Is a dual mesh really necessary?

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Subjects with parallel 40+ year histories

Finite Element Method

M. Zlámal, *On the finite element method.* Numer. Math., 12, 1968, pp. 394-409.

Clifford analysis

D. Hestenes, *Multivector calculus.* J. Math. Anal. and Appl., 24:2, 1968, pp. 313-325.

More recent developments

M. Desbrun, A. N. Hirani, M. Leok and J. E. Marsden, *Discrete exterior calculus.* arXiv:math/0508341v2 [math.DG]. 2005.

P. Bochev and J. Hyman, *Principles of Mimetic Discretizations of Differential Operators.* Compatible Spatial Discretizations, Springer, 2006, pp. 89-119.

D. N. Arnold, R. S. Falk and R. Winther, *Finite element exterior calculus, homological techniques, and applications.* Acta Numerica, 15, 2006, pp. 1-155.

Topics

- Related previous work
  - Geometric Calculus / Clifford analysis.
  - Compatible discretization
- Finite Element Geometric Calculus (FEGC)
  - FEGC via Hodge Decomposition
  - Related estimates

Grassmann, Clifford and Geometric Algebras

Clifford algebras can be constructed on any vector space with a quadratic form including tangent spaces on orientable manifolds with a metric.

Geometric Algebra, a "unified language for physics and engineering," uses the exterior (Grassmann) product and the geometric (Clifford) product, as well as contractions, on the same underlying tangent space, with the quadratic form defined by the metric.

$$xy = x \wedge y + x \cdot y.$$

(Porteous 1995; Lounesto 1997; Lasenby, Lasenby and Doran 2000; Cnops 2002)

Clifford analysis and Geometric Calculus

Clifford algebras are a natural setting for Dirac operators, such as the vector derivative.

Clifford analysis (CA) studies the Dirac operator and its kernel in various contexts, including smooth manifolds, finding structures, functions and relationships analogous to complex analysis.

Geometric Calculus (GC) encompasses both Clifford analysis and the use of exterior derivatives and differential forms on embedded orientable manifolds with arbitrary metric signatures.

(Sobczyk 1992; Delanghe 2001; Cnops 2002; Doran and Lasenby 2003; Eastwood and Ryan 2007)

Some successes of Clifford analysis

Generalizations of the Cauchy-Riemann operator, the Cauchy integral theorem and holomorphic function theory (Lounesto 1997; Cnops 2002; Gürlebeck, K. Habetha, and W. Sprössig 2008; others).

Solution of Kato's square root problem (McIntosh 1985; Auscher, Hofmann, Lacey, McIntosh and Tchamitchian 2002; Axelsson, Keith and McIntosh 2006).

Generalized series expansions, generating functions, kernels, and special functions including orthogonal polynomials (Delanghe 2001; Gürlebeck, K. Habetha, and W. Sprössig 2008; Malonek and Tomaz 2008; others).

Maxwell's equations (Chantaveerod, Seagar, and Angkaew 2007; Krausshar, Caçắo and Constales 2009; Constales, Faustino and Krausshar 2011; others) and Navier Stokes' equations (Gürlebeck,, A. Hommel, U. Kähler 2006; Krausshar and Constales 2008; others).

Discrete Clifford analysis

Mostly uses finite difference methods, lattices and umbral calculus. Lattice frameworks for the Dirac-Kahler operator date to the 1980s (Becher and Joos 1982; Jourjine 1986).

Gürlebeck and Sprössig worked on finite differences and discrete Dirac operators (1997 and later).

The PhD thesis of Nelson Faustino (2009) combines the finite element exterior algebra with umbral calculus and discrete Dirac operators, including operators on lattices (Forgy and Schreiber 2004; Faustino and Kähler 2007; Faustino, Kähler and Sommen 2007).

These methods have been further developed by Faustino and by the Clifford research group at Ghent University (Brackx, De Schepper, Sommen and Van de Voorde 2009; Faustino 2010; Faustino and Ren (in proof)).

Geometric Calculus and Clifford analysis on cell complexes

Multivectors provide a natural data structure for simplices and other cells, chains, complexes, and mixed grade differential forms (Lundholm and Svensson 2009).

It has been known for quite some time how Geometric Calculus and Clifford analysis, relate to differential forms (Hestenes 1993) and to cell complexes.
The directed integral can be defined as the limit of a sum defined on cell complexes.
The vector derivative can be defined as a limit of a directed integral over the boundary of a simplex, in such a way that Stokes' theorem holds.

(Sobczyk 1992; Cnops 2002; Doran and Lasenby 2003; Sobczyk and Sanchez 2008).

