Practical computation with Clifford algebras

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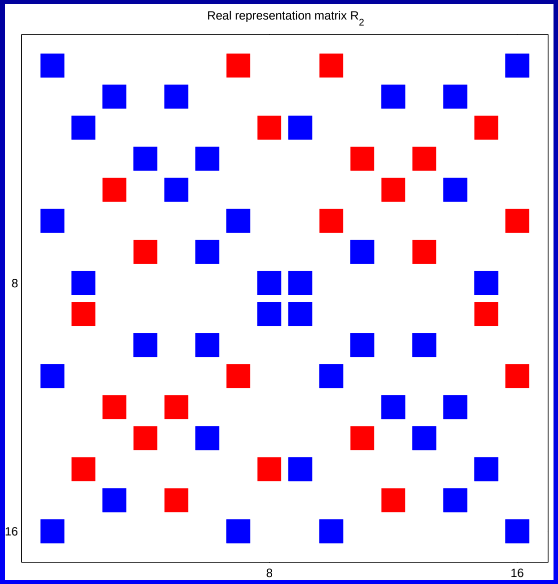


Agenda

- Motivation
- Clifford algebras
- Notation
- Representations
- Functions

GluCat

(Lounesto et al. 1987; Lounesto 1992; Raja 1996; Bangerth et al.; Karmesin et al.; Siek et al.)

- Generic library of universal Clifford algebra templates
- C++ template library for use with other libraries such as *deal.II* and *POOMA*
- For details, see `http://glucat.sf.net` and the thesis, "Practical computation with Clifford algebras"

Motivation

- The vector derivative ∇
- Integral formulae in $\mathbb{R}^n$
- Methods for PDEs

The vector derivative ∇

(Hestenes and Sobczyck 1984)

- Applies on left or right
- Multiplies like a vector under the *geometric product*
- May have a left or a right inverse
- Can have a functional calculus with $f(\nabla)$ etc.
- $\ker \nabla$ and $\nabla \ker$ define *monogenic functions*
- Maxwell's equations can be written as

$$\nabla F = J = \widetilde{F} \nabla$$

Integral formulae preliminaries

(Hestenes and Sobczyck 1984)

- The surface area of a unit ball in n dimensions is

$$\omega_n := \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

- Define the Green's function

$$g_n(x) := \frac{x}{\omega_n \left|x\right|^n}$$

- A *Lyapunov surface* is a closed surface in $\mathbb{R}^n$ which has a Hölder continuous outward normal ν

Integral formulae in $\mathbb{R}^n$

(Hestenes and Sobczyck 1984)

- Let Ω be a domain which is bounded by a piecewise Lyapunov surface $\partial\Omega$. Then $\forall u \in C^1(\Omega, \mathbb{R}_{n,0}) \cap C(\overline{\Omega}, \mathbb{R}_{n,0})$

  Stoke's theorem: $$\int_\Omega \nabla u(y) dy = \int_{\partial\Omega} u(y)\, |d\sigma_y|$$

  Cauchy-Borel-Pompeiu formula:

  $$\int_\Omega g_n(x-y)\nabla u(y) dy - \int_{\partial\Omega} g_n(x-y)\nu(y)u(y)\, |d\sigma_y| = \begin{cases} u(x) & (x \in \Omega), \\ 0 & (x \in \mathbb{R}^n \setminus \overline{\Omega}) \end{cases}$$

Methods for PDEs

- Appropriate vector derivative (Obolashvili 1998)
- Operator calculus and quadrature

