\theta \in \left(0, n\Theta_n^{\left(\frac{d}{2}, \frac{d}{2}-1\right)}\right)$, *for any* $\mathrm{y} \in \mathbb{S}^d$,

$$\sum_{\mathrm{x}_k \in \mathsf{S}\left(\mathrm{y}, \frac{\theta}{n}\right)} w_k \leqslant \frac{K(1)}{K^2\left(\cos\frac{\theta}{n}\right)}$$

$$= \frac{\omega_d}{\mathcal{D}(d,n)} \left(\widetilde{P}_n^{\left(\frac{d}{2}, \frac{d}{2}-1\right)}\left(\cos\frac{\theta}{n}\right)\right)^{-2}.$$

Monotonicity of $\widetilde{P}_n^{(1,0)}(\cos\theta/n)$?

Sequence of $\widetilde{\boldsymbol{P}}_{\boldsymbol{n}}^{(1,0)}(\cos\boldsymbol{\theta}/\boldsymbol{n})$ seems monotonic to the first zero:



Conjectures on Jacobi polynomials

Conjecture 1. *For* $\alpha > -1$, $\beta > -1$, *if for* $\theta \in (0, \Theta_1^{(\alpha,\beta)}]$ *we have*

$$\widetilde{P}_1^{(\alpha,\beta)}(\cos\theta) < \widetilde{P}_2^{(\alpha,\beta)}\left(\cos\frac{\theta}{2}\right) \tag{1}$$

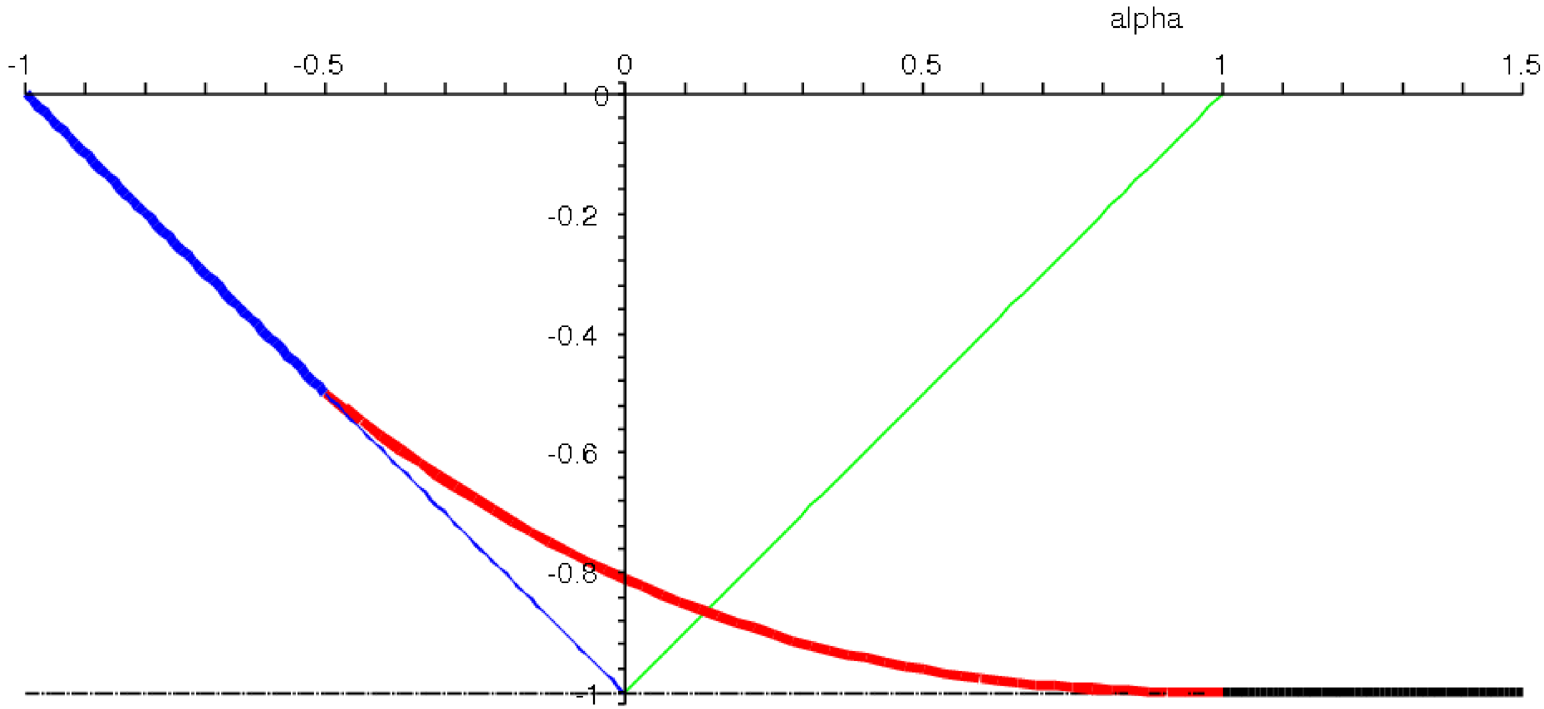
then for $n \geqslant 1$, $\theta \in (0, n\Theta_n^{(\alpha,\beta)}]$, *we have*

$$\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) < \widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right) \tag{2}$$

and therefore

$$n\Theta_n^{(\alpha,\beta)} < (n+1)\Theta_{n+1}^{(\alpha,\beta)}. \tag{3}$$

Where does premise (1) hold?



