Partitions of the unit sphere into regions of equal area and small diameter

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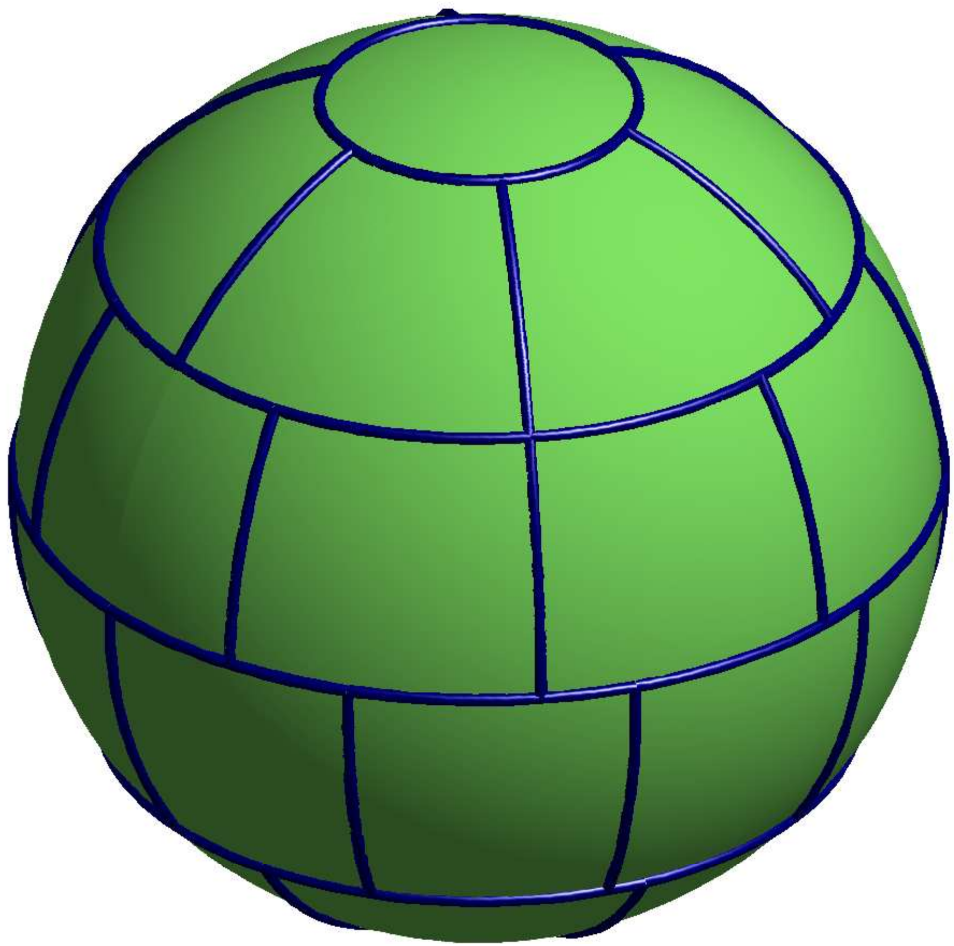
Partition of $\mathbb{S}^2$ into 33 regions of equal area



Outline of talk

- The sphere, partitions, diameter bounds
- Precedents, Stolarsky's assertion
- The Feige-Schechtman algorithm
- The Recursive Zonal Equal Area algorithm
- Outline of proof of bounds
- Numerical results

The unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$

Definition 1. *The unit sphere* $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ *is*

$$\mathbb{S}^d := \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; \sum_{k=1}^{d+1} x_k^2 = 1 \right\}.$$

Definition 2. Spherical polar coordinates *describe a point* p *of* $\mathbb{S}^d$ *using one longitude,* $p_1 \in [0, 2\pi]$, *and* $d-1$ *colatitudes,* $p_i \in [0, \pi]$, *for* $i \in \{2, \ldots, d\}$.

Equal-area partitions of $\mathbb{S}^d$

Definition 3. *An* equal area partition *of* $\mathbb{S}^d$ *is a nonempty finite set* $\mathcal{P}$ *of Lebesgue measurable subsets of* $\mathbb{S}^d$, *such that*

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where σ *is the Lebesgue area measure on* $\mathbb{S}^d$.

Diameter bounded sets of partitions

Definition 4. *The* diameter *of a region* $\boldsymbol{R} \subset \mathbb{R}^{\boldsymbol{d+1}}$ *is defined by*

$$\operatorname{diam} \boldsymbol{R} := \sup\{\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}\},$$

where $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y})$ *is the* $\mathbb{R}^{\boldsymbol{d+1}}$ *Euclidean distance* $\left\| \underline{\boldsymbol{x}} - \underline{\boldsymbol{y}} \right\|$.