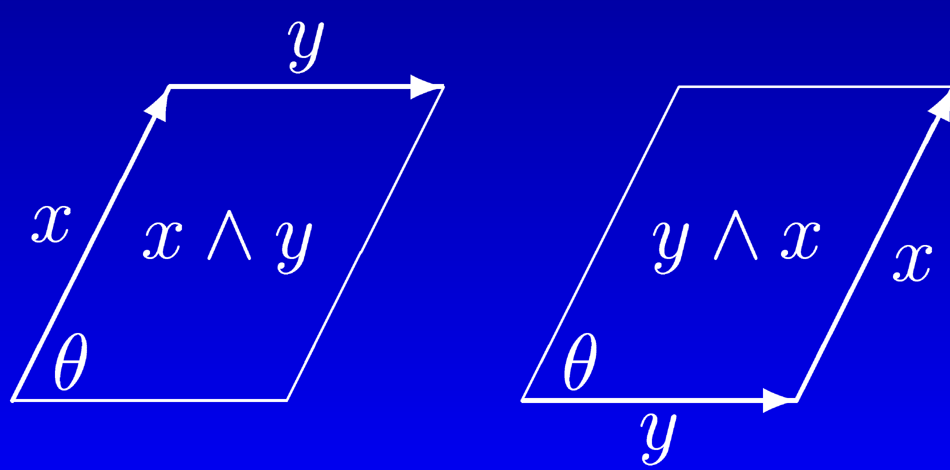
  (Gürlebeck and Sprößig 1997)
- Discrete operators (Gürlebeck and Sprößig 1997)
- Function spaces
  - Galerkin method
  - Clifford wavelets (Mitrea 1994)

Outer product

(Lasenby and Doran 1999; Lounesto 1997)

- For x and y in $\mathbb{R}^n$, making angle θ, the *outer* product $x \wedge y$ is a *directed area* in the plane of x and y:

$$|x \wedge y| = |x|\,|y| \sin\theta$$



Properties of outer product

(Lasenby and Doran 1999)

- The outer product is *anticommutative* on vectors:

$$x \wedge y = -y \wedge x \qquad \forall x, y \in \mathbb{R}^n$$

- and *distributive*:

$$x \wedge (y + z) = x \wedge y + x \wedge z \qquad \forall x, y, z$$

- $x \wedge y$ is a $bivector$ for vectors $x \neq y$
- Bivectors form a linear space

The geometric product

(Lasenby and Doran 1999)

- The *geometric* product of vectors in $\mathbb{R}^n$ is:

$$xy = x \cdot y + x \wedge y \qquad \forall x, y \in \mathbb{R}^n$$

- Sum of scalar and bivector
- Encodes the angle between x and y

$$yx = y \cdot x + y \wedge x = x \cdot y - x \wedge y$$

$$x \cdot y = \frac{1}{2}(xy + yx)$$

$$x \wedge y = \frac{1}{2}(xy - yx) \qquad \forall x, y \in \mathbb{R}^n$$

Multivectors

(Lasenby and Doran 1999)

- A vector space closed under the geometric product is a *Clifford algebra*
- Elements are called *multivectors*
- A multivector is a 0-vector (scalar), plus a 1-vector (vector), plus a 2-vector (bivector), plus $\ldots$ an n-vector (pseudoscalar)
- Formal definition uses *quadratic forms*

Quadratic forms

(Lounesto 1997)

For vector space $\mathbb{V}$ over field $\mathbb{F}$, characteristic $\neq 2$:

- Map $f : \mathbb{V} \to \mathbb{F}$, with

$$f(\lambda x) = \lambda^2 f(x), \forall \lambda \in \mathbb{F}, x \in \mathbb{V}$$

- $f(x) = b(x, x)$, where

$$b : \mathbb{V} \times \mathbb{V} \to \mathbb{F}, \text{given by}$$

$$b(x, y) := \frac{1}{2}\left(f(x + y) - f(x) - f(y)\right)$$

is a symmetric bilinear form

Quadratic spaces, Clifford maps

(Porteous 1995; Lounesto 1997)

- A *quadratic space* is the pair $(\mathbb{V}, f)$, where f is a quadratic form on $\mathbb{V}$
- A *Clifford map* is a vector space homomorphism

$$\varphi : \mathbb{V} \to \mathbb{A}$$

where $\mathbb{A}$ is an associative algebra, and

$$(\varphi v)^2 = f(v) \qquad \forall v \in \mathbb{V}$$

Universal Clifford algebras

(Lounesto 1997)

- The *universal Clifford algebra* $Cl(f)$ for the quadratic space $(\mathbb{V}, f)$ is the algebra generated by the image of the Clifford map φ_f such that $Cl(f)$ is the universal initial object such that $\forall$ suitable algebras $\mathbb{A}$ with Clifford map $\varphi_{\mathbb{A}}$ $\exists$ a homomorphism

$$\rho_{\mathbb{A}} : Cl(f) \to \mathbb{A}$$

$$\varphi_{\mathbb{A}} = \rho_{\mathbb{A}} \circ \varphi_f$$

PDEs and quadratic forms

(Sobolev 1964)

