Approximation of the square root and logarithm functions in Clifford algebras: what to do in case of negative eigenvalues?

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Overview

Functions in Clifford algebras are a special case of matrix functions, as can be seen via representation theory. The square root and logarithm functions, in particular, pose problems for the author of a general purpose library of Clifford algebra functions. This is partly because the *principal* square root and logarithm of a matrix do not exist for a matrix containing a negative eigenvalue [5].

Functions in Clifford algebras

For f analytic in $\Omega \subset \mathbb{C}$, x in a Clifford algebra,

$$f(\mathrm{x}) := \frac{1}{2\pi i}\int_{\partial\Omega} f(z)\,(z-\mathrm{x})^{-1}\,dz,$$

where the spectrum $\Lambda(\rho\mathrm{x}) \subset \Omega$, with $\rho\mathrm{x}$ the matrix representing x (adapted from [5]).

Problems

1. Define the square root and logarithm of a multivector in the case where the matrix representation has negative eigenvalues.
2. Predict or detect negative eigenvalues.

Definitions of sqrt and log

When the matrix representing x has a negative eigenvalue and no imaginary eigenvalues, define

$$\mathtt{sqrt}(\mathrm{x}) := \frac{1+\iota}{\sqrt{2}}\mathtt{sqrt}(-\iota\mathrm{x}),$$
$$\mathtt{log}(\mathrm{x}) := \mathtt{log}(-\iota\mathrm{x}) + \iota\frac{\pi}{2},$$

where $\iota^2 = -1$ and $\iota\mathrm{x} = \mathrm{x}\iota$.

When x also has imaginary eigenvalues, the real matrix representing $-\iota\mathrm{x}$ has negative eigenvalues. Find some ϕ such that $\exp(\iota\phi)\mathrm{x}$ does not have negative eigenvalues, and define

$$\mathtt{sqrt}(\mathrm{x}) := \exp\left(-\iota\frac{\phi}{2}\right)\mathtt{sqrt}(\exp(\iota\phi)\mathrm{x}),$$
$$\mathtt{log}(\mathrm{x}) := \mathtt{log}(\exp(\iota\phi)\mathrm{x}) - \iota\phi.$$

Matrix representations of Clifford algebras

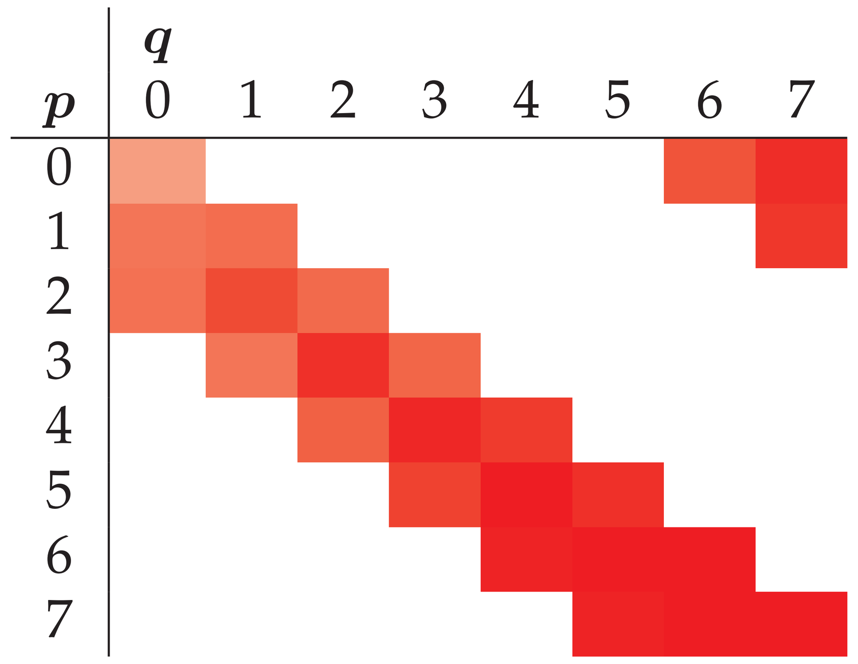
Each Clifford algebra $\mathbb{R}_{p,q}$ is generated by $n = p + q$ anticommuting generators, p of which square to 1 and q of which square to -1; and is isomorphic to a matrix algebra over $\mathbb{R}$, ${}^2\mathbb{R} := \mathbb{R} + \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or ${}^2\mathbb{H}$ per the following table, with periodicity of 8 [6, 7, 8, 9]. The $\mathbb{R}$ and ${}^2\mathbb{R}$ matrix algebras are highlighted in red.