Definition 5. *A set* $\boldsymbol{\Xi}$ *of partitions of* $\mathbb{S}^{\boldsymbol{d}} \subset \mathbb{R}^{\boldsymbol{d+1}}$ *is* diameter-bounded *with* diameter bound $\boldsymbol{K} \in \mathbb{R}_{+}$ *if for all* $\boldsymbol{\mathcal{P}} \in \boldsymbol{\Xi}$, *for each* $\boldsymbol{R} \in \boldsymbol{\mathcal{P}}$,

$$\operatorname{diam} \boldsymbol{R} \leqslant \boldsymbol{K} \left|\boldsymbol{\mathcal{P}}\right|^{-\boldsymbol{1}/\boldsymbol{d}}.$$

Precedents

The $\mathbf{EQ}$ partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Ed Saff, and on Ian Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Alexander (1972) uses the existence of a diameter-bounded set of equal-area partitions of $\mathbb{S}^2$ to analyze the maximum sum of distances between points. Alexander (1972) suggests a construction different from Zhou (1995).

Equal-area partitions of $\mathbb{S}^2$ used in the geosciences and astronomy do not have a proven bound on the diameter of regions.

Stolarsky's assertion

Stolarsky (1973) asserts the existence of a diameter-bounded set of equal-area partitions of $\mathbb{S}^d$ for all d, but offers no construction or existence proof.

Beck and Chen (1987) quotes Stolarsky. Bourgain and Lindenstrauss (1988) quotes Beck and Chen.

Wagner (1993) implies the existence of an $\mathbf{EQ}$-like construction for $\mathbb{S}^d$. Bourgain and Lindenstrauss (1993) gives a partial construction.

Feige and Schechtman (2002) gives a construction which proves Stolarsky's assertion.

Spherical caps, zones, and collars

The *spherical cap* $S(p, \theta) \in \mathbb{S}^d$ is

$$S(p, \theta) := \left\{ q \in \mathbb{S}^d \mid \underline{p} \cdot \underline{q} \geqslant \cos(\theta) \right\}.$$

For $d > 1$, a *zone* can be described by

$$Z(a, b) := \left\{ p \in \mathbb{S}^d \mid p_d \in [a, b] \right\},$$

where $0 \leqslant a < b \leqslant \pi$.

$Z(0, b)$ is a North polar cap and $Z(a, \pi)$ is a South polar cap.

If $0 < a < b < \pi$, $Z(a, b)$ is a *collar*.

Area of a spherical cap

For $d > 1$, the area of a spherical cap of spherical radius θ is

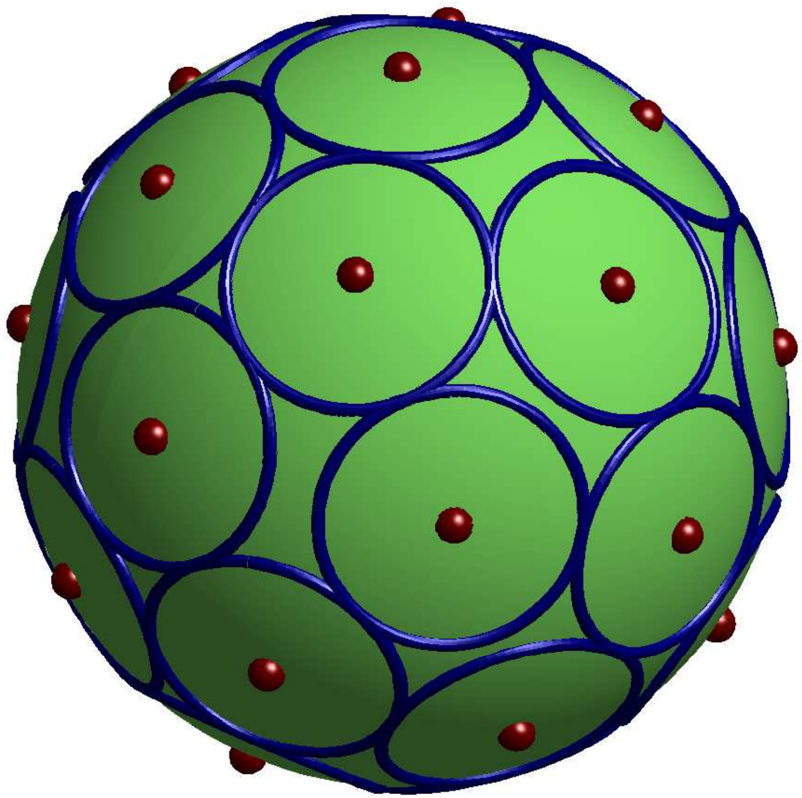
$$\mathcal{V}(\theta) := \sigma\big(S(p, \theta)\big) = \omega \int_0^\theta (\sin \xi)^{d-1} d\xi,$$

where $\omega = \sigma(\mathbb{S}^{d-1})$.