- A second order PDE with constant coefficients determines a real symmetric matrix, eg.

$$f(x) := 3\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2} - 2\frac{\partial^2 u}{\partial x_2^2}$$

$$= D^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} Du(x) \text{ where}$$

$$D^T := [\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}]$$

Real quadratic forms

(Lounesto 1997)

- A real symmetric matrix determines a real quadratic form, eg.

$$f(x) := 3x_1^2 + 2x_1x_2 - 2x_2^2$$
$$= x^T B x \text{ where}$$

$$B := \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

Canonical quadratic forms

(Sobolev 1964; Lipschutz 1968)

- A real symmetric matrix can be diagonalized
- *Sylvester's theorem* implies $\exists$ unique *canonical quadratic form* $\phi(x)$, eg.

$$B := \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = S^T Q S, \text{ with}$$

$$S := \begin{bmatrix} \sqrt{3} & \sqrt{\frac{1}{3}} \\ 0 & \sqrt{\frac{7}{3}} \end{bmatrix}, Q := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so we can define $\phi(x) := x^T Q x = x_1^2 - x_2^2$

Notation for integer sets

- For $S \subseteq \mathbb{Z}$, define

$$S^- := \{k \in S : k < 0\}$$
$$S^+ := \{k \in S : k > 0\}$$
$$S^\pm := S^- \cup S^+$$

$$\sum_{k \in S} f_k := \sum_{\substack{k=\min S \\ k \in S}}^{\max S} f_k \quad \prod_{k \in S} f_k := \prod_{\substack{k=\min S \\ k \in S}}^{\max S} f_k$$

- For $m \leqslant n \in \mathbb{Z}$, define

$$[m \ldots n]^\pm := \{m, m+1, \ldots, n-1, n\}^\pm$$

Real Clifford algebras $\mathbb{R}_{p,q}$

(Porteous 1995)

- The real quadratic space $\mathbb{R}^{p,q}$ is $\mathbb{R}^{p+q}$ with

$$\phi(x) := -\sum_{k=-q}^{-1} {x_k}^2 + \sum_{k=1}^{p} {x_k}^2$$

- $\forall\ p, q \in \mathbb{N}\ \exists\ \mathbb{R}_{p,q}$: the real universal Clifford algebra for $\mathbb{R}^{p,q}$
- $\mathbb{R}_{p,q}$ is isomorphic to some matrix algebra over one of: $\mathbb{R}, {}^2\mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{H}$
- For example, $\mathbb{R}_{1,1} \cong \mathbb{R}(2)$

Notation for Clifford algebras

(Hestenes and Sobczyck 1984; Wene 1992; Ashdown)

- For finite $S \subset \mathbb{Z}$, $p, q \in \mathbb{N}$ define

$$\mathbb{R}^{[-q \ldots p]^{\pm}} := \mathbb{R}^{p,q}$$

$$\mathbb{R}_S := \left(\mathbb{R}^S, \mathbb{R}_{\sharp S^+, \sharp S^-}, \varphi_S\right)$$

$$\mathbb{R}_{[-q \ldots p]^{\pm}} := \left(\mathbb{R}^{p,q}, \mathbb{R}_{p,q}, \varphi_{p,q}\right)$$

$$\text{where } \gamma_S : S \rightarrow \mathbb{R}^S$$

$$\varphi_S : \mathbb{R}^S \rightarrow \mathbb{R}_{\sharp S^+, \sharp S^-}$$

$$\left(\varphi_S(\gamma_S)_k\right)^2 = \phi_S\left((\gamma_S)_k\right) = \operatorname{sign} k$$

Real frame groups

(Braden 1985; Lam and Smith 1989)

- A *frame* is an ordered basis $(\gamma_{-q}, \ldots, \gamma_p)$ for $\mathbb{R}^{p,q}$ which puts quadratic form into canonical form ϕ
- $\forall\ p, q\ \exists$ real *frame group* $\mathbb{GR}_{p,q}$
- The frame and -1 generate the real frame group via the *real frame group map*

$$g : \{-q, \ldots, p\}^{\pm} \to \mathbb{GR}_{p,q}$$

$$(g_k)^2 = \phi\gamma_k,\ g_k g_m = -1 g_m g_k\ \forall k \neq m$$

Canonical products

(Bergdolt 1996; Lounesto 1997; Dorst 2001)

