CAN COMPATIBLE DISCRETIZATION, FINITE ELEMENT METHODS, AND DISCRETE CLIFFORD ANALYSIS BE FRUITFULLY COMBINED?

PAUL LEOPARDI

Abstract. This paper describes work in progress, towards the formulation, implementation and testing of compatible discretization of differential equations, using a combination of Finite Element Exterior Calculus and discrete Geometric Calculus / Clifford analysis. Much work has been done in the two seemingly separate areas of the Finite Element Method and Geometric Calculus for over 42 years, and the first part of this paper briefly describes some of this work. The combination of the two methods could be called Finite Element Geometric Calculus (FEGC). The second part of the paper gives a tentative description of what FEGC might reasonably be expected to look like, if it were to be developed.

1. Introduction

In view of the 42-plus year histories of both the Finite Element Method (e.g. Zlámal [58]) and Geometric Calculus / Clifford analysis (e.g. Hestenes [30]), it is somewhat surprising that, as far as I know, no systematic attempt has so far been made to combine these subject areas to produce new methods for the solution of differential equations. This is especially surprising in view of the development of Finite Element Exterior Calculus (FEEC), which is based on differential forms, and therefore inherits the structure of Grassmann's exterior algebra.

In 2010, at the 4th conference on Applied Geometric Algebras in Computer Science and Engineering (AGACSE 2010) in Amsterdam, both D. Hestenes [32] and C. Doran [18] called for Geometric Calculus to be applied to the Finite Element Method. This paper is not a response to that call, but rather an outline of the features one might reasonably expect to see in the methods of Finite Element Geometric Calculus, if such methods are ever developed.

2. Related previous work

Related previous work falls into two categories: (1) compatible discretization; and (2) Geometric Calculus / Clifford analysis. This work is briefly described below.

2.1. **Compatible discretization.** Many physical quantities can be formulated in terms of a variational principle, that is, in terms of a trajectory in a suitably defined abstract space which makes some functional of the motion stationary, usually at a maximum or a minimum. The prototypical example of such a principle is Hamilton's Principle of Stationary Action [54, Sect. 1.8] [25, Sect. 10.2]. Noether's Theorem [47] [54, Chap. 3] [25, Sect. 20.1] states that certain symmetries in the equations describing a variational principle give rise to quantities which are conserved by the motion. Simply put, symmetries are equivalent to conservation

Date: Submitted 27 January 2012.

2000 *Mathematics Subject Classification.* 65N30,58A10,15A66.

Key words and phrases. Finite Element Exterior Calculus, Geometric Calculus, Clifford analysis.

laws. Noether's Theorem has also been generalized to cover some non-conservative systems [54, Sect. 3.12].

The idea of compatible (or *mimetic*) discretization [1, 5] is to create a discrete description of a physical phenomenon which preserves many or all of the same conservation laws which are obeyed by the continuous description given by a differential equation. Thus if a method using compatible discretization can calculate a conserved quantity accurately, the accuracy is maintained by the incorporation of the conservation law into the discretization.

Some of the tools of compatible discretization include (1) the continuous description of the physical phenomenon using equations involving differential forms on manifolds; (2) the analysis of the symmetries of the equations; and (3) discretization by dividing the manifold into cells, chains and complexes, with corresponding differential forms.

A number of compatible discretization methods, such as that of Desbrun et al. [16], are based on the use of differential forms and on fundamental objects called simplicial *chains* and *cochains*. Roughly speaking, these are discrete objects which correspond in some continuous limit to domains of integration and to differential forms, respectively. Various concepts of chains and cochains find their origin in homology theory and the foundations of geometry (e.g. Whitney [56], Eilenberg [20]).

Related work on compatible discretization includes the work of Bochev and Hyman on a discrete cochain approach to mimetic discretization [5], the work of Mansfield and Quispel on variational complexes for the finite element method [44], and the work of Harrison on chainlets, extending the domain of integration from smooth manifolds to soap bubbles and fractals [28, 29].