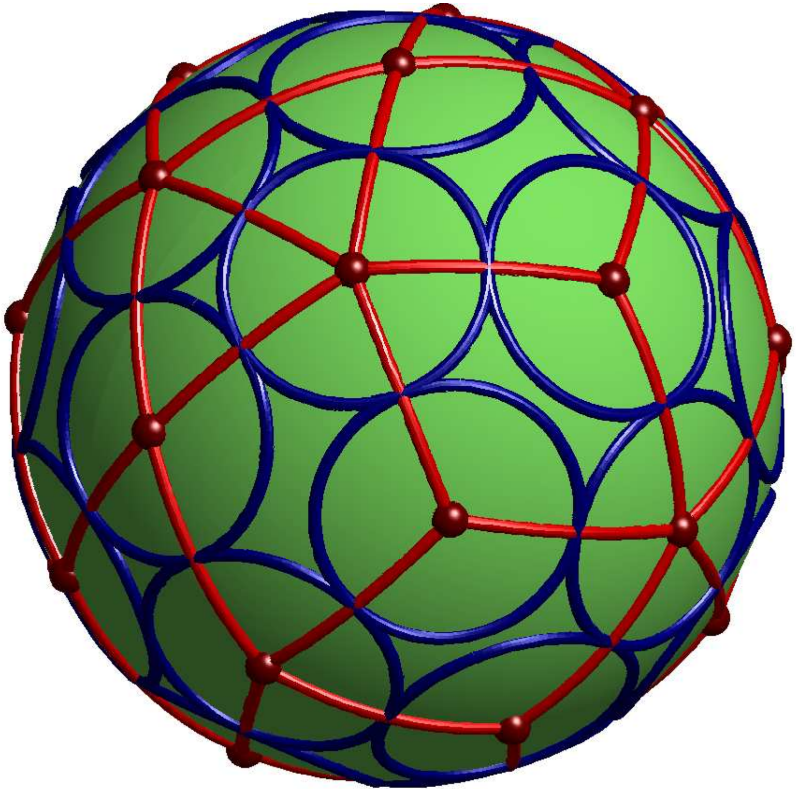
Outline of the Feige-Schechtman algorithm

1. Find spherical radius $\boldsymbol{\theta_c}$ of caps
2. Create optimal packing of caps of spherical radius $\boldsymbol{\theta_c}$
3. Create graph of kissing caps
4. Create directed tree from graph
5. Create Voronoi tessellation
6. Move area from V-cells towards root of tree
7. Split adjusted cells

