# Approximating the square root and logarithm functions in Clifford algebras: what to do in the case of negative eigenvalues?

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The square and the exponential functions are commonplace in Geometric Algebra. The square of a vector is the scalar value of its quadratic form [11, Chapter 14] (or its negative, depending on convention). The exponential of a bivector is a rotor [2, Sections 4.2, 11.3]. In general, a rotor is an element of the spin group corresponding to the Clifford algebra of the quadratic space being studied [1, Section IV]. The spin group is a subgroup of the invertible elements of the even subalgebra of this Clifford algebra.

We are therefore led to the study of the square and the exponential functions as applied to general multivectors. This, then leads to the question of how to define inverse functions for the square and the exponential functions, in other words, the square root and logarithm functions. In particular, the importance of the numerical calculation of the logarithm of a multivector has long been recognized, dating back at least as far as the CLICAL package [10]. More recently, Wareham, Cameron and Lasenby have used a logarithm function $\ell(\cdot)$ to interpolate between displacement rotors in Conformal Geometric Algebra [13].

Functions in Clifford algebras are a special case of matrix functions, as can be seen via representation theory. CLICAL calculates such functions by using the QR algorithm [10, Section 3]. The square root and logarithm functions pose problems for the author of a general purpose library of Clifford algebra functions, partly because the *principal* square root and principal logarithm of a matrix do not exist for a matrix containing a negative eigenvalue [7, Section 1.7]. The principal square root and principal logarithm of a real matrix have the desirable property that they are both real matrices.

We are thus led to the following problems:

1. Define the square root and logarithm of a multivector in the case where the matrix representation has negative eigenvalues.

2. Predict or detect negative eigenvalues.

The analogous situation for real numbers is very instructive. A negative real number does not have a real square root or a real logarithm, but in both cases there is a solution if the real line is incorporated into the complex plane. In this case, for $x < 0$ and complex $c \neq 0$,

$$\sqrt{x} = \sqrt{1/c}\,\sqrt{cx},$$
$$\log(x) = \log(cx) - \log c,$$

as can be seen by squaring both sides of the first equation, and exponentiating both sides of the second equation. This works regardless of which branch of the multivalued square root and logarithm function are taken.

The general multivector case is only a little more complicated. Multivectors do not commute in general, but each real Clifford algebra $\mathcal{A}$ is a subalgebra of a (possibly larger) real finite-dimensional real Clifford algebra $\mathcal{C}$, containing the element $\iota$ as a pseudoscalar, such that $\iota^2 = -1$ and such that the two dimensional subalgebra generated by $\iota$ is the centre $Z(\mathcal{C})$ of the algebra $\mathcal{C}$; that is every element of $\mathcal{C}$ commutes with every element of $Z(\mathcal{C})$. Thus $\mathcal{C}$ is isomorphic to an algebra over the complex field.

In this case, for $\mathbf{x} \in \mathcal{A}$ and any $c \in Z(\mathcal{C})$ with $c \neq 0$, if $c\mathbf{x}$ has no negative eigenvalues, then the principal square root and principal logarithm of $c\mathbf{x}$ are well defined. We can therefore define

$$\begin{aligned}\mathtt{sqrt}(\mathbf{x}) &:= \sqrt{1/c}\,\sqrt{c\mathbf{x}},\\ \mathtt{log}(\mathbf{x}) &:= \log(c\mathbf{x}) - \log c,\end{aligned}$$

where the square root and logarithm of $c\mathbf{x}$ on the right hand side are the principal square root and logarithm respectively, and where the square root and logarithm in $Z(\mathcal{C})$ are defined via the isomorphism with the complex field.

Even if $\mathbf{x} \in \mathcal{A}$, this definition may yield $\mathtt{sqrt}(\mathbf{x}) \notin \mathcal{A}$ or $\mathtt{log}(\mathbf{x}) \notin \mathcal{A}$. This may not be desirable, but it is perfectly in accord with the situation in the case where $x$ is a negative real number and neither $\sqrt{x}$ nor $\log(x)$ are real. In fact, even if the principal square root and logarithm of $\mathbf{x}$ exist, while they may be represented by real matrices of the same size as $\mathbf{x}$, they may still not be members of $\mathcal{A}$. The question of the geometric interpretation and use of such values is open.

More precise definitions of the resulting square root and logarithm functions are as follows:

When the matrix representing $\mathbf{x}$ has a negative eigenvalue and no imaginary eigenvalues, we can simply define

$$\begin{aligned}\mathtt{sqrt}(\mathbf{x}) &:= \frac{1+\iota}{\sqrt{2}}\sqrt{-\iota\mathbf{x}},\\ \mathtt{log}(\mathbf{x}) &:= \log(-\iota\mathbf{x}) + \iota\frac{\pi}{2},\end{aligned}$$

where $\iota^2 = -1$ and $\iota\mathbf{x} = \mathbf{x}\iota$. Multiplication of $X$ by $-\iota$ rotates the eigenvalues in the complex plane by $-\frac{\pi}{2}$. The matrix representing the multivector $-\iota\mathbf{x}$ thus has no negative eigenvalues because the matrix representing $\mathbf{x}$ has no imaginary eigenvalues.