Finite Element Exterior Calculus (FEEC).. The Finite Element Method is a method for solving certain types of boundary problems based on partial differential equations. The original problem in a Hilbert space of functions is put into variational form, and is mapped into a problem defined on a finite dimensional function space, whose basis consists of functions supported in small regions, such as simplices [8, Chap. II, Sect. 4] [34, Chap. 8].

The theory of Finite Element Exterior Calculus (FEEC) [1, 2] is based on Hilbert complexes, which are cochain complexes, such that the relevant vector spaces are Hilbert spaces. In the case of the de Rham complex, FEEC uses the Hodge theory of Riemannian manifolds, specifically Hodge decomposition, the exterior derivative and differential forms.

In a recent paper [2], D. Arnold, R. Falk and R. Winther show that the numerical stability of the FEEC discretization depends on the existence of a bounded cochain projection from a Hilbert complex to a subcomplex. The FEEC discretization uses smoothed projections to obtain this numerical stability [1], especially in the case of the de Rham complex: "By combining the canonical interpolation operators onto the standard finite element spaces of exterior calculus with a suitable smoothing operator one can obtain modified operators with desirable properties. More precisely, these modified interpolation operators are projections, they commute with the exterior derivative, and they are uniformly L^2 bounded. This is in contrast to the canonical interpolation operators, defined directly from the degrees of freedom, which are only defined for functions with higher order regularity." [12]

Another approach to FEEC discretization, in the case of hypersurfaces is that of Holst and Stern, which uses *Variational Crimes* rather than smoothing [33].

Applications to Maxwell's equations. D. White, J. Koning and R. Rieben [55] recently successfully formulated, implemented and tested a high order finite element compatible discretization method for Maxwell's electromagnetic equations based on the concepts of FEEC.

More recently, M. Costabel and A. McIntosh have produced regularity results for certain integral operators [14] which can be used to explain the convergence of compatible discretization methods for Maxwell eigenvalue problems [6].

Other recent applications of compatible discretization methods to Maxwell's equations include Tonti's finite formulation of the electromagnetic field [53], Kangas, Tarhasaari and Kettunen's use of Whitney's finite element theory [57, 36] and Stern, Tong, Desbrun and Marsden's combination of compatible discretization with variational integration, using a Lagrangian action principle [52].

2.2. **Geometric Calculus and Clifford analysis (GC/CA).** Clifford algebras can be used to describe the motion and spatial relationship of objects in space. In general, they can be constructed on any vector space with a quadratic form [41, Chap. 14] [48], including tangent spaces on orientable manifolds with a metric [13, Chap. 2].

The theory of exterior calculus uses exterior differential forms, based on Grassmann's exterior algebra. Grassmann and Clifford algebras are intimately related. Essentially, given a metric, a Clifford algebra can be defined on the same vector space as a Grassmann algebra using the same basis elements but a different multiplication rule [41, Chap. 14]. Geometric algebra provides a "unified language" for physics and engineering [39], based on multivectors, which supports Grassmann's exterior product, and left and right contractions as well as the Clifford product.

Clifford algebras are a natural setting for Dirac operators, such as the vector derivative [50, 13, 19]. The subject area of Clifford analysis studies the Dirac operator and its kernel in various contexts, including smooth manifolds [15]. Geometric Calculus encompasses both Clifford analysis and the use of exterior derivatives and differential forms on embedded orientable manifolds with arbitrary metric signatures [17, Chap. 6].

Clifford analysis has traditionally proceeded by finding structures, functions and relationships in the Clifford algebra setting analogous to those found in complex analysis. To date, this has been remarkably successful, resulting in generalizations of the Cauchy-Riemann operator, the Cauchy integral theorem and holomorphic function theory [41, Chap. 20] [13, 27]. Generalized series expansions, generating functions, kernels, and special functions including orthogonal polynomials have also been studied [15] [27, Chap. IV] [43]. This study has been accompanied by the study of the Clifford formulation and solution of a number of equations, including Maxwell's equations [11, 38] and the Navier Stokes' equations [37].