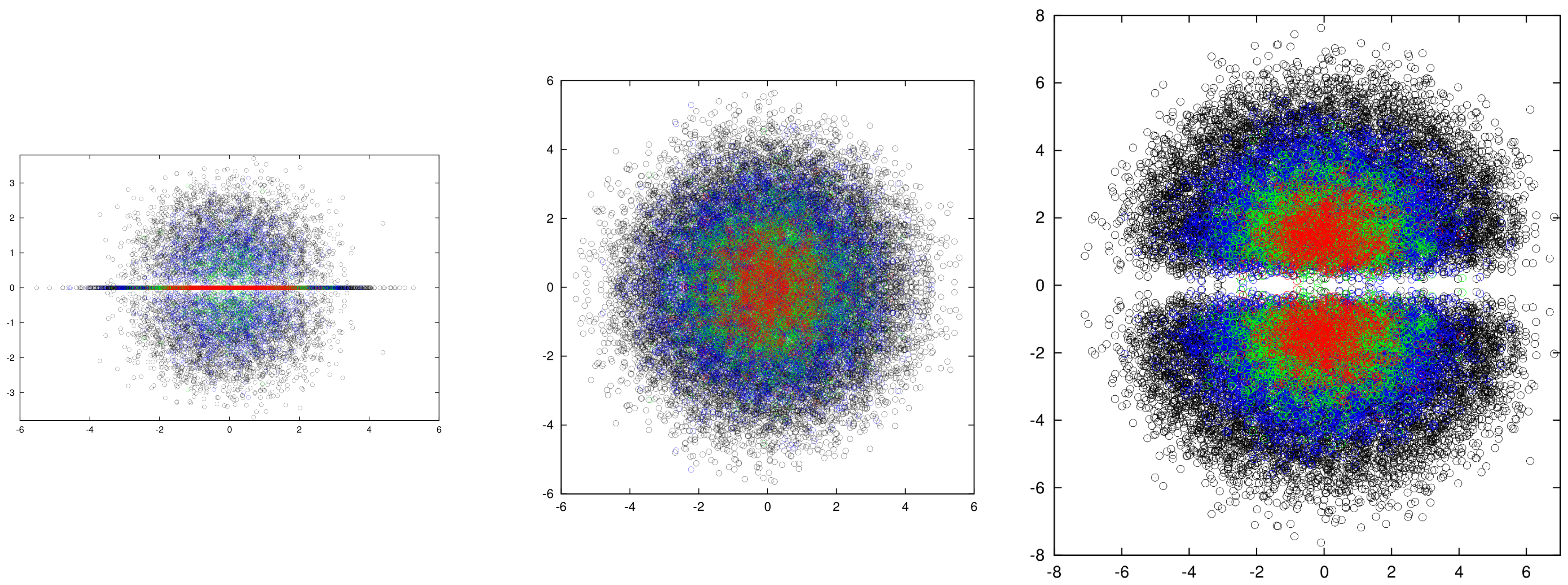
p \ q	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$
1	${}^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
2	$\mathbb{R}(2)$	${}^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	${}^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$
5	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
6	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(64)$	$\mathbb{R}(128)$

A real matrix representation is obtained by representing each complex or quaternion value as a real matrix.

Predicting negative eigenvalues?

In Clifford algebras with a faithful irreducible *complex* or *quaternion* representation, a multivector with independent $N(0,1)$ random coefficients is *unlikely* to have a negative eigenvalue. In large Clifford algebras with an irreducible real representation, such a random multivector is very likely to have a negative eigenvalue. The table at right illustrates this. Probability is denoted by shades of red. This phenomenon is a direct consequence of the eigenvalue density of the Ginibre ensembles [1, 2, 3, 4].





Real | Complex | Quaternion

Eigenvalue density of real matrix representations of Ginibre ensembles

Detecting negative eigenvalues

Trying to predict negative eigenvalues using the p and q of $\mathbb{R}_{p,q}$ is futile. Negative eigenvalues are always possible, since $\mathbb{R}_{p,q}$ contains $\mathbb{R}_{p',q'}$ for all $p' \leqslant p$ and $q' \leqslant q$.

The eigenvalue densities of the Ginibre ensembles simply make testing more complicated.

In the absence of an efficient algorithm to detect negative eigenvalues only, it is safest to use a standard algorithm to find all eigenvalues. If the real Schur factorization is used this allows the `sqrt` and `log` algorithms to operate on triangular matrices only [5].

Further problem

Devise an algorithm which detects negative eigenvalues only, and is more efficient than standard eigenvalue algorithms.

References

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