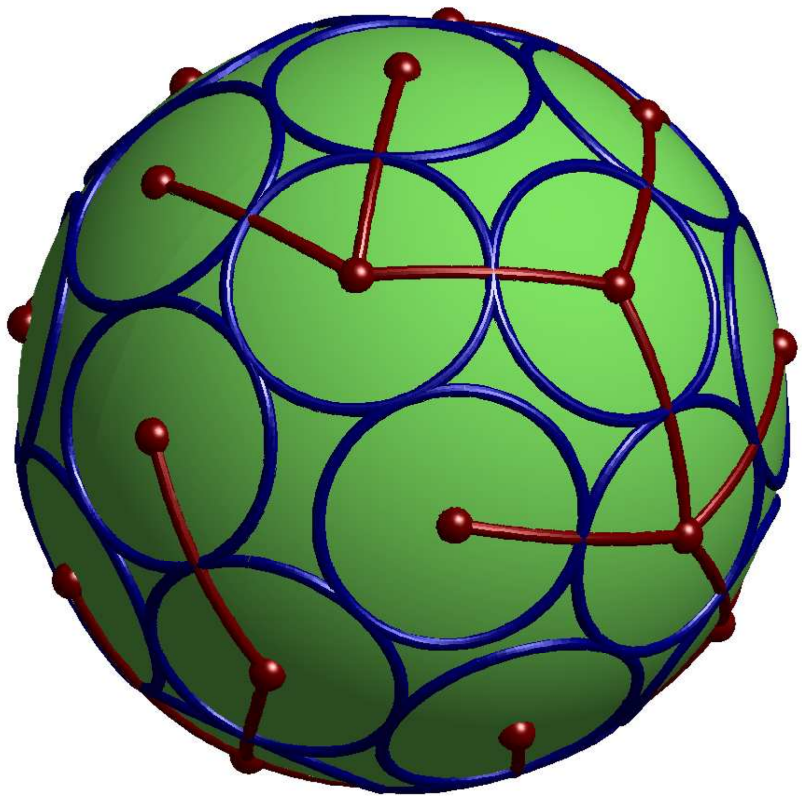
2. Create optimal packing of caps



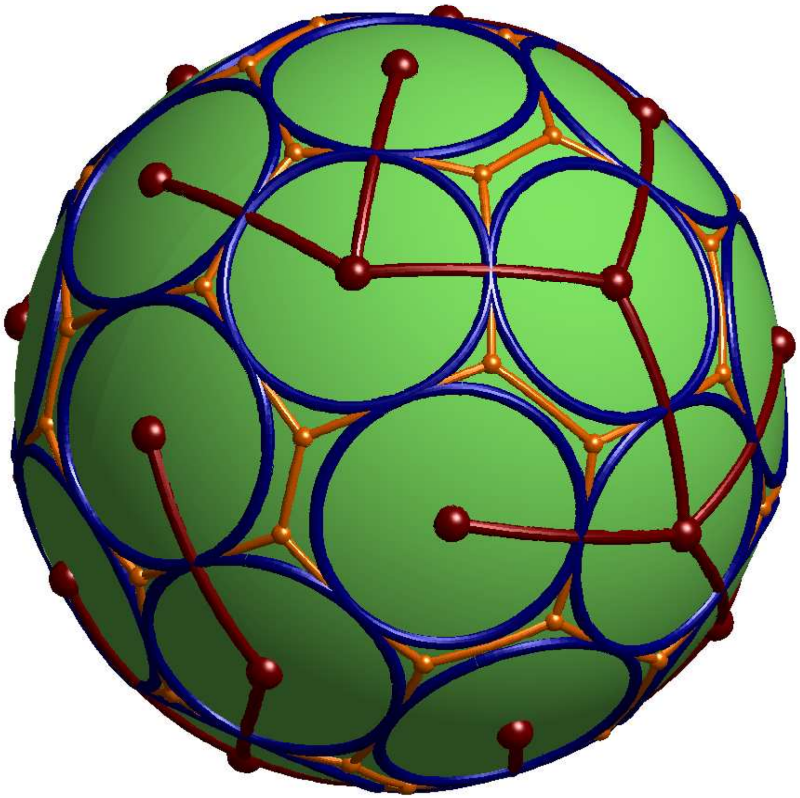
3. Create graph of kissing caps



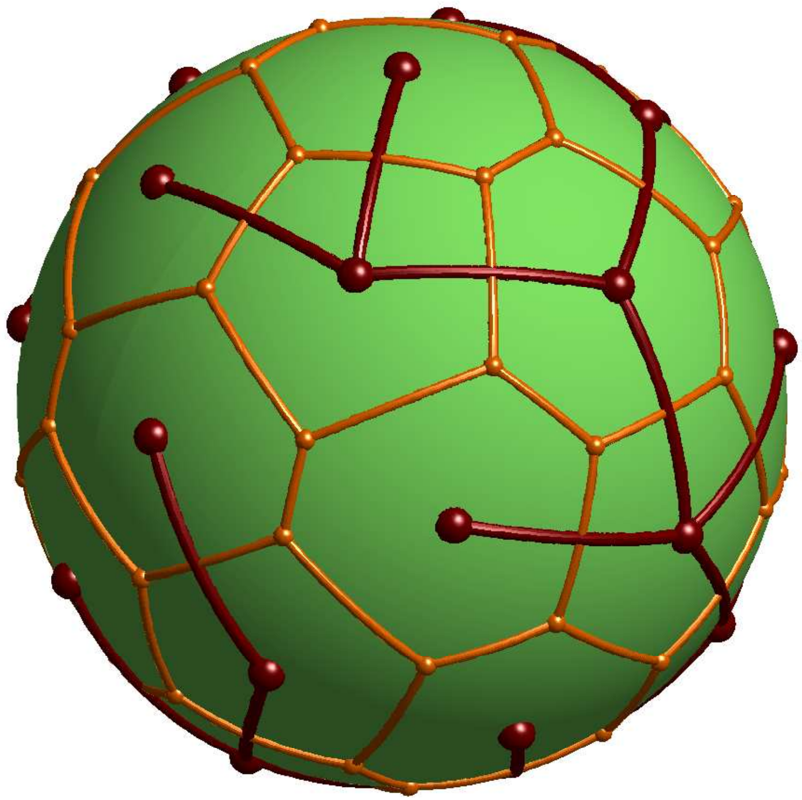
4. Create directed tree from graph



5. Create Voronoi tessellation



6. Move area from V-cells towards root



Outline of proof the F-S bound

- Packing radius is $\theta_c = \mathrm{O}(N^{-1/d})$.
- V-cells are in caps of spherical radius $2\theta_c$.
- Each V-cell has area larger than target area.
- Area is moved from V-cells of kissing packing caps.
- Adjusted cells are in caps of spherical radius $4\theta_c$.
- So Euclidean diameter is bounded above by