- The real frame group $\mathbb{GR}_{p,q}$ has order 2^{p+q+1}
- Each member w can be expressed as the canonically ordered product

$$w = (-1)^a \prod_{k=-p,\ k\neq 0}^{q} {g_k}^{b_k}$$

where $a, b_k = 0, 1$

Index sets

(Lounesto 1997; Dorst 2001)

For finite $S \subset \mathbb{Z}^{\pm}$ define the *index map* η_S

$$\eta_S : P(S) \to \mathbb{GR}_S$$

$$(\eta_S)_T = \prod_{k \in S} {g_k}^{(\chi_T)_k}, \text{ where}$$

- $P(S)$ is the *power set* of S
- χ_T is the characteristic function of T
- Drop subscript S when it is understood. eg.

$\eta_{\{-1,2\}} = \eta_{\{-1\}}\eta_{\{2\}}$

Clifford algebra of frame group

(Braden 1985; Lam and Smith 1989; Lounesto 1997; Dorst 2001)

- For finite $S \subset \mathbb{Z}^{\pm}$ define the map α_S

$$\alpha_S : \mathbb{GR}_S \to \mathbb{R}_S$$

$$\alpha_S 1 := 1, \qquad \alpha_S(-1) := -1$$

$$\alpha_S g_k := \varphi_S \gamma_k, \qquad \alpha_S(gh) := \alpha_S g \alpha_S h$$

- $\alpha_S(\eta_S)_T = \prod_{k \in S} (\varphi_S \gamma_k)^{(\chi_T)_k}$
- eg. $\alpha \eta_{\{-1,2\}} = (\varphi \gamma_{-1})(\varphi \gamma_2)$

Generators and basis elements

(Lounesto 1997; Dorst 2001)

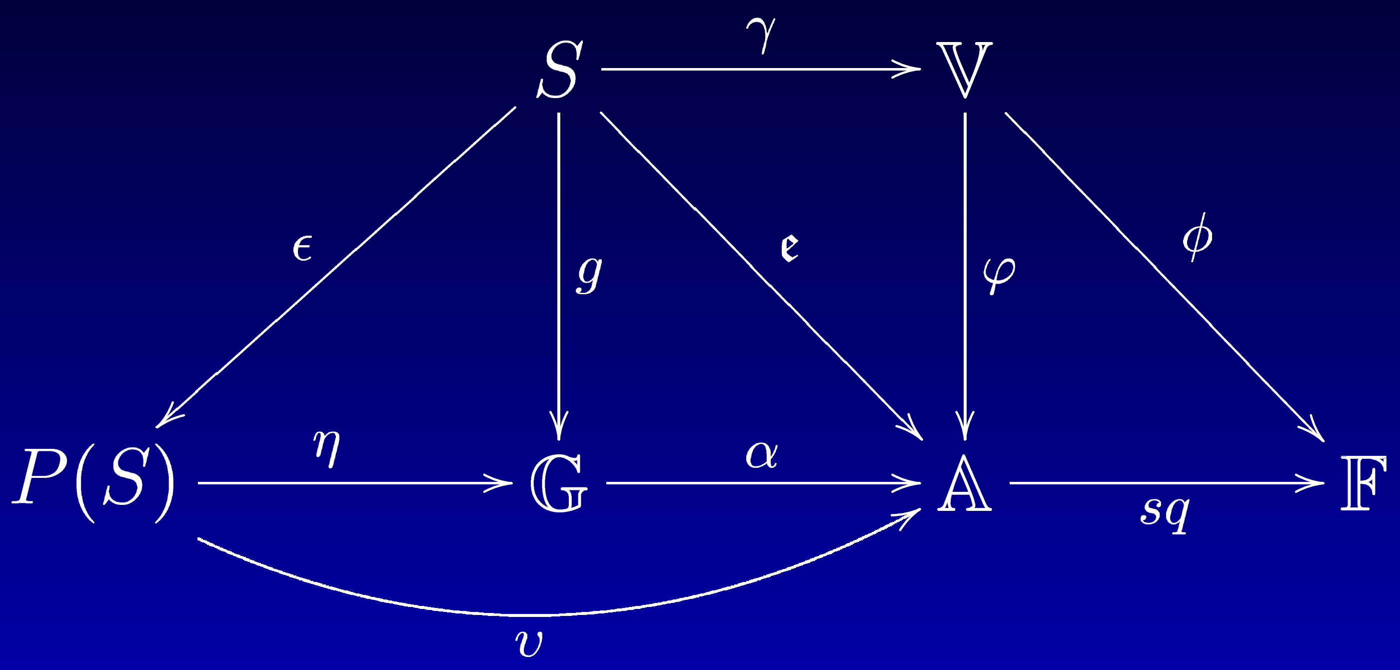
- For finite $S \subset \mathbb{Z}^{\pm}$ define the map $\mathfrak{e}_S$