Discrete Clifford analysis. Theoretical frameworks for discrete versions of Geometric Calculus and Clifford analysis have more recently been developed, concentrating on finite difference methods and umbral calculus. The PhD thesis of Nelson Faustino [22] provides one such framework. The thesis combines the ideas of finite element exterior algebra with various types of discrete Dirac operators, including operators on lattices [21, 24]. Similar frameworks for the Dirac-Kahler operator date to the 1980s [3, 35]. Researchers at the Clifford research group at Ghent University in Belgium have also recently published a paper aimed at further development of the theory of discrete Clifford analysis [7]. The systematic study of the discrete counterparts to the operators, spaces and domains encountered in Clifford analysis also includes work by Gürlebeck and Sprössig [26, Chap. 5].

Geometric Calculus and Clifford analysis on cell complexes. Multivectors provide a natural data structure for simplices and other cells, chains, complexes, and mixed grade differential forms [50, 51, 42]. It has also been known for quite some time how Geometric Calculus and Clifford analysis, relate to differential forms [31] and to cell complexes [50, 17, 51]. In fact the Dirac operator is often constructed in the context of geometric integration, with the directed integral defined as the limit of a sum defined on cell complexes, and the vector derivative is defined as a limit of a directed integral over the boundary of a simplex, in such a way that Stokes' theorem holds [50, Sect. 5] [13, Chap. 3]. This turns out to be one of the starting points for the examination of Finite Element Methods in the context of CA/GC.

3. Finite Element Geometric Calculus

If Finite Element Geometric Calculus (FEGC) existed today, what might we reasonably expect it to look like? Essentially, it would combine the techniques of Finite Element Exterior Calculus (FEEC) with those of Geometric Calculus / Clifford analysis on manifolds (GC/CA) on a fundamental level.

The advantages of FEGC over FEEC alone could stem from the advantages of GC/CA over the use of differential forms in differential geometry and the formulation and solution of differential equations. Arguably, these advantages would be closely related to each other, and could include:

(1) a unified treatment of problems in Euclidean, Projective and Conformal geometries;
(2) a more natural treatment of problems involving Dirac-type operators and their inverses;
(3) a more natural treatment of problems involving multivector fields, especially mixed-grade fields, rather than treating these as collections of homogeneous differential forms;
(4) a different and possibly more natural treatment of the metric, as embodied in Clifford algebras on tangent or cotangent bundles;
(5) a more general and natural formulation of problems involving generalized Stokes' theorems, Green's functions and Cauchy integral formulas;
(6) greater economy of expression of some problems; and
(7) greater geometrical insight on the formulation of some problems.

Ideally, the problems which could be addressed by FEGC would include those currently treated by numerical methods for GC/CA, as well as the problems treated by FEEC. The problems which would initially yield the most insight on how to develop FEGC, could be those currently treated by both methods. Such problems include boundary and initial value problems, such as the Poisson problem, Stokes' equations, Maxwell's equations, and the equations of elasticity.

A seemingly straightforward method of taking a first step towards FEGC would be to discretize boundary value problems by using Hodge decomposition followed by the existing techniques of FEEC. Rather than just decomposing the Hodge Laplacian, problems involving the multivector-valued fields and Dirac operators would be addressed by decomposing the Hodge Dirac operator into operators defined in terms of the exterior derivative and Hodge star.