$$8\theta_c = \mathrm{O}(N^{-1/d}).$$

Key properties of the EQ partition of $\mathbb{S}^d$

The *recursive zonal equal area* partition of $\mathbb{S}^d$ into N regions is denoted as $\mathbf{EQ}(d, N)$.

The set of partitions $\mathbf{EQ}(d) := \{\mathbf{EQ}(d, N) \mid N \in \mathbb{N}_+\}$.

The $\mathbf{EQ}$ partition satisfies:

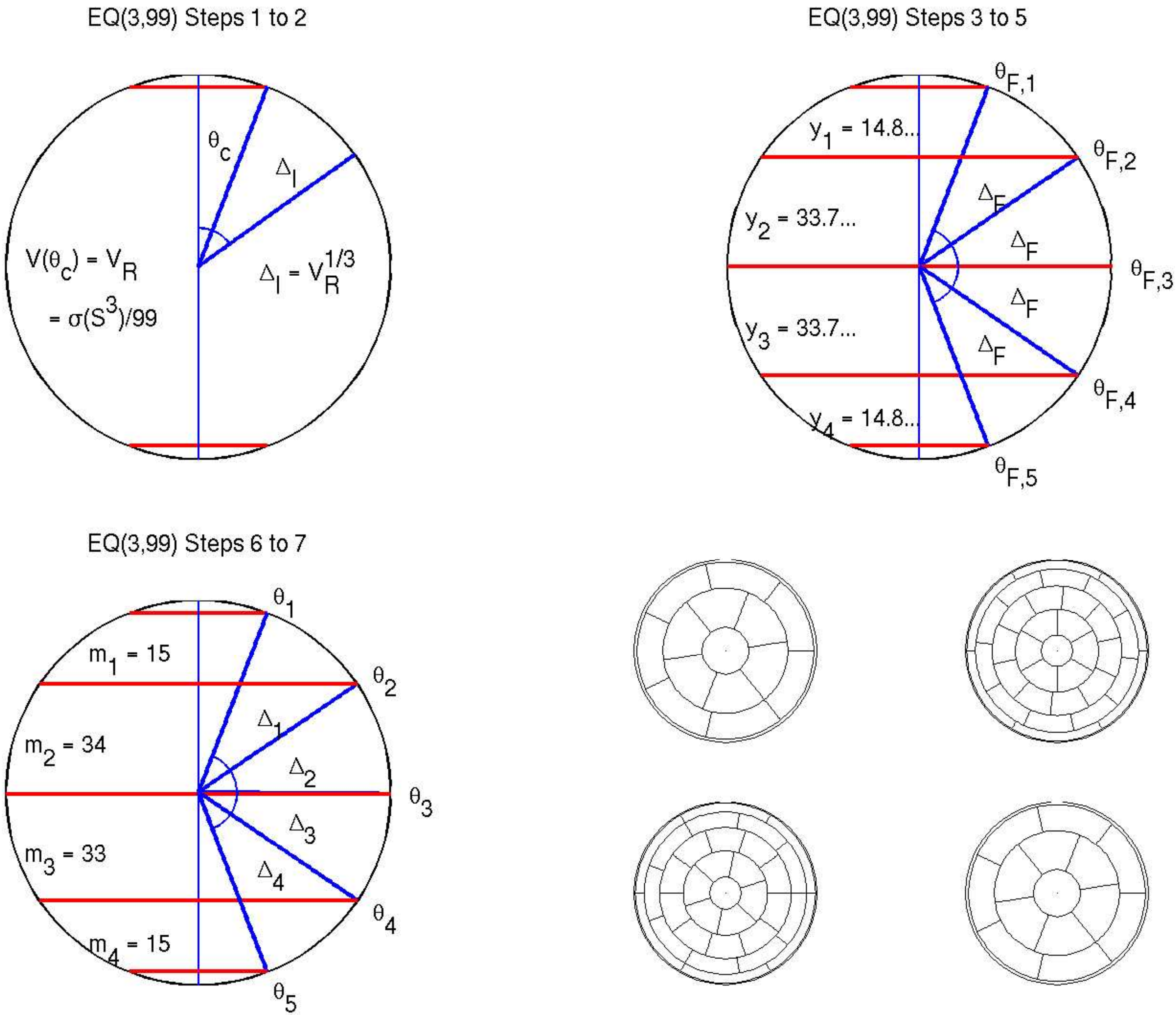
Theorem 1. *For* $N \geqslant 1$, $\mathbf{EQ}(d, N)$ *is an equal-area partition.*