$$\mathfrak{e}_S : S \to \mathbb{R}_S$$
$$(\mathfrak{e}_S)_k := \alpha_S(\gamma_S)_k$$

- For finite $S \subset \mathbb{Z}^{\pm}$ define the map υ_S

$$\upsilon_S : P(S) \to \mathbb{R}_S$$
$$(\upsilon_S)_T := \alpha_S(\eta_S)_T$$

Summary diagram



Vector derivative

(Lasenby and Doran 1999; Lounesto 1997)

Given a frame $(\gamma_{-q}, \ldots, \gamma_p)$ for $\mathbb{R}^{p,q}$, the *vector derivative* ∇ can be defined using the *reciprocal frame*

$$\mathfrak{e}^k \mathfrak{e}_m = \delta^k_m$$

$$\nabla := \sum_{k \in [-q \ldots p]^{\pm}} \mathfrak{e}^k \partial_k$$

$$= -\sum_{k=-q}^{-1} \mathfrak{e}_k \partial_k + \sum_{k=1}^{p} \mathfrak{e}_k \partial_k$$

Laplacian $= \nabla^2$

(Lounesto 1997)

- The sign convention for ϕ used here agrees with Lounesto 1997

$$\nabla^2 = -\sum_{k=-q}^{-1} \partial_k^2 + \sum_{k=1}^{p} \partial_k^2 \qquad = \Delta$$

- With the opposite sign convention, we would have $\nabla^2 = -\Delta$

Subalgebras

(Porteous 1995)

- If A is a Clifford algebra for a real quadratic space X and W is a linear subspace of X, then the subalgebra generated by W is a Clifford algebra for W.
- A subset of a canonical frame of a real Clifford algebra spans a subspace and generates a subalgebra which is a Clifford algebra for this subspace.

$$T \subseteq \text{ finite } S \subset \mathbb{Z}^{\pm} \Rightarrow \mathbb{R}_T \sqsubseteq \mathbb{R}_S$$

Isomorphisms

$$\sharp T^{+} = \sharp S^{+}, \sharp T^{-} = \sharp S^{-} \Rightarrow \mathbb{R}_T \equiv \mathbb{R}_S$$

- *Proof.* The following diagram commutes

$$\begin{array}{ccccc}
T & \xrightarrow{\gamma_T} & R^T & \xrightarrow{\varphi_T} & R_{\sharp T^{+}, \sharp T^{-}} \\
f \downarrow & & \nu_{TS} \downarrow & & \Vert \\
S & \xrightarrow[\gamma_S]{} & R^S & \xrightarrow[\varphi_S]{} & R_{\sharp S^{+}, \sharp S^{-}}
\end{array}$$

The map ν_{TS} is a quadratic space isomorphism []

Framed inner product, norm

(Gilbert and Murray 1991)

- We can treat $\mathbb{R}_S$ as an inner product space.
- $\{(\upsilon_S)_T : T \subseteq S\}$ is a $2^{\sharp S}$ element basis for $\mathbb{R}_S$
- For $x = \sum_{T \subseteq S} x_T \upsilon_T, y = \sum_{T \subseteq S} y_T \upsilon_T \in \mathbb{R}_S$
  - The *real framed inner product* is

$$x \bullet y := \sum_{T \subseteq S} x_T y_T$$

  - The *real framed norm* is

$$\|x\|_{RF} := (x \bullet x)^{\frac{1}{2}}$$

Framed notation

- For finite $S \subset \mathbb{Z}^{\pm}$ define the *framed notation*

$$\{\mathtt{k}_1, \mathtt{k}_2, \ldots\} := (\upsilon_S)_{\{k_1, k_2, \ldots\}}$$