This approach seems promising for Maxwell's equations, which can be expressed in terms of a Dirac operator. In general, the method may encounter obstacles in higher dimensions, similar to those mentioned by Boffi et al [6]. Also, the process of decomposition itself may sacrifice geometric insight, and possibly invertibility, and might be better delayed to as late as possible, or eventually eliminated. This idea of late decomposition leads to a "notional commutative diagram": instead of Hodge decomposition of problem P into problem Q followed by FEEC discretization into problem Q_h, it may be possible to perform "FEGC discretization" into problem P_h followed by "discrete Hodge decomposition" into problem Q_h:

$$\begin{tikzcd}
P \arrow[rr, Rightarrow, "\text{Hodge decomposition}"] \arrow[d, dotted, "\text{FEGC discretization (?)}"'] & & Q \arrow[d, Rightarrow, "\text{FEEC discretization}"] \\
P_h\ \text{(?)} \arrow[rr, dotted, "\text{discrete Hodge decomposition (?)}"'] & & Q_h
\end{tikzcd}$$

One possible guide to what "FEGC discretization" might look like is to take an existing finite element space defined on cells, and ensure that Stokes' theorem holds exactly for the appropriate Dirac operator on each cell. The simplest case would be in for the vector derivative in Euclidean space, with each cell a simplex.

Following Cnops [13], we have, for a compact k-dimensional submanifold C of an m-dimensional manifold M, with boundary ∂C, and multivector-valued functions f and g,

$$\int_C \overline{f(x)} dM_k(x) g(x) \simeq \sum_j \overline{f(y_j)} v_k(T_j) g(y_j),$$

for some y_j near T_j, where

$$v_k(T) := \frac{1}{k!}(x_1 - x_0) \wedge \ldots \wedge (x_k - x_0),$$

for the k-simplex T with vertices $x_0, \ldots, x_k$, where $\overline{x}$ is the main anti-involution of x in the relevant Clifford algebra, and where dM_k is defined via oriented k dimensional surface elements in M, or alternatively, via differential forms, or via Lebesgue measure. See Cnops [13, (3.6)] for details. Also Stokes' theorem for the vector derivative, V_M on M, gives us

$$\int_{\partial C} \overline{f(x)} dM_{m-1}(x) g(x) = \int_C \overline{V_M f(x)} dM_m(x) g(x) + (-1)^m \int_C \overline{f(x)} dM_m(x) V_M g(x).$$

Setting $g \equiv 1$, so that $V_M g \equiv 0$, gives us

$$\int_{\partial C} \overline{f(x)} dM_{m-1}(x) = \int_C \overline{V_M f(x)} dM_m(x).$$

On a single m-dimensional simplex T with vertices $x_0, \ldots x_m$, and boundary ∂T consisting of faces $S_0, \ldots S_m$, we obtain

$$\sum_{j=0}^{m} \overline{f(y_j)} v_{m-1}(S_j) \simeq \overline{V_M f(y)} v_m(T), \tag{1}$$

for some y near T and y_j near S_j. We can use this to define the discrete vector derivative V_E of a multivector-valued affine function f on an m-simplex T in Euclidean space as:

$$V_E f(y) := \overline{v_m}(T)^{-1} \sum_{j=0}^{m} \overline{v_{m-1}}(S_j) \sum_{i \neq j} f(x_i)/m.$$

for any y in T, with x_i and S_j as per (1) above. We must then verify that this definition agrees with the usual definitions, and that Stokes' theorem holds for T as well as in the limit. Thus a function which is piecewise affine on simplices has a discrete vector derivative which is piecewise constant on these same simplices. Also note that the vector derivative takes even grade multivectors to odd grade and vice-versa, as a consequence of the $\mathbb{Z}_2$ grading of the Clifford algebra.

This exercise could be repeated with more sophisticated and higher order elements, such as Whitney [57], Raviart-Thomas [49], and Nédélec [45, 46]. This would yield pairs of function spaces, which could then be compared to the appropriate direct sums of the spaces obtained by decomposition followed by discretization.

Such an exercise could also be attempted for the the spinor Dirac and Hodge-Dirac operators on manifolds [13, Chap. 3]

The bulk of the theoretical work in the development of FEGC discretization may be in proving consistency and stability, and in proving bounds for rates of convergence for each such pair of function spaces.

