Applications of equal area partitions of the unit sphere

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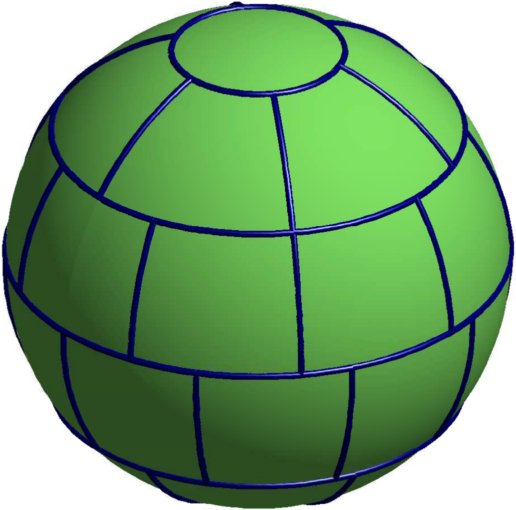
Outline of talk

- Some applications of equal area partitions of the unit sphere
- Overview of properties of the EQ codes
- Construction of the EQ codes
- Details of some properties of the EQ codes

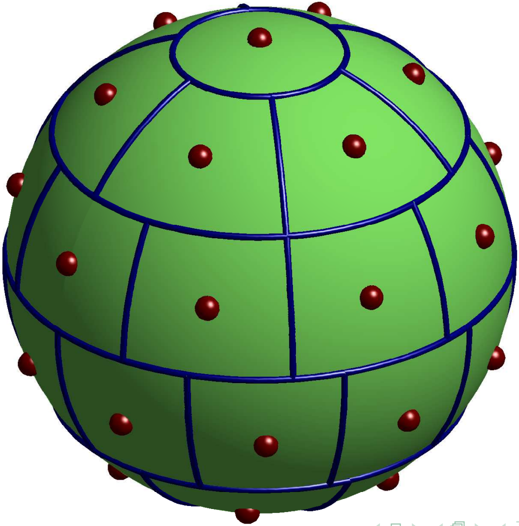
The partition EQ(2,33) on $\mathbb{S}^2 \subset \mathbb{R}^3$

EQ partitions: Recursive Zonal Equal Area partitions of the sphere, $\bigcup \mathbf{EQ}(\mathbf{d}, \mathcal{N}) = \mathbb{S}^{\mathbf{d}}$, with $|\mathbf{EQ}(\mathbf{d}, \mathcal{N})| = \mathcal{N}$.



The spherical code EQP(2,33) on $\mathbb{S}^2$

EQ codes: The Recursive Zonal Equal Area spherical codes, $\mathbf{EQP(d,\mathcal{N})} \subset \mathbb{S}^{\mathbf{d}}$, with $|\mathbf{EQP(d,\mathcal{N})}| = \mathcal{N}$.



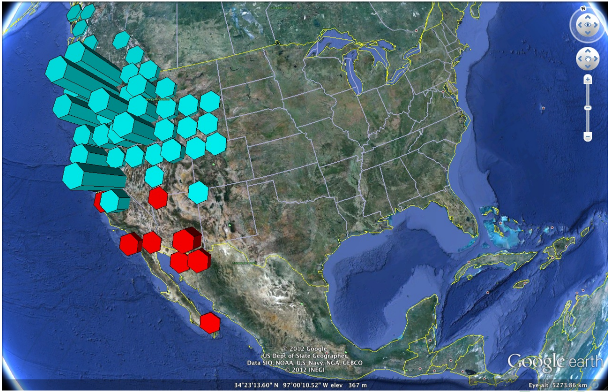
Range of applications of sphere partitions

The paper "A partition of the unit sphere into regions of equal area and small diameter" (ETNA 2006) has about 90 citations in Google Scholar, 40 in Web of Science.