  - eg. $\{-\mathtt{1}, \mathtt{2}\} = \upsilon_{\{-1,2\}} = \{-\mathtt{1}\}\{\mathtt{2}\}$
  - eg. $1 + 2\{-\mathtt{3}\} + 4\{-\mathtt{2}\} + 5\{-\mathtt{3}, -\mathtt{2}, \mathtt{2}\} = 1 + 2\upsilon_{\{-3\}} + 4\upsilon_{\{-2\}} + 5\upsilon_{\{-3,-2,2\}}$

- In hand calculation, it is convenient to use an *underline notation*:

$$\underline{k_1, k_2, \ldots} := \{\mathtt{k}_1, \mathtt{k}_2, \ldots\} = (\upsilon_S)_{\{k_1, k_2, \ldots\}}$$

eg. $\underline{-1, 2} = \{-\mathtt{1}, \mathtt{2}\} = \upsilon_{\{-1,2\}} = \underline{-1}\ \underline{2}$

Framed representation

For finite $S \subset \mathbb{Z}^{\pm}$ define

- *lexicogaphical ordering*

$$T < U \subseteq S := (\min T < \min U) \vee$$
$$\left(\exists k \in S : (\forall m < k, m \in T \Leftrightarrow m \in U) \wedge k \notin T \wedge k \in U\right)$$

- $\mathrm{setvalue}_S : P(S) \to \mathbb{N}$

$$\mathrm{setvalue}_S\, T \subseteq S := 2^{-\min S^-} \sum_{k \in T^-} 2^k + 2^{-\min S^- - 1} \sum_{k \in T^-} 2^k$$

- $\mathrm{coord} : \mathbb{R}_S \to \mathbb{R}^{2^{\max S^+ - \min S^-}}$

$$\left(\mathrm{coord} \sum_{T \subseteq S} x_T v_T\right)_{\mathrm{setvalue}_S\, T} := x_T$$

Matrix representations

(Cartan and Study 1908; Porteous 1995; Lounesto 1997)

- Each real Clifford algebra $\mathbb{R}_{p,q}$ can be *represented* by an endomorphism algebra over a finite dimensional module over one of: $\mathbb{R}, {}^2\mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{H}$.
- *Representation map* ρ, *representation matrix* R

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\text{coord}} & \mathbb{V}1 \\ \rho \big\downarrow & & \big\downarrow R \\ \mathbb{M} & \xrightarrow[\text{reshape}]{} & \mathbb{V}2 \end{array}$$

Matrix representations of $\mathbb{R}_{p,q}$

(Cartan and Study 1908; Porteous 1995; Lounesto 1997)

$p \downarrow$ \ $q \rightarrow$	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$
1	${}^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$
2	$\mathbb{R}(2)$	${}^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	${}^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$
5	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	${}^2\mathbb{H}(32)$
6	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(64)$	${}^2\mathbb{R}(128)$	$\mathbb{R}(256)$

- Notation $\mathbb{R}(N)$ means $\mathbb{M}_{N,N}(\mathbb{R})$, i.e. $\mathbb{R}^{N\times N}$
- Periodicity of 8 and other symmetries

Real representations of $\mathbb{R}_{p,q}$

(Cartan and Study 1908; Porteous 1995; Lounesto 1997)

p ↓ \ q →	0	1	2	3	4	5	6	7	8
0	**1**	2	4	8	8	8	**8**	16	**16**
1	2	**2**	4	8	16	16	16	**16**	32
2	**2**	4	**4**	8	16	32	32	32	**32**
3	4	**4**	8	**8**	16	32	64	64	64
4	8	8	**8**	16	**16**	32	64	128	128
5	16	16	16	**16**	32	**32**	64	128	256
6	16	32	32	32	**32**	64	**64**	128	256
7	16	32	64	64	64	**64**	128	**128**	256
8	**16**	32	64	128	128	128	**128**	256	**256**

- The table lists N for $\mathbb{R}(N)$
- $\mathbb{R}_{p,q}$ chooses generators from $\mathbb{R}_{P,Q}$ with **red** N