Theorem 2. *For* $d \geqslant 1$, $\mathbf{EQ}(d)$ *is diameter-bounded.*

Outline of the EQ algorithm

if $N = 1$ **then**
There is a single region which is the whole sphere;
else if $d = 1$ **then**
Divide the circle into N equal segments;
else
Divide the sphere into zones, each the same area as an integer number of regions:
1. Determine the colatitudes of polar caps,
2. Determine an ideal collar angle,
3. Determine an ideal number of collars,
4. Determine the actual number of collars,
5. Create a list of the ideal number of regions in each collar,
6. Create a list of the actual number of regions in each collar,
7. Create a list of colatitudes of each zone;

Partition each spherical collar into regions of equal area,
using the EQ algorithm for dimension $d - 1$;
endif .


EQ(3,99) Steps 1 to 2
θc
ΔI
V(θc) = VR
= σ(S3)/99
ΔI = VR1/3
EQ(3,99) Steps 3 to 5
θF,1
y1 = 14.8...
θF,2
ΔF
y2 = 33.7...
ΔF
θF,3
ΔF
y3 = 33.7...
ΔF
θF,4
y4 = 14.8...
θF,5
EQ(3,99) Steps 6 to 7
θ1
m1 = 15
θ2
Δ1
m2 = 34
Δ2
θ3
Δ3
m3 = 33
Δ4
θ4
m4 = 15
θ5

Rounding the number of regions per collar

Similarly to Zhou (1995), given the sequence y_i for n collars, with

$$\sum_{i=1}^{n} y_i = N - 2,$$

define the sequences a and m by: $a_0 := 0$,
and for $i \in \{1, \ldots, n\}$,

$$m_i := \mathrm{round}(y_i + a_{i-1}), \quad a_i := \sum_{j=1}^{i} (y_j - m_j).$$

Then m_i is the required number of regions in collar i, and
$a_i \in [-1/2, 1/2)$ and $a_n = 0$.

Geometry of regions

Each region R in collar i of $\mathrm{EQ}(d, N)$ is of the form

$$R = R_{d-1} \times [\theta_i, \theta_{i+1}],$$

in spherical polar coordinates, where
$R_{d-1} = [t_1, b_1] \times \ldots \times [t_{d-1}, b_{d-1}]$, with $t, b \in \mathbb{S}^{d-1}$.

We can show that

$$\operatorname{diam} R \leqslant \sqrt{\Delta_i^2 + w_i^2 (\operatorname{diam} R_{d-1})^2},$$

where $\Delta_i := \theta_{i+1} - \theta_i$ and $w_i := \max_{\xi \in [\theta_i, \theta_{i+1}]} \sin \xi$.

The inductive step

Define

$$P_i := w_i\, m_i^{\frac{-1}{d-1}}.$$

Assuming that $\mathbf{EQ}(d-1)$ has diameter bound κ, we have

$$\operatorname{diam} R \leqslant \sqrt{\left(\max_{i\in\{1,\ldots,n\}} \Delta_i\right)^2 + \kappa^2 \left(\max_{i\in\{1,\ldots,n\}} P_i\right)^2}.$$

Cap, Δ, P bounds

We can use properties and estimates of $\mathcal{V}$ to show that:

- There is a constant $K_c > 0$ such that for $N > 1$, the diameter of each polar cap of $\mathrm{EQ}(d, N)$ is bounded by $K_c N^{-1/d}$.
- For $d > 1$, there are constants $K_\Delta > 0, C_P > 0$, $N_\Delta, N_P \in \mathbb{N}$ such that for $\mathrm{EQ}(d, N)$ with $N > \max(N_\Delta, N_P)$,

$$\max_{i \in \{1,\ldots,n\}} \Delta \leqslant K_\Delta N^{-1/d},$$

$$\max_{i \in \{1,\ldots,n\}} P \leqslant C_P N^{-1/d}.$$

Outline of proof of Theorem 2

Assume that $N > 2$ and $d > 1$.
Define $N_H := \max(N_\Delta, N_P)$.

