An abstract Hodge–Dirac operator and its stable discretization

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Subjects with parallel 45+ year histories

Finite Element Method

M. Zlámal, *On the finite element method.* Numer. Math., 12, 1968, pp. 394-409.

Clifford analysis

D. Hestenes, *Multivector calculus.* J. Math. Anal. and Appl., 24:2, 1968, pp. 313-325.

Topics

- Finite Element Exterior Calculus
- Discretization of the Hodge–Dirac operator
- Discretization of the Hodge Laplacian
- Numerical examples
- Further considerations

Finite Element Method

The Finite Element Method solves boundary value problems based on partial differential equations.

The original problem in a Hilbert space of functions is put into variational form, and is mapped into a problem defined on a finite dimensional function space, whose basis consists of functions supported in small regions, such as simplices.

(Iserles 1996; Braess 2001).

Finite element exterior calculus (FEEC)

FEEC is based on the Finite Element Method over Hilbert complexes. These are cochain complexes where the relevant vector spaces are Hilbert spaces.

For the de Rham complex, FEEC uses Hodge decomposition, the exterior derivative and differential forms.

The numerical stability of the FEEC discretization depends on the existence of a bounded cochain projection from a Hilbert complex to a subcomplex. FEEC uses smoothed projections to obtain this numerical stability.

(Arnold, Falk and Winther 2006, 2010; Christiansen and Winther 2008)

An abstract Hodge–Dirac problem: setting

Let $\mathbf{d}$ be a closed, densely defined nilpotent operator on the Hilbert space $\boldsymbol{W}$, with domain $\boldsymbol{V}$, and closed range $\mathfrak{B}$.

In $\boldsymbol{V}$, we use the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{V}} := \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \mathbf{d}\boldsymbol{u}, \mathbf{d}\boldsymbol{v} \rangle.$$

We have the orthogonal Hodge decomposition

$$\begin{aligned} \boldsymbol{W} &= \mathfrak{B} \oplus \mathfrak{H} \oplus \mathfrak{B}^*, \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathfrak{B}} \oplus \boldsymbol{u}_{\mathfrak{H}} \oplus \boldsymbol{u}_{\mathfrak{B}^*}, \quad \forall \boldsymbol{u} \in \boldsymbol{W}. \end{aligned}$$

where $\mathfrak{Z} = \mathfrak{B} \oplus \mathfrak{H}$ is the null space of $\mathbf{d}$.

Example: Hilbert complexes

If (M, g) is an oriented, compact Riemannian manifold, then each space of smooth k-forms has an L^2-inner product,

$$\langle u, v\rangle_{L^2\Omega^k(M)} = \int_M \langle u, v\rangle_g \operatorname{vol}_g = \int_M u \wedge \star_g v.$$

This gives an adjoint operator $\mathrm{d}_k^* : \Omega^k(M) \to \Omega^{k-1}(M)$ for each k, e.g.

$$0 \rightleftarrows \Omega^0(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^2(U) \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3(U) \rightleftarrows 0$$

$$0 \rightleftarrows C^\infty(U) \underset{\mathrm{div}}{\overset{\mathrm{grad}}{\rightleftarrows}} \mathfrak{X}(U) \underset{\mathrm{curl}}{\overset{\mathrm{curl}}{\rightleftarrows}} \mathfrak{X}(U) \underset{\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} C^\infty(M) \rightleftarrows 0$$

The Hodge decomposition says that each $f \in L^2\Omega^k(M)$ can be orthogonally decomposed as $f = \mathrm{d}\alpha + \mathrm{d}^*\beta + \gamma$, where $\mathrm{d}\gamma = 0$, $\mathrm{d}^*\gamma = 0$.

An abstract Hodge–Dirac problem

We want to solve the problem $\mathbf{D}\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{f}_{\mathfrak{H}}$,
where $\mathbf{D} := \mathrm{d} + \mathrm{d}^*$ is the *abstract Hodge–Dirac operator*.