Citations appear in Geophysical Journal International, Global Change Biology, IEEE Transactions on Audio Speech and Language Processing, Journal of Approximation Theory, Journal of the Atmospheric Sciences, Journal of Computational Chemistry, Journal of Differential Equations, Mathematics of Computation, Radio Science, RNA Journal, and elsewhere.

Histograms on the sphere (1)

Arrigo, et al. "Quantitative visualization of biological data in Google Earth using R2G2, an R CRAN package."
(Molecular Ecology Resources 2012)
R2G2 uses equal area partitions of fixed size (e.g. 500, 10 000, 20 000) as bins for histograms displayed via Google Earth.



Histograms on the sphere (2)

Strong and Magnusdottir, "Tropospheric Rossby Wave Breaking and the NAO/NAM." (J. Atmos. Sci. 2008).

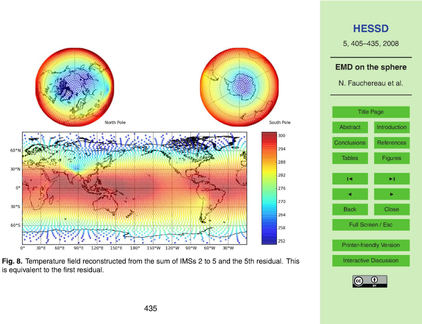
Strong and Maberly, "The influence of atmospheric wave dynamics on interannual variation in the surface temperature of lakes in the English Lake District." (Global Change Biology 2013).

Each paper uses an equal area partition on the sphere to partition the Northern Hemisphere into 400 or 800 bins (respectively) to estimate statistics related to Rossby wave breaking (RWB), specifically, the relative frequency of anticyclonic RWB centroids.

Empirical mode decomposition

Fauchereau et al. "Empirical Mode Decomposition on the sphere: application to the spatial scales of surface temperature variations." (Hydrol. Earth Syst. Sci. 2008).

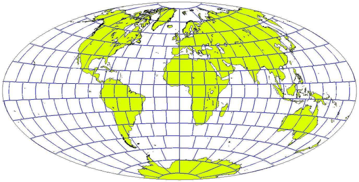
Uses 6500 points generated by an equal area partition to implement an Empirical Mode Decomposition to find the length scales of spatial variations in temperature on the surface of the earth.



Load balancing of parallel computation

Mozdzynski et al. "A PGAS implementation by co-design of the ECMWF Integrated Forecasting System (IFS)."
(2012 SC Companion).
The European Centre for Medium Range Weather Forecasting uses equal area partitions of the sphere to improve the efficiency of the Integrated Forecast System.

eq_regions partitioning T799 251 tasks



```
N_REGIONS( 1)=  1
N_REGIONS( 2)=  7
N_REGIONS( 3)= 12
N_REGIONS( 4)= 17
N_REGIONS( 5)= 22
N_REGIONS( 6)= 25
N_REGIONS( 7)= 28
N_REGIONS( 8)= 27
N_REGIONS( 9)= 28
N_REGIONS(10)= 25
N_REGIONS(11)= 22
N_REGIONS(12)= 17
N_REGIONS(13)= 12
N_REGIONS(14)=  7
N_REGIONS(15)=  1
```

Solution schemes for equations – a non-application

Staniforth and Thuburn, "Horizontal grids for global weather and climate prediction models: a review."
(Quarterly Journal of the Royal Meteorological Society 2012).

Classes equal area partitions as a type of "reduced grid."
Curvature of region boundaries near the poles increases the error of finite differencing if polar coordinates are used naively
(Williamson and Browning 1974, Williamson 2007).

Fadeev, "Algorithm for reduced grid generation on a sphere for a global finite-difference atmospheric model."
(Computational Mathematics and Mathematical Physics 2013).
Generates a reduced grid by minimizing the interpolation error of a function, instead of using an equal area partition.