When $\mathbf{x}$ also has imaginary eigenvalues, the real matrix representing $-\iota\mathbf{x}$ has negative eigenvalues. In this case, we find some real $\phi$ such that $\mathtt{exp}(\iota\phi)\mathbf{x}$ has no negative eigenvalues, and define

$$\begin{aligned}\mathtt{sqrt}(\mathbf{x}) &:= \mathtt{exp}\Big(-\iota\frac{\phi}{2}\Big)\sqrt{\mathtt{exp}(\iota\phi)\mathbf{x}},\\ \mathtt{log}(\mathbf{x}) &:= \log\big(\mathtt{exp}(\iota\phi)\mathbf{x}\big) - \iota\phi.\end{aligned}$$

A simple example is the case $\mathbb{R}_{1,0}$, generated by $\mathrm{e}_1$, where $\mathrm{e}_1^2 = 1$. The eigenvalues of the matrix representing $\mathrm{e}_1$ in the usual real representation of $\mathbb{R}_{1,0}$ are -1 and 1. Thus $\mathrm{e}_1$ does not have a principal square root or principal logarithm. If we embed $\mathbb{R}_{1,0}$ into $\mathbb{R}_{3,0}$, generated by $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$, where $\mathrm{e}_1^2 = \mathrm{e}_2^2 = \mathrm{e}_3^2 = 1$, we can define

$$\begin{aligned}\mathtt{sqrt}(\mathrm{e}_1) &:= \frac{1}{2} + \frac{1}{2}\,\mathrm{e}_1 - \frac{1}{2}\,\mathrm{e}_2\,\mathrm{e}_3 + \frac{1}{2}\,\mathrm{e}_1\,\mathrm{e}_2\,\mathrm{e}_3,\\ \mathtt{log}(\mathrm{e}_1) &:= -\frac{\pi}{2}\,\mathrm{e}_2\,\mathrm{e}_3 + \frac{\pi}{2}\,\mathrm{e}_1\,\mathrm{e}_2\,\mathrm{e}_3,\end{aligned}$$

and verify that $\mathtt{sqrt}(\mathrm{e}_1) \times \mathtt{sqrt}(\mathrm{e}_1) = e_1$ and $\exp(\mathtt{log}(\mathrm{e}_1)) = e_1$.

Another example in $\mathbb{R}_{3,0}$ is the vector $v := -2\,\mathrm{e}_1 + 2\,\mathrm{e}_2 - 3\,\mathrm{e}_3$, whose matrix representation has eigenvalues approximately equal to $-4.12311$ and $4.12311$. Since $\mathbb{R}_{3,0}$ is isomorphic to a complex matrix algebra, $\mathtt{sqrt}(v)$ and $\mathtt{log}(v)$ are both contained in $\mathbb{R}_{3,0}$. We find that

$$\begin{aligned}\mathtt{sqrt}(v) &:\simeq 1.015 - 0.4925\,\mathrm{e}_1 + 0.4925\,\mathrm{e}_2 - 0.7387\,\mathrm{e}_3\\ &\qquad + 0.7387\,\mathrm{e}_1\,\mathrm{e}_2 + 0.4925\,\mathrm{e}_1\,\mathrm{e}_3 + 0.4925\,\mathrm{e}_2\,\mathrm{e}_3 + 1.015\,\mathrm{e}_1\,\mathrm{e}_2\,\mathrm{e}_3,\\ \mathtt{log}(v) &:\simeq 1.417 + 1.143\,\mathrm{e}_1\,\mathrm{e}_2 + 0.7619\,\mathrm{e}_1\,\mathrm{e}_3 + 0.7619\,\mathrm{e}_2\,\mathrm{e}_3 + 1.571\,\mathrm{e}_1\,\mathrm{e}_2\,\mathrm{e}_3\,.\end{aligned}$$