Consider the following mixed variational problem:
Find $(\boldsymbol{u}, \boldsymbol{p}) \in \boldsymbol{V} \times \mathfrak{H}$ satisfying

$$\begin{aligned} \langle \mathrm{d}\boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \mathrm{d}\boldsymbol{v} \rangle + \langle \boldsymbol{p}, \boldsymbol{v} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle, && \forall \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \boldsymbol{u}, \boldsymbol{q} \rangle &= 0, && \forall \boldsymbol{q} \in \mathfrak{H}. \end{aligned} \tag{1}$$

To show that this problem is well-posed, it suffices to prove the inf-sup condition for the symmetric bilinear form

$$\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{p}; \boldsymbol{v}, \boldsymbol{q}) := \langle \mathrm{d}\boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \mathrm{d}\boldsymbol{v} \rangle + \langle \boldsymbol{p}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{q} \rangle$$

on $\boldsymbol{V} \times \mathfrak{H}$.

The problem is well-posed

Theorem 2.4 (LS 2014)

There exists a constant $\gamma > 0$, depending only on the Poincaré constant c_P, such that for all non-zero $(u, p) \in V \times \mathfrak{H}$, there exists a non-zero $(v, q) \in V \times \mathfrak{H}$ satisfying

$$B(u, p; v, q) \geq \gamma(\|u\|_V + \|p\|)(\|v\|_V + \|q\|).$$

Proof.

(hint). Consider the test functions

$$v := \rho + p + du, \quad q := u_{\mathfrak{H}},$$

where $\rho \in \mathfrak{Z}^{\perp}$ is the unique element such that $\mathrm{d}\rho = u_{\mathfrak{B}}$. □

A corresponding discrete problem

Suppose $V_h \subset V$ is a Hilbert subspace, with a bounded projection $\pi_h : V \to V_h$, such that $\pi_h d = d\pi_h$.

Consider the discrete problem:

Find $(u_h, p_h) \in V_h \times \mathfrak{H}_h$ satisfying

$$\begin{aligned} \langle \mathrm{d}u_h, v_h \rangle + \langle u_h, \mathrm{d}v_h \rangle + \langle p_h, v_h \rangle &= \langle f, v_h \rangle, & \forall v_h &\in V_h, \\ \langle u_h, q_h \rangle &= 0, & \forall q_h &\in \mathfrak{H}_h. \end{aligned} \tag{2}$$

This problem is well-posed, with a discrete inf-sup condition, where the constant γ_h depends only on c_P and the norm $\|\pi_h\|$.

An error estimate

Theorem 3.4 (LS 2014)

Let (u, p) be the solution to (1) and (u_h, p_h) be the solution to (2). If the projections π_h are V-bounded uniformly, independently of h, then the error can be estimated by

$$\begin{aligned} &\|u - u_h\|_V + \|p - p_h\| \\ &\leqslant C \left(\inf_{v \in V_h} \|u - v\|_V + \inf_{q \in V_h} \|p - q\|_V + \mu \inf_{v \in V_h} \|P_{\mathfrak{B}} u - v\|_V \right), \end{aligned}$$

where $\mu := \|(1 - \pi_h) P_{\mathfrak{H}}\|$.

The Hodge–Laplace problem

The *abstract Hodge–Laplace operator* is $\mathbf{L} = \mathbf{D}^2 = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$, defined on the domain $\mathcal{D}(\mathbf{L}) = \mathbf{D}^{-1}(\boldsymbol{V} \cap \boldsymbol{V}^*) \subset \boldsymbol{V} \cap \boldsymbol{V}^*$ with kernel $\mathcal{N}(\mathbf{L}) = \mathcal{N}(\mathbf{D}) = \mathfrak{H}$.

The Hodge–Laplace problem is:

Given $\boldsymbol{f} \in \boldsymbol{W}$, find $(\boldsymbol{u}, \boldsymbol{p}) \in \left(\mathcal{D}(\mathbf{L}) \cap \mathcal{N}(\mathbf{L})^\perp\right) \oplus \mathcal{N}(\boldsymbol{L})$ such that

$$\mathbf{L}\boldsymbol{u} + \boldsymbol{p} = \boldsymbol{f}.$$

To solve this, we can solve $\mathbf{D}\boldsymbol{w} + \boldsymbol{p} = \boldsymbol{f}$, and then solve $\mathbf{D}\boldsymbol{u} = \boldsymbol{w}$.