Geometric properties

For $\mathbf{EQP(d,\mathcal{N})}$

Good:

- Centre points of regions of diameter $= \mathbf{O(\mathcal{N}^{-1/d})}$,
- Mesh norm (covering radius) $= \mathbf{O(\mathcal{N}^{-1/d})}$,
- Minimum distance and packing radius $= \mathbf{\Omega(\mathcal{N}^{-1/d})}$.

Bad:

- Mesh ratio $= \mathbf{\Omega(\sqrt{d})}$,
- Packing density $\leqslant \frac{\pi^{d/2}}{2^d\ \Gamma(d/2+1)}$ as $\mathcal{N} \to \infty$.

Approximation properties

Not so bad?

- Normalized spherical cap discrepancy $= \mathbf{O}(\mathcal{N}^{-1/d})$,
- Normalized $\mathbf{s}$-energy

$$\mathbf{E_s} = \begin{cases} \mathbf{I_s} \pm \mathbf{O}(\mathcal{N}^{-1/d}) & \mathbf{0 < s < d-1} \\ \mathbf{I_s} \pm \mathbf{O}(\mathcal{N}^{-1/d} \log \mathcal{N}) & \mathbf{s = d-1} \\ \mathbf{I_s} \pm \mathbf{O}(\mathcal{N}^{s/d-1}) & \mathbf{d-1 < s < d} \\ \mathbf{O}(\log \mathcal{N}) & \mathbf{s = d} \\ \mathbf{O}(\mathcal{N}^{s/d-1}) & \mathbf{s > d.} \end{cases}$$

Ugly:

- Cannot be used for polynomial interpolation:
  proven for large enough $\mathcal{N}$, conjectured for small $\mathcal{N}$.

Some precedents

The **EQ** partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Saff, and on Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Separation without equidistribution: Hamkins (1996) and Hamkins and Zeger (1997) constructed $\mathbb{S}^d$ codes with asymptotically optimal packing density.

Equidistibution without separation: Many constructions for $\mathbb{S}^2$, eg. mapped Hammersley, Halton, $(\mathbf{t}, \mathbf{s})$ etc. sequences.
Feige and Schechtman (2002) constructed a diameter bounded equal area partition of $\mathbb{S}^d$. Put one point in each region.

Equal-area partitions of $\mathbb{S}^d \subset \mathbb{R}^d$

An *equal area partition* of $\mathbb{S}^d \subset \mathbb{R}^d$ is a finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\lambda_d(R) = \frac{\lambda_d(\mathbb{S}^d)}{|\mathcal{P}|},$$

where λ_d is the Lebesgue area measure on $\mathbb{S}^d$.

Diameter bounded sets of partitions

The *diameter* of a region $\mathbf{R} \subset \mathbb{R}^{\mathbf{d+1}}$ is defined by

$$\mathbf{diam\,R} := \mathbf{sup}\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in \mathbf{R}\}.$$

A set $\mathbf{\Xi}$ of partitions of $\mathbb{S}^{\mathbf{d}} \subset \mathbb{R}^{\mathbf{d+1}}$ is *diameter-bounded* with *diameter bound* $\mathbf{K} \in \mathbb{R}_+$ if for all $\mathcal{P} \in \mathbf{\Xi}$, for each $\mathbf{R} \in \mathcal{P}$,

$$\mathbf{diam\,R} \leqslant \mathbf{K}\,|\mathcal{P}|^{-\mathbf{1/d}}.$$

Key properties of the **EQ** partition of $\mathbb{S}^d$

$\mathbf{EQ}(d, \mathcal{N})$ is the *recursive zonal equal area* partition of $\mathbb{S}^d$ into $\mathcal{N}$ regions.

The set of partitions $\mathbf{EQ}(d) := \{\mathbf{EQ}(d, \mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+\}$.