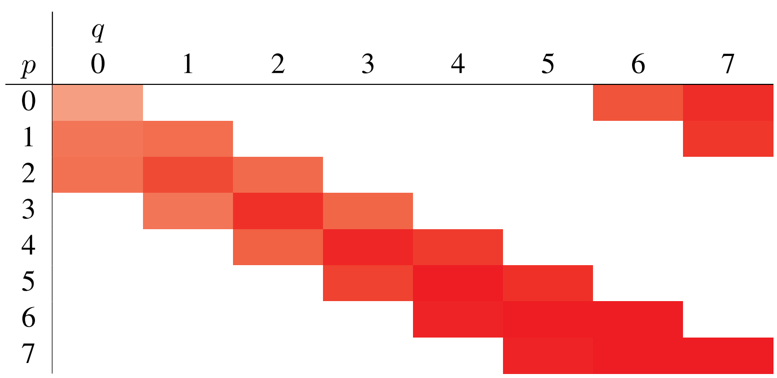
Each Clifford algebra $\mathbb{R}_{p,q}$ is generated by $n = p + q$ anticommuting generators, $p$ of which square to 1 and $q$ of which square to -1; and is isomorphic to a matrix algebra over $\mathbb{R}$, ${}^2\mathbb{R} := \mathbb{R} + \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or ${}^2\mathbb{H}$ per the following table, with periodicity of 8 [8, 9, 11, 12]. The $\mathbb{R}$ and ${}^2\mathbb{R}$ matrix algebras are highlighted in red.

| $p$ \ $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | ${}^2\mathbb{H}$ | $\mathbb{H}(2)$ | $\mathbb{C}(4)$ | $\mathbb{R}(8)$ | ${}^2\mathbb{R}(8)$ |
| 1 | ${}^2\mathbb{R}$ | $\mathbb{R}(2)$ | $\mathbb{C}(2)$ | $\mathbb{H}(2)$ | ${}^2\mathbb{H}(2)$ | $\mathbb{H}(4)$ | $\mathbb{C}(8)$ | $\mathbb{R}(16)$ |
| 2 | $\mathbb{R}(2)$ | ${}^2\mathbb{R}(2)$ | $\mathbb{R}(4)$ | $\mathbb{C}(4)$ | $\mathbb{H}(4)$ | ${}^2\mathbb{H}(4)$ | $\mathbb{H}(8)$ | $\mathbb{C}(16)$ |
| 3 | $\mathbb{C}(2)$ | $\mathbb{R}(4)$ | ${}^2\mathbb{R}(4)$ | $\mathbb{R}(8)$ | $\mathbb{C}(8)$ | $\mathbb{H}(8)$ | ${}^2\mathbb{H}(8)$ | $\mathbb{H}(16)$ |
| 4 | $\mathbb{H}(2)$ | $\mathbb{C}(4)$ | $\mathbb{R}(8)$ | ${}^2\mathbb{R}(8)$ | $\mathbb{R}(16)$ | $\mathbb{C}(16)$ | $\mathbb{H}(16)$ | ${}^2\mathbb{H}(16)$ |
| 5 | ${}^2\mathbb{H}(2)$ | $\mathbb{H}(4)$ | $\mathbb{C}(8)$ | $\mathbb{R}(16)$ | ${}^2\mathbb{R}(16)$ | $\mathbb{R}(32)$ | $\mathbb{C}(32)$ | $\mathbb{H}(32)$ |
| 6 | $\mathbb{H}(4)$ | ${}^2\mathbb{H}(4)$ | $\mathbb{H}(8)$ | $\mathbb{C}(16)$ | $\mathbb{R}(32)$ | ${}^2\mathbb{R}(32)$ | $\mathbb{R}(64)$ | $\mathbb{C}(64)$ |
| 7 | $\mathbb{C}(8)$ | $\mathbb{H}(8)$ | ${}^2\mathbb{H}(8)$ | $\mathbb{H}(16)$ | $\mathbb{C}(32)$ | $\mathbb{R}(32)$ | ${}^2\mathbb{R}(64)$ | $\mathbb{R}(128)$ |

A real matrix representation is obtained by representing each complex or quaternion value as a real matrix. Each complex or quaternion matrix entry can also be represented as a real matrix, giving a real matrix representation.

If the distribution of coefficients for a particular set of multivectors is known, the likelihood of negative eigenvalues can be predicted, at least in the following special case. In Clifford algebras with a faithful irreducible *complex* or *quaternion* representation, a multivector with independent $N(0,1)$ random coefficients is *unlikely* to have a negative eigenvalue. In large Clifford algebras with a faithful irreducible *real* representation, a multivector with independent $N(0,1)$ random coefficients is *very likely* to have a negative eigenvalue.

The table below illustrates this. Probability is denoted by shades of red.




This phenomenon is a direct consequence of the eigenvalue density of the Ginibre ensembles [3, 4, 5, 6].

In general, though, trying to predict negative eigenvalues using the $p$ and $q$ of $\mathbb{R}_{p,q}$ is futile. Negative eigenvalues are always possible, since $\mathbb{R}_{p,q}$ contains $\mathbb{R}_{p',q'}$ for all $p' \leqslant p$ and $q' \leqslant q$. The eigenvalue densities of the Ginibre ensembles simply make testing more complicated.

In the absence of an efficient algorithm to detect negative eigenvalues only, it is safest to use a standard algorithm to find all eigenvalues.

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