Complex representations of $\mathbb{R}_{p,q}$

(Cartan and Study 1908; Porteous 1995; Lounesto 1997)

p ↓ \ q →	0	1	2	3	4	5	6	7	8
0	1	**1**	2	4	4	**4**	8	16	16
1	2	2	**2**	4	8	8	**8**	16	32
2	2	4	4	**4**	8	16	16	**16**	32
3	**2**	4	8	8	**8**	16	32	32	**32**
4	4	**4**	8	16	16	**16**	32	64	64
5	8	8	**8**	16	32	32	**32**	64	128
6	8	16	16	**16**	32	64	64	**64**	128
7	**8**	16	32	32	**32**	64	128	128	**128**
8	16	**16**	32	64	64	**64**	128	256	256

- The table lists N for $\mathbb{C}(N)$
- $\mathbb{R}_{p,q}$ chooses generators from $\mathbb{R}_{P,Q}$ with **red** N

Neutral matrix representations

(Cartan and Study 1908; Porteous 1995; Lounesto 1997)

- *Representation map* ρ_m, *representation matrix* R_m

$$
\begin{array}{ccc}
\mathbb{R}_{m,m} & \xrightarrow{\text{coord}} & \mathbb{R}^{4^m} \\
\rho_m \downarrow & & \downarrow R_m \\
\mathbb{R}(2^m) & \xrightarrow[\text{reshape}]{} & \mathbb{R}^{4^m}
\end{array}
$$

Representations of generators

(Porteous 1995; Lounesto 1997)

Proposition. *For each finite* m, $\mathbb{R}_{m,m} \cong \mathbb{R}(2^m)$

Proof. By induction, giving explicit representations of generators. $\rho_0 := \emptyset$, For $m > 0$:

$$\rho_m\{\mathtt{-m}\} := \begin{bmatrix} 0 & -I_{m-1} \\ I_{m-1} & 0 \end{bmatrix} \quad \rho_m\{\mathtt{m}\} := \begin{bmatrix} 0 & I_{m-1} \\ I_{m-1} & 0 \end{bmatrix}$$

$$\rho_m\{\mathtt{k}\} := \begin{bmatrix} \rho_{m-1}\{\mathtt{k}\} & 0 \\ 0 & -\rho_{m-1}\{\mathtt{k}\} \end{bmatrix} \quad -m < k < m, k \neq 0$$

where I_{m-1} here is the unit of $\mathbb{R}(2^{m-1})$, for $m > 0$.

[]

Generator sets, frame group

(Braden 1985; Lam and Smith 1989; Porteous 1995; Bergdolt 1996)

- The generator set S_m for the matrix representation of $\mathbb{R}_{m,m}$ is as follows:

$$S_0 := \emptyset, \quad \text{For } m > 0:$$

$$S_m := \{\rho_m\{\mathtt{k}\} : -m < k < m, k \neq 0\}$$

- The generator set S_m generates a matrix representation of the real frame group: $\langle S_m \rangle \cong \mathbb{GR}_{m,m}$

Matrix inner product, norm

(Golub and van Loan 1996)

- We can treat $\mathbb{R}(N)$ as an inner product space.
- For $A, B \in \mathbb{R}(N)$,
  - The *normalised Frobenius inner product* is:

$$A \bullet B := \frac{1}{N} \sum A \star B \quad \text{where}$$

$$\left(\sum A \star B\right)_{j,k} := \sum_{j=1}^{N} \sum_{k=1}^{N} A_{j,k} B_{j,k}$$

  - The *normalised Frobenius norm* is:

$$||A||_{NF} := \sqrt{A \bullet A} = \sqrt{\frac{1}{N} \sum A \star A} = \frac{1}{\sqrt{N}} ||A||_F$$

where $||A||_F$ is the *Frobenius norm* of A.