Maxwell's equations in Geometric Calculus

Using differential forms in Minkowski space:

$$d \lrcorner G = J,$$
$$d \wedge F = 0.$$

In $\mathcal{C}\ell_{3,1}$, using $\partial := \nabla - \frac{1}{c}\mathbf{e}_4 \frac{\partial}{\partial t}$, where ∇ is a Dirac operator in $\mathbb{R}^3$:

$$\partial \lrcorner \mathbf{G} = \mathbf{J},$$
$$\partial \wedge \mathbf{F} = 0.$$

In a vacuum,

$$\partial \mathbf{F} = \mathbf{J},$$
$$\partial \mathbf{A} = -\mathbf{F}.$$

(Lounesto 1997)

Variational principles and Noether's theorem

Variational principle: Abstract trajectory that makes some functional stationary.

Example: Hamilton's Principle of Stationary Action.

Noether's Theorem: *Symmetries* in the equations describing a variational principle are equivalent to *conservation laws*.

(Noether 1918; Vujanovic and Jones 1989; Frankel 2004)

Principles of compatible discretization

Compatible (or *mimetic*) discretization creates a discrete description of a physical phenomenon which preserves many or all of the same conservation laws which are obeyed by the continuous description given by a differential equation.

If a method using compatible discretization can calculate a conserved quantity accurately, the accuracy is maintained by the incorporation of the conservation law into the discretization.

(Hiptmair 2001; Bochev and Hyman 2006; Arnold Falk and Winther 2006; He and Teixeira 2006)

Some aspects of compatible discretization

These include

1. the continuous description of the physical phenomenon using equations involving differential forms on manifolds;
2. the analysis of the symmetries of the equations; and
3. discretization by dividing the manifold into cells, chains and complexes, with corresponding differential forms.

Simplicial *chains* and *cochains* are discrete objects which correspond in some continuous limit to domains of integration and to differential forms, respectively.
Various concepts of chains and cochains arise in homology theory and the foundations of geometry (Whitney 1937; Eilenberg 1944).

(Hiptmair 2001; Bochev and Hyman 2006; Arnold Falk and Winther 2006; He and Teixeira 2006)

Discrete Exterior Calculus

These compatible discretization methods focus on the discretization of operators via simplicial complexes.

Often both a primal and a *dual mesh* is used, to simplify or even diagonalize the discrete Hodge star operator.

Much effort is put into the definition of the dual mesh and the Hodge star, to optimize performance for particular problems.

(Hiptmair 2001; Bochev and Hyman 2006; Arnold Falk and Winther 2006; He and Teixeira 2006)

Finite Element Method

The Finite Element Method solves boundary value problems based on partial differential equations.

The original problem in a Hilbert space of functions is put into variational form, and is mapped into a problem defined on a finite dimensional function space, whose basis consists of functions supported in small regions, such as simplices.

(Iserles 1996; Braess 2001).

Finite Element Exterior Calculus (FEEC)

FEEC discretizes spaces rather than operators. It is based on the Finite Element Method over Hilbert complexes. These are cochain complexes where the relevant vector spaces are Hilbert spaces.

For the de Rham complex, FEEC uses Hodge decomposition, the exterior derivative and differential forms.

The numerical stability of the FEEC discretization depends on the existence of a bounded cochain projection from a Hilbert complex to a subcomplex. FEEC uses smoothed projections to obtain this numerical stability.

(Arnold, Falk and Winther 2006, 2010; Christiansen and Winther 2008)

Applications to Maxwell's equations

White, Koning and Rieben recently (2006) successfully formulated, implemented and tested a high order finite element compatible discretization method for Maxwell's electromagnetic equations based on the concepts of FEEC.

This method is incorporated into the the EMSolve package from Lawrence Livermore National Laboratory.

What would FEGC look like?

Finite Element Geometric Calculus (FEGC) should combine the techniques of Finite Element Exterior Calculus (FEEC) with those of Geometric Calculus / Clifford analysis on manifolds (GC/CA) on a fundamental level.

The problems to be addressed by FEGC could include those currently treated by numerical methods for GC/CA, as well as the problems treated by FEEC.

Problems currently treated by both methods, such as the Poisson problem, Stokes' equations, Maxwell's equations, and the equations of elasticity, should initially yield the most insight on how to develop FEGC.