The **EQ** partition satisfies:

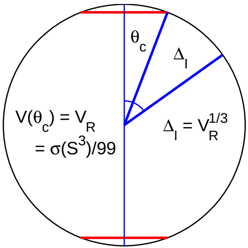
Theorem 1

For $d \geqslant 1$, $\mathcal{N} \geqslant 1$, $\mathbf{EQ}(d, \mathcal{N})$ *is an equal-area partition.*

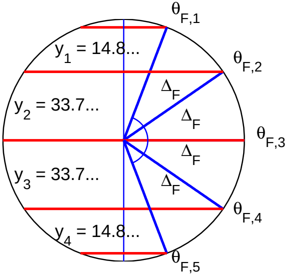
Theorem 2

For $d \geqslant 1$, $\mathbf{EQ}(d)$ *is diameter-bounded.*

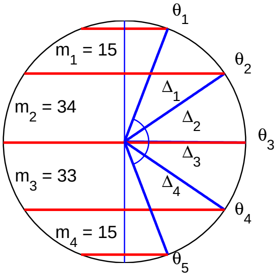
EQ(3,99) Steps 1 to 2

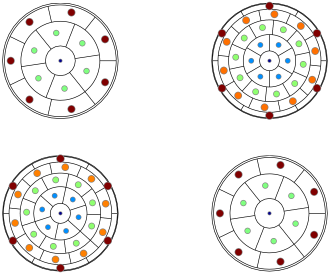


EQ(3,99) Steps 3 to 5



EQ(3,99) Steps 6 to 7





Minimum distance and packing radius

The *minimum distance* of $\mathbf{X} := \{\mathbf{x_1}, \ldots, \mathbf{x}_{\mathcal{N}}\} \subset \mathbb{S}^{\mathbf{d}}$ is

$$\text{min dist } \mathbf{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \mathbf{X}} \|\mathbf{x} - \mathbf{y}\|,$$

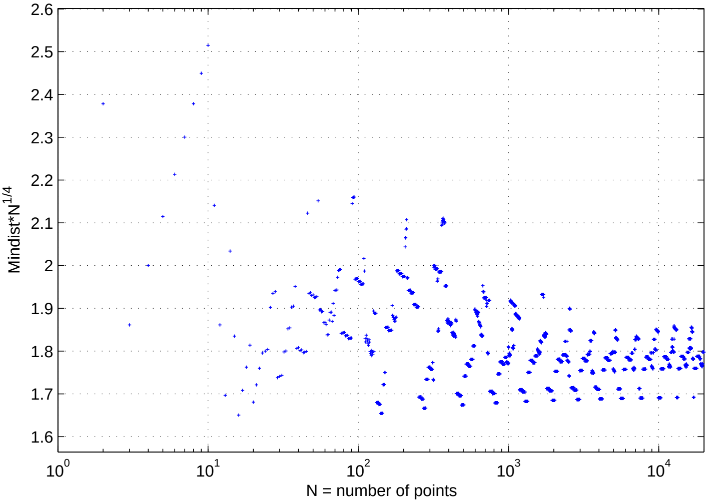
and the *packing radius* of $\mathbf{X}$ is

$$\text{prad } \mathbf{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \mathbf{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y})/2.$$

It can be shown that $\quad$ min dist $\mathbf{EQP}(\mathbf{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\mathbf{d}})$,

and therefore $\quad$ prad $\mathbf{EQP}(\mathbf{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\mathbf{d}})$.

Minimum distance of EQP(4) codes



Normalized spherical cap discrepancy

We use the probability measure $\sigma := \lambda_d / \lambda_d(\mathbb{S}^d)$.

For $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ the *normalized spherical cap discrepancy* is

$$\operatorname{disc} X := \sup_{y \in \mathbb{S}^d} \sup_{\theta \in [0,\pi]} \left| \frac{|X \cap S(y,\theta)|}{\mathcal{N}} - \sigma\big(S(y,\theta)\big) \right|.$$

It can be shown that

$$\operatorname{disc} \ \mathrm{EQP}(d, \mathcal{N}) = O(\mathcal{N}^{-1/d}).$$

For EQSP Matlab code

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

`http://eqsp.sourceforge.net`