In the case of FEEC discretization, where smoothed projection operators are used [1, 12], the correspondence with FEGC discretization seems less straightforward, and a different approach may be needed to deal with the need for smoothing. The role of variational crimes [33] in FEGC, may also be worth examining in detail, especially in the case of the spinor Dirac and Hodge-Dirac operators.

A more fundamental question in this context is: what is the role of Hilbert complexes in FEGC, given that the Dirac operator ∇ does not, in general, have the property that $\nabla \circ \nabla = 0$? Related questions: What replaces Hilbert complexes as the fundamental concept of FEGC? What replaces Hilbert projections?

Explicit calculation with Grassmann and Clifford algebras may also be useful in the implementation of a FEGC scheme. One way to investigate this would be to interfacing Geometric Algebra packages and libraries, such as the GluCat library and PyCliCal [40] with FEEC libraries, such as FEMSTER/EMSolve [10, 55], FEniCS [23] and PyDEC [4].

Meanwhile, conformal geometric algebra has been used in the formulation and solution of deformation problems, using Finite Element methods, by researchers in TU Darmstadt [9].

Acknowledgements

This paper was originally presented at the 9th International Conference on Clifford Algebras and their Applications (ICCA 9), in Weimar, July 2011. Most of the work for this paper was conducted at the Australian National University in Canberra, with portions at AGACSE 2010 in Amsterdam, and during a visit to R. S. Krausshar in Paderborn. The author wishes to thank C. Doran and especially R. S. Krausshar for fruitful discussions and encouragement. The support of the Australian Research Council, the Australian Mathematical Sciences Institute, and the Australian National University is gratefully acknowledged.

References

[1] D. N. Arnold, R. S. Falk and R. Winther, *Finite element exterior calculus, homological techniques, and applications.* Acta Numerica, 15, 2006, pp. 1-155.

[2] D. N. Arnold, R. S. Falk and R. Winther, *Finite element exterior calculus: from Hodge theory to numerical stability.* Bull. Amer. Math. Soc. (N.S.), 47, 2010, pp. 281-354.

[3] P. Becher and H Joos, *The Dirac-Kahler equation and fermions on the lattice.* Zeitschrift für Physik C Particles and Fields, 15:4, 1982 pp. 343-365.

[4] N. Bell, A. N. Hirani, *PyDEC: Software and Algorithms for Discretization of Exterior Calculus* arXiv:1103.3076v1 [cs.NA], 2011.

[5] P. Bochev and J. Hyman, *Principles of Mimetic Discretizations of Differential Operators.* Compatible Spatial Discretizations, Springer, 2006, pp. 89-119.

[6] D. Boffi, M. Costabel, M. Dauge, L. Demkowicz, R. Hiptmair, *Discrete compactness for the p-version of discrete differential forms.* arXiv:0909.5079v4 [math.NA], 2009.

[7] F. Brackx, H. De Schepper, F. Sommen and L. Van de Voorde, *Discrete Clifford analysis: a germ of function theory.* Hypercomplex Analysis, Birkhauser, 2009, pp. 37-53.

[8] D. Braess, *Finite Elements. Theory, Fast Solvers and Applications in Solid Mechanics.* Cambridge University Press, 2nd ed., 2001.

[9] E. Brendel, T. Kalbe, D. Hildenbrand, and M. Schfer, *Simulation of elastic rods using conformal geometric algebra.* International Symposium on Frontiers of Computational Science, Nagoya, Japan, 2008.

[10] P. Castillo, R. Rieben, and D. White, *FEMSTER: An object-oriented class library of higher order discrete differential forms.* ACM Trans. Math. Software 31, 2005, pp. 425-457.

[11] A. Chantaveerod, A. D. Seagar, and T. Angkaew, *Calculation of Electromagnetic Field with Integral Equation Based on Clifford Algebra.* PIERS Proceedings, 62 - 71, August 27-30, Prague, Czech Republic, 2007.

[12] S .H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus.* Math. Comp. 77, no. 262, 2008, pp. 813-829.