Possible advantages of FEGC over FEEC

1. Unified treatment of problems in Euclidean, Projective and Conformal geometries.
2. More natural treatment of problems involving Dirac-type operators and their inverses.
3. More natural treatment of problems involving multivector fields, especially mixed-grade fields, rather than treating these as collections of homogeneous differential forms.
4. Different and possibly more natural treatment of the metric, via Clifford algebras on tangent or cotangent bundles.
5. More general and natural formulation of problems involving Stokes' theorems, Green's functions, Cauchy integral formulas.
6. Greater economy of expression of some problems.
7. Greater geometrical insight in formulating some problems.

FEGC via Hodge Decomposition

A viable approach is to discretize boundary value problems involving the multivector-valued fields and Dirac operators by using Hodge decomposition followed by the existing techniques of FEEC.

The Hodge Dirac operator is decomposed into operators defined in terms of the exterior derivative and Hodge star.

Using the weak formulation eliminates the explicit use of the Hodge star in favour of the use of inner products.

This approach to discretizing the Hodge Dirac operator essentially works in the same way as FEEC.

An abstract Hodge-Dirac problem: setting

Let (W, d) be a closed Hilbert complex with domain complex (V, d). In V, we use the inner product

$$\langle u, v \rangle_V := \langle u, v \rangle + \langle du, dv \rangle.$$

We use the orthogonal Hodge decomposition

$$W = \mathfrak{B} \oplus \mathfrak{H} \oplus \mathfrak{B}^*,$$
$$u = u_{\mathfrak{B}} \oplus u_{\mathfrak{H}} \oplus u_{\mathfrak{B}^*}, \quad \forall u \in W.$$

where $V \in W$ is the domain of d, $\mathfrak{B}$ is the range of d, and $\mathfrak{Z}$ is the null space of d.

An abstract Hodge-Dirac problem

We want to solve the problem $(d + d^*)u = f - f_{\mathfrak{H}}$, where $d + d^*$ is the abstract Hodge-Dirac operator.
Consider the following mixed variational problem:
Find $(u, p) \in V \times \mathfrak{H}$ satisfying

$$\begin{aligned} \langle du, v\rangle + \langle u, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, \quad \forall v \in V \\ \langle u, q\rangle &= 0, \quad \forall q \in \mathfrak{H}. \end{aligned} \tag{1}$$

To show that this problem is well-posed, it suffices to prove the inf-sup condition for the symmetric bilinear form

$$B(u, p; v, q) := \langle du, v\rangle + \langle u, dv\rangle + \langle p, v\rangle + \langle u, q\rangle$$

on $V \times \mathfrak{H}$.

The problem is well-posed

Theorem 1

There exists a constant $\gamma > 0$, depending only on the Poincaré constant c_P, such that for all non-zero $(u, p) \in V \times \mathfrak{H}$, there exists non-zero $(v, q) \in V \times \mathfrak{H}$ such that

$$B(u, p; v, q) \geq \gamma(\|u\|_V + \|p\|)(\|v\|_V + \|q\|).$$

Proof (hint). Consider the test functions

$$v := \rho + p + du, q := u_{\mathfrak{H}},$$

where $\rho \in \mathfrak{Z}^{\perp}$ is the unique element such that $d\rho = u_{\mathfrak{B}}$.

A corresponding discrete problem

Suppose $V_h \subset V$ is a Hilbert subcomplex, with a bounded cochain projection $\pi_h : V \to V_h$.

Consider the discrete problem:

Find $(u_h, p_h) \in V_h \times \mathfrak{H}_h$ satisfying

$$\begin{aligned} \langle du_h, v_h \rangle + \langle u_h, dv_h \rangle + \langle p_h, v_h \rangle &= \langle f, v_h \rangle, \quad \forall v_h \in V_h \\ \langle u_h, q_h \rangle &= 0, \quad \forall q_h \in \mathfrak{H}_h. \end{aligned} \tag{2}$$

This problem is well-posed, with a discrete inf-sup condition, where the constant γ_h depends only on c_P and the norm $\|\pi_h\|$.

An error estimate

Theorem 2

Let (u, p) *be the solution to* (1) *and* (u_h, p_h) *be the solution to* (2). *If the projections* π_h *are* V*-bounded uniformly, independently of* h, *then the error can be estimated by*

$$\|u - u_h\|_V + \|p - p_h\| \leq C\left(\inf_{v\in V_h}\|u - v\| + \inf_{q\in V_h}\|p - q\| + \mu\inf_{v\in V_h}\|P_{\mathfrak{B}}u - v\|\right),$$

where

$$\mu := \sup_{r\in\mathfrak{H},\|r\|=1}\|(1 - \pi_h)r\|,$$

and where C *depends only on the Poincaré constant* c_P.