Then if $d \geqslant 1$, if $\mathrm{EQ}(d-1)$ has diameter bound κ, and if $N > N_H$, we have $\operatorname{maxdiam}(d, N) \leqslant K_H N^{-1/d}$, where $K_H := \max\left(K_c, \sqrt{K_\Delta^2 + \kappa^2 C_P^2}\right)$.

The diameter of any region is bounded by 2.
Therefore for $N \leqslant N_H$, $\operatorname{maxdiam}(d, N) \leqslant K_L N^{-1/d}$, where $K_L := 2 N_H^{1/d}$.

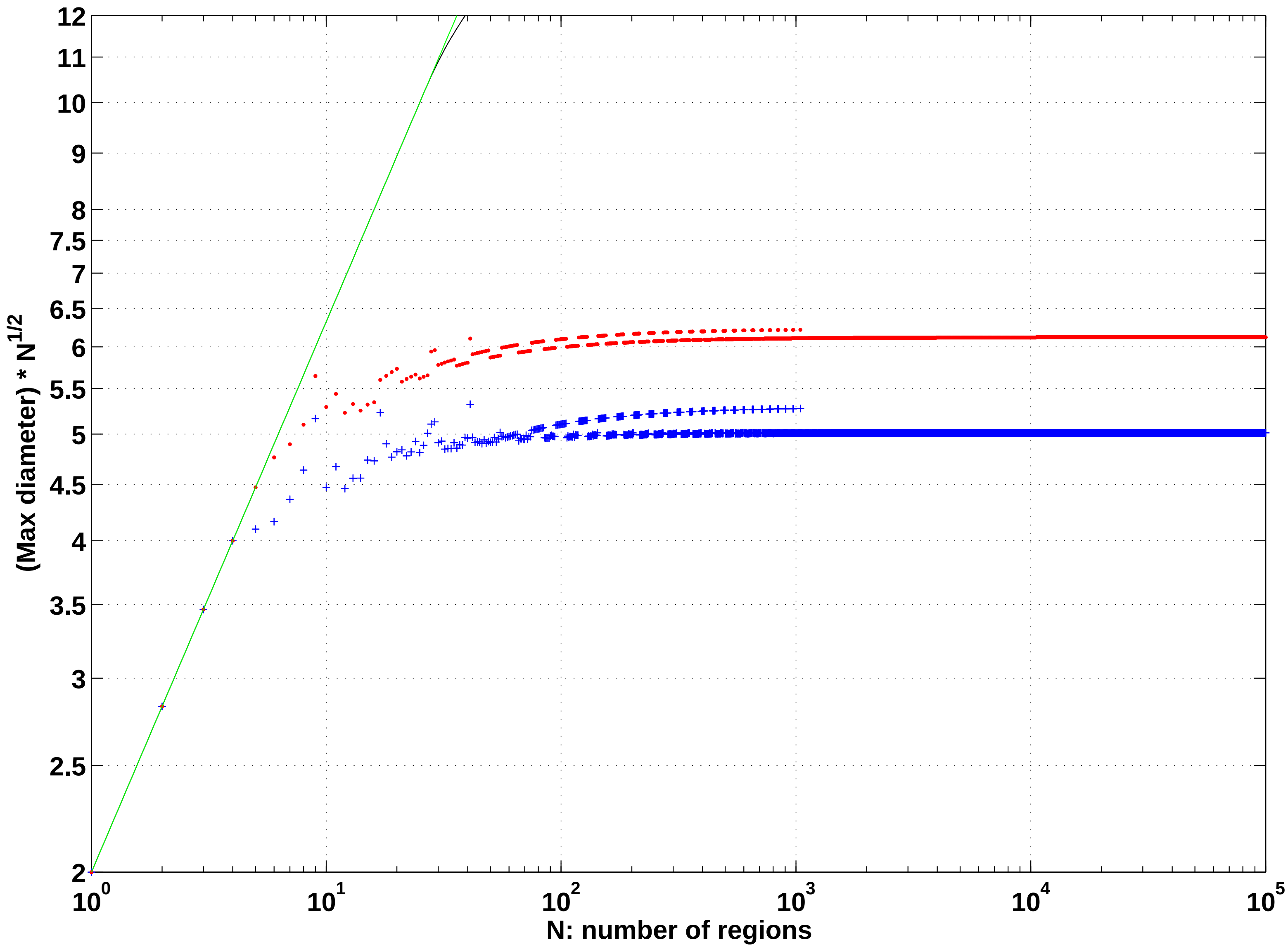
$\mathrm{EQ}(1, N)$ consists of N equal segments, so $\mathrm{EQ}(1)$ has diameter bound 2π. The result follows by induction.