[13] J. Cnops, *An Introduction to Dirac Operators on Manifolds.* Progress in Mathematical Physics, Birkhäuser, Boston, 2002.

[14] M. Costabel and A. McIntosh, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains.* Math. Zeit., 265:2, 2010, pp. 297-320.

[15] R. Delanghe, *Clifford Analysis: History and Perspective.* Computational Methods And Function Theory, 1:1, 2001, pp. 107-154.

[16] M. Desbrun, A. N. Hirani, M. Leok and J. E. Marsden, *Discrete exterior calculus.* arXiv:math/0508341v2 [math.DG]. 2005.

[17] C. Doran and A. Lasenby, *Geometric Algebra for Physicists.* Cambridge University Press, 2003.

[18] C. Doran, *Thoughts from the front line: Current issues in real-time graphics and areas where Geometric Algebra can help.* AGACSE 2010, Amsterdam, 2010 (unpublished).

[19] M. G. Eastwood and J. Ryan, *Aspects of Dirac Operators in Analysis.* Milan J. Math. 75, 2007, pp. 91-116.

[20] S. Eilenberg, *Singular Homology Theory.* The Annals of Mathematics Second Series, 45:3, 1944, pp. 407-447.

[21] N. Faustino, U. Kahler and F. Sommen, *Discrete Dirac operators in Clifford analysis.* Advances in Applied Clifford Algebras, 17, 2007, pp. 451-467.

[22] N. Faustino, *Discrete Clifford analysis.* PhD thesis, Universidade de Aveiro, 2009.

[23] FEniCS, FEniCS Project.
`http://www.fenics.org`

[24] E. Forgy and U. Schreiber, *Discrete differential geometry on n-diamond complexes.*
`http://www-stud.uni-essen.de/~sb0264/p4a.pdf`

[25] T. Frankel, *The Geometry of Physics: an Introduction.* 2nd Edition, Cambridge University Press, 2004.

[26] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers.* Wiley, New York, 1997.

[27] K. Gürlebeck, K. Habetha, and W. Sprössig, *Holomorphic functions in the plane and n-dimensional space.* Birkhäuser, Basel, 2008.

[28] J. Harrison, *Geometric Hodge star operator with applications to the theorems of Gauss and Green.* Mathematical Proceedings of the Cambridge Philosophical Society, 140:1, 2006, pp. 135-155.

[29] J. Harrison, *Differential complexes and exterior calculus.* arXiv:math-ph/0601015v4, 2006.

[30] D. Hestenes, *Multivector calculus.* J. Math. Anal. and Appl., 24:2, 1968, pp. 313-325.

[31] D. Hestenes, *Differential forms in Geometric Calculus.* In F. Brackx and R. Delanghe (editors) Third International Conference on Clifford Algebras and their Applications in Mathematical Physics, Kluwer, 1993.

[32] D. Hestenes, *The shape of differential geometry.* AGACSE 2010, Amsterdam, 2010 (unpublished).

[33] M. Holst and A. Stern, *Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces.* arXiv:1005.4455v1 [math.NA], 2010.

[34] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations.* Cambridge, Cambridge University Press, 1996.

[35] A.N. Jourjine, *Dimensional phase transitions: Spinors, gauge fields, and background gravity on a cell complex.* Phys. Rev. D 34, 1986, pp. 3058-3068.

[36] J. Kangas, T. Tarhasaari and L. Kettunen, *Reading Whitney and finite elements with hindsight.* IEEE Transactions on Magnetics, 43:4, 2007, pp. 1157-1160.

[37] R. S. Krausshar and D. Constales, *On the Navier-Stokes' equation with free convection in three dimensional unbounded triangular channels.* Mathematical Methods in the Applied Sciences, 31:6, 2008, pp. 735-751. Mathematical Methods in the Applied Sciences, 31:6, 2008, 735-751.