The mixed variational form

The mixed variational form of the Hodge–Laplace problem is:
Find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{p}) \in \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{\mathfrak{H}}$ such that

$$\begin{aligned} \langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle - \langle \boldsymbol{u}, \mathbf{d}\boldsymbol{\tau} \rangle &= \mathbf{0}, && \forall \boldsymbol{\tau} \in \boldsymbol{V}, \\ \langle \mathbf{d}\boldsymbol{\sigma}, \boldsymbol{v} \rangle + \langle \mathbf{d}\boldsymbol{u}, \mathbf{d}\boldsymbol{v} \rangle + \langle \boldsymbol{p}, \boldsymbol{v} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle, && \forall \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \boldsymbol{u}, \boldsymbol{q} \rangle &= \mathbf{0}, && \forall \boldsymbol{q} \in \boldsymbol{\mathfrak{H}}. \end{aligned} \tag{3}$$

The discrete Hodge–Laplace problem

The corresponding discrete Hodge–Laplace mixed variational problem is:

Find $(\boldsymbol{\sigma_h}, \boldsymbol{u_h}, \boldsymbol{p_h}) \in \boldsymbol{V_h} \times \boldsymbol{V_h} \times \mathfrak{H}_h$ such that

$$\begin{aligned} \langle \boldsymbol{\sigma_h}, \boldsymbol{\tau_h} \rangle - \langle \boldsymbol{u_h}, \mathbf{d}\boldsymbol{\tau_h} \rangle &= \mathbf{0}, && \forall \boldsymbol{\tau_h} \in \boldsymbol{V_h}, \\ \langle \mathbf{d}\boldsymbol{\sigma_h}, \boldsymbol{v_h} \rangle + \langle \mathbf{d}\boldsymbol{u_h}, \mathbf{d}\boldsymbol{v_h} \rangle + \langle \boldsymbol{p_h}, \boldsymbol{v_h} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v_h} \rangle, && \forall \boldsymbol{v_h} \in \boldsymbol{V_h}, \\ \langle \boldsymbol{u_h}, \boldsymbol{q_h} \rangle &= \mathbf{0}, && \forall \boldsymbol{q_h} \in \mathfrak{H}_h. \end{aligned} \tag{4}$$

We can solve the discrete Hodge–Laplace problem by first finding the solution $(\boldsymbol{w_h}, \boldsymbol{p_h}) \in \boldsymbol{V_h} \times \mathfrak{H}_h$ of the Hodge–Dirac problem for $\boldsymbol{f}$, then finding the solution $(\boldsymbol{u_h}, \mathbf{0}) \in \boldsymbol{V_h} \times \mathfrak{H}_h$ of the Hodge–Dirac problem for $\boldsymbol{w_h}$. Then $(\boldsymbol{w_h} - \mathbf{d}\boldsymbol{u_h}, \boldsymbol{u_h}, \boldsymbol{p_h})$ solves (4).

Relationship to the discrete Hodge Laplacian

Theorem 4.4 (LS 2014)

Under the hypotheses of Theorem 3.4, if $(\sigma, u, p) \in V \times V \times \mathfrak{H}$ solves (3) and $(\sigma_h, u_h, p_h) \in V_h \times V_h \times \mathfrak{H}_h$ solves (4), then we have the error estimate

$$\begin{aligned} &\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ &\qquad \leq C\biggl(\inf_{\tau \in V_h} \|\sigma - \tau\|_V + \inf_{v \in V_h} \|u - v\|_V + \inf_{q \in V_h} \|p - q\|_V \\ &\qquad\qquad + \mu \inf_{v \in V_h} \|P_{\mathfrak{B}} u - v\|_V \biggr). \end{aligned}$$