$$(3\alpha^2 + 2\alpha\beta - \beta^2 + 9\alpha + \beta + 4)\sqrt{\frac{\beta+1}{\alpha+\beta+2}} + (\alpha+\beta)^2 + 3\alpha + 7\beta + 4 = 0.$$

Partial results in $[-1/2, 1/2]^2$

So far proved:

$$n\Theta_n^{(\alpha,\beta)} < (n+1)\Theta_{n+1}^{(\alpha,\beta)} \quad \text{for } n \geqslant 1,\ (\alpha,\beta) \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)^2$$

(Sturm comparison or Gatteschi (1987)),

$$\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) < \widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right)$$

for $n \geqslant 1,\ \theta \in (0,\pi),\ (\alpha,\beta) \in \left\{\left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}$

(Koumandos 2005).

Weaker result for $\alpha \geqslant \beta > -1/2$

Theorem 2. *For* $n \geqslant 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, *we have*

$$\left(2n \sin\frac{\theta}{2n}\right)^{\alpha-\beta} \left(n \sin\frac{\theta}{n}\right)^{\beta+\frac{1}{2}} \widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) < \left((2n+2)\sin\frac{\theta}{2n+2}\right)^{\alpha-\beta} \left((n+1)\sin\frac{\theta}{n+1}\right)^{\beta+\frac{1}{2}} \widetilde{P}_{n+1}^{(\alpha,\beta)}\left(\cos\frac{\theta}{n+1}\right).$$

Proved by Sturm comparison using

$$F_n^{(\alpha,\beta)}(\theta) := \frac{1}{n^2}\left(\frac{\frac{1}{4}-\alpha^2}{4\sin^2\frac{\theta}{2n}} + \frac{\frac{1}{4}-\beta^2}{4\cos^2\frac{\theta}{2n}}\right) + \left(1+\frac{\alpha+\beta+1}{2n}\right)^2,$$

$$V_n^{(\alpha,\beta)}(\theta) := \left(2n\sin\frac{\theta}{2n}\right)^{\alpha+\frac{1}{2}} \left(\cos\frac{\theta}{2n}\right)^{\beta+\frac{1}{2}} \widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right),$$

$$\frac{\partial^2}{\partial\theta^2} V_n^{(\alpha,\beta)}(\theta) + F_n^{(\alpha,\beta)}(\theta)\, V_n^{(\alpha,\beta)}(\theta) = 0.$$

Application to Property (R): 1

From Lemma 1 and Conjecture 1 immediately follows:

Conjecture 2. *Let* $Q := (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *of strength* $2n$.

Then for $\theta \in (0, \Theta_1^{(\frac{d}{2}, \frac{d}{2}-1)})$, *for any* $\mathbf{y} \in \mathbb{S}^d$,

$$\sum_{\mathbf{x}_k \in \mathsf{S}\left(\mathbf{y}, \frac{\theta}{n}\right)} w_k \leqslant \frac{\omega_d}{\mathcal{D}(d, n)} \left(\widetilde{P}_1^{(\frac{d}{2}, \frac{d}{2}-1)}(\cos\theta)\right)^{-2}.$$

Application to Property (R): 2

Conjecture 3. *For* $t \geqslant t_0 \geqslant 2$, *let* $Q = (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *which is exact on* $\mathbb{P}_t(\mathbb{S}^d)$.
Then for $\phi \in (0, \pi)$, *for any* $\mathrm{y} \in \mathbb{S}^d$, *we have*

$$\sum_{\mathrm{x}_k \in S\left(\mathrm{y}, \frac{\phi}{t}\right)} w_k \leqslant c_1\, t^{-d} \leqslant c_1\, c_2\, \sigma\left(S\left(\mathrm{y}, \frac{\phi}{t}\right)\right),$$

where

$$c_1 := 2^{d-1}\, \omega_d\, d!\, \left(\widetilde{P}_1^{\left(\frac{d}{2}, \frac{d}{2}-1\right)}\left(\cos\frac{\phi}{2}\right)\right)^{-2},$$

$$c_2 := \frac{d}{\omega_{d-1}} \left(\operatorname{sinc}\frac{\phi}{t_0}\right)^{-d+1} \phi^{-d}.$$

Application to Property (R): 3

Lemma 1 and our weaker result, Theorem 2, give us only:

Theorem 3. *With the same conditions and notation as Conjecture 3, for* $\phi \in (0, \pi)$, *for any* $\mathbf{y} \in \mathbb{S}^d$, *we have*

$$\sum_{\mathbf{x}_k \in S\left(\mathbf{y}, \frac{\phi}{t}\right)} w_k \leqslant c_3\, t^{-d} \leqslant c_3\, c_2\, \sigma\left(S\left(\mathbf{y}, \frac{\phi}{t}\right)\right),$$

where

$$c_3 := c_1 \left(\operatorname{sinc} \frac{\phi}{2}\right)^{-d-1},$$

c_1, c_2 *as per Conjecture 3*.