[38] R. S. Krausshar, I. Cação and D. Constales, *Explicit representations of the regular solutions of the time-harmonic Maxwell equations combined with the radial symmetric Euler operator.* Mathematical Methods in the Applied Sciences, 32:1, 2009, pp. 1-11.

[39] J. Lasenby, A. Lasenby and C. Doran, *A Unified Mathematical Language for Physics and Engineering in the 21st Century.* Philosophical Transactions: Mathematical, Physical and Engineering Sciences, 358:1765, 2000, pp. 21-39.

[40] P. Leopardi, *GluCat: Generic library of universal Clifford algebra templates*, 2001-2011.

http://glucat.sourceforge.net

[41] P. Lounesto, *Clifford algebras and spinors.* Cambridge University Press, 1997.

[42] D. Lundholm and L. Svensson, *Clifford algebra, geometric algebra and applications.* Arxiv preprint arXiv:0907.5356v1 [math-ph], 2009.

[43] H. R. Malonek, G. Tomaz, *Bernoulli polynomials and Pascal matrices in the context of Clifford analysis.* Discrete Applied Mathematics, 157:4, 2009, pp. 838-847.

[44] E. L. Mansfield and G. R. W. Quispel, *Towards a variational complex for the finite element method.* Group theory and numerical analysis, CRM Proc. Lecture Notes, 39, Amer. Math. Soc., Providence, RI, 2005, pp. 207-232.

[45] J. C. Nédélec, Mixed finite elements in R^3. Numer. Math. 35, 1980, pp. 315-341.

[46] J. C. Nédélec, A new family of mixed finite elements in R^3. Numer. Math. 50, 1986, pp. 57-81.

[47] E. Noether, *Invariante Variationsprobleme.* Nachr. v. d. Ges. d. Wiss. zu Gttingen 1918, pp. 235-257. English translation: M. A. Tavel, Transport Theory and Statistical Mechanics 1(3), 1971, pp. 183-207.

[48] I. R. Porteous, *Clifford algebras and the classical groups.* Cambridge University Press, 1995.

[49] P. Raviart and J. Thomas, *A mixed finite element method for 2nd order elliptic problems.* Mathematical Aspects of the Finite Element Method, I. Galligani and E. Mayera, eds. Lecture Notes in Mathematics, vol. 606. Springer Verlag, Berlin, Germany, 1977, pp. 293-315.

[50] G. Sobczyk, *Simplicial Calculus with Geometric Algebra.* Clifford Algebras and Their Applications in Mathematical Physics, A. Micali, R. Boudet, and J. Helmstetter (eds), Kluwer Academic Publishers, Dordrecht, 1992.

[51] G. Sobczyk and O. L. Sanchez, *Fundamental Theorem of Calculus.* Arxiv preprint arXiv:0809.4526, 2008.

[52] A. Stern, Y. Tong, M. Desbrun, and J. E. Marsden, *Variational integrators for Maxwell's equations with sources.* PIERS Proceedings, July 2-6, Cambridge, USA, 2008, pp. 443-447.

[53] E. Tonti, *Finite formulation of the electromagnetic field.* Progress In Electromagnetics Research, PIER 32, 2001, pp. 1-44.

[54] B. D. Vujanovic and S. E. Jones, *Variational Methods in Nonconservative Phenomena.* Academic Press, 1989.

[55] D. White, J. Koning, and R. Rieben, *Development and application of compatible discretizations of Maxwell's equations,* Compatible Discretization of Partial Differential Equations. Springer-Verlag, 2006, pp. 209-234.

[56] H. Whitney, *On Products in a Complex.* Proceedings of the National Academy of Sciences of the USA, 23:5, 1937, pp. 285-291.

[57] H. Whitney, *Geometric Integration Theory.* Princeton University Press, Princeton, NJ, 1957.

[58] M. Zlámal, *On the finite element method.* Numer. Math., 12, 1968, pp. 394-409.

Australian National University, Canberra, Australia
E-mail address: paul.leopard@anu.edu.au