Diameter bound constants

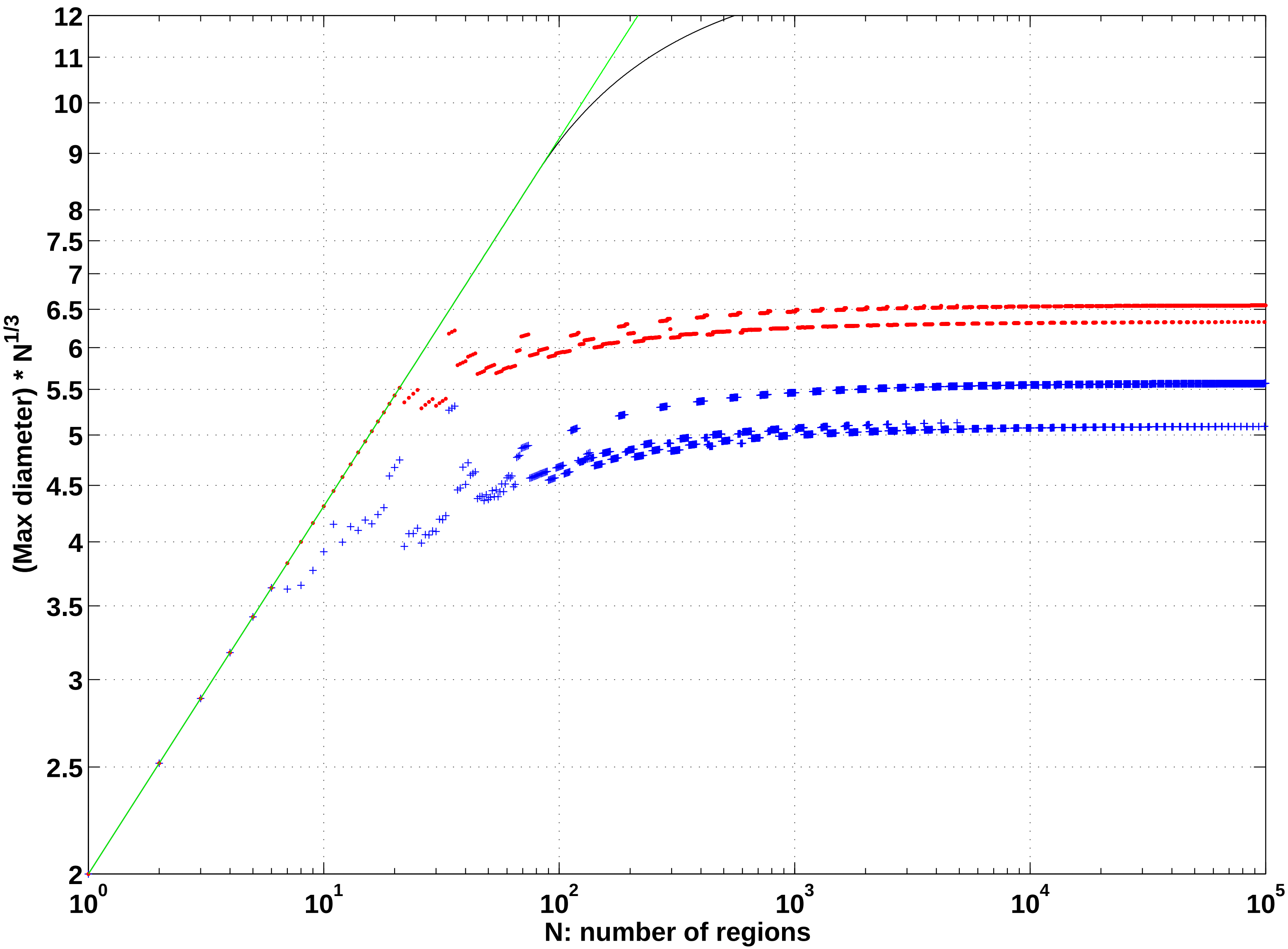
d	K_{FS}	K_d (l. b.)	K_d $(N \to \infty)$
2	16.0	8.9	12.8
3	13.4	10.4	25.1
4	12.2	11.5	48.3
5	11.4	12.3	90.7
6	10.9	13.0	166.3
7	10.5	13.6	297.4
8	10.3	14.2	519.7
9	10.0	14.7	888.1
10	9.9	15.1	1486.0

Zhou obtains $K_2 \leqslant 7$ for his (1995) algorithm.

Diameter bounds for $\mathbb{S}^2$



Diameter bounds for $\mathbb{S}^3$



Diameter bounds for $\mathbb{S}^4$