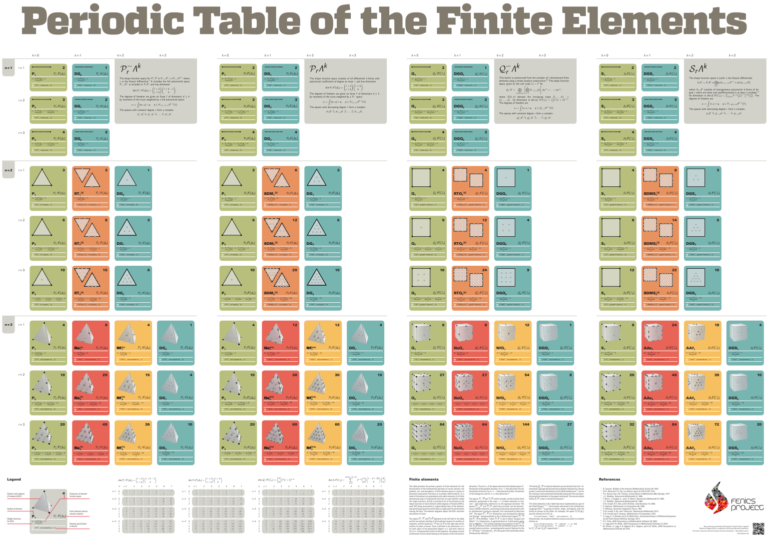
The periodic table of finite elements

The elements commonly used in finite element exterior calculus also yield a stable discretization of the Hodge–Dirac problem.

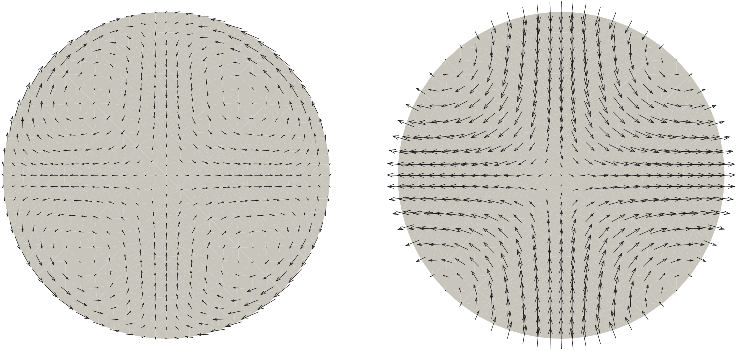
These elements, including the $\mathcal{P}_r$ and $\mathcal{P}_r^-$ families of piecewise-polynomial differential forms on simplicial meshes (Arnold Falk and Winther, 2006, 2010) and the more recent $\mathcal{S}_r$ family on cubical meshes (Arnold, 2013; Arnold and Awanou, 2014) give subcomplexes of the L^2 de Rham complex with bounded commuting projections.

The $\mathcal{P}_r$ and $\mathcal{P}_r^-$ families have been implemented in *FEniCS* and can be used to solve the discrete Hodge–Dirac problem.

(Arnold, Falk and Winther 2006, 2010; Arnold 2013; Arnold and Awanou, 2014; Logg et al. 2012)


Periodic Table of the Finite Elements
Legend
Finite elements
References

The Hodge–Dirac problem allows us to find a vector field with prescribed divergence and curl:



Divergence-free vector fields on the unit disk with curl $x_1 x_2$.
Left: natural boundary conditions (zero normal component).
Right: essential boundary conditions (zero tangential component).

Further considerations

On an embedded Riemannian manifold, subdivision into Euclidean simplices introduces geometric errors. Holst and Stern (2012) have addressed this with their work on *geometric variational crimes*.

This idea has been extended to adaptive mixed methods by Holst, Mihalik and Szypowski (2014) but the case of Dirac operators on Riemannian manifolds is yet to be tried.

Future research could include: numerical examples with more dimensions, more realistic geometries and boundary conditions, perturbed operators, eigenvalue problems, functional calculus, different metric signatures, rougher domains . . .

(Holst and Stern 2012, Holst, Mihalik and Szypowski 2014)

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