Basis matrices are orthonormal

(Porteous 1995)

Lemma. *The basis matrices for* $\mathbb{R}_{m,m}$ *are orthonormal with respect to the normalised Frobenius inner product* $\bullet$*:*

$$A \bullet B = \begin{cases} 1 & A = B \\ 0 & B \neq B \end{cases} \qquad \forall A, B \in \langle S_m \rangle$$

Proof. (Sketch) Use induction on m. If $m = 0$ then $\langle S_m \rangle = \emptyset$, so conclusion is trivial. Inductive step uses generator sets and representation of generators. []

Corollary. *The real representation matrix* R_m *is* 2^m *times an orthogonal matrix.*

Inverse representation maps

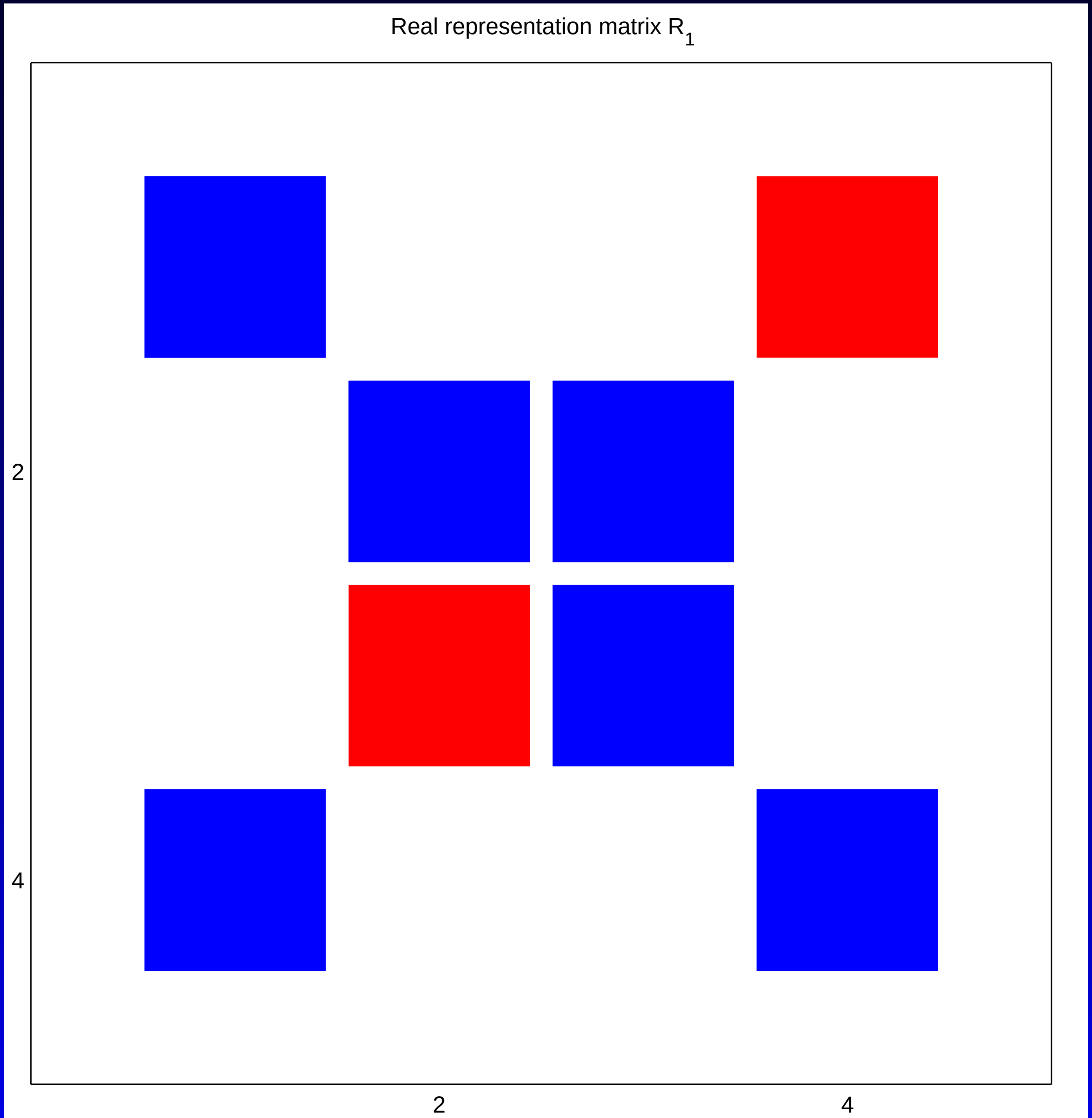
Corollary. *For* $a, b \in \mathbb{R}_{m,m}$

$$\rho a \bullet \rho v_S = a \bullet v_S, \quad \rho a \bullet \rho b = a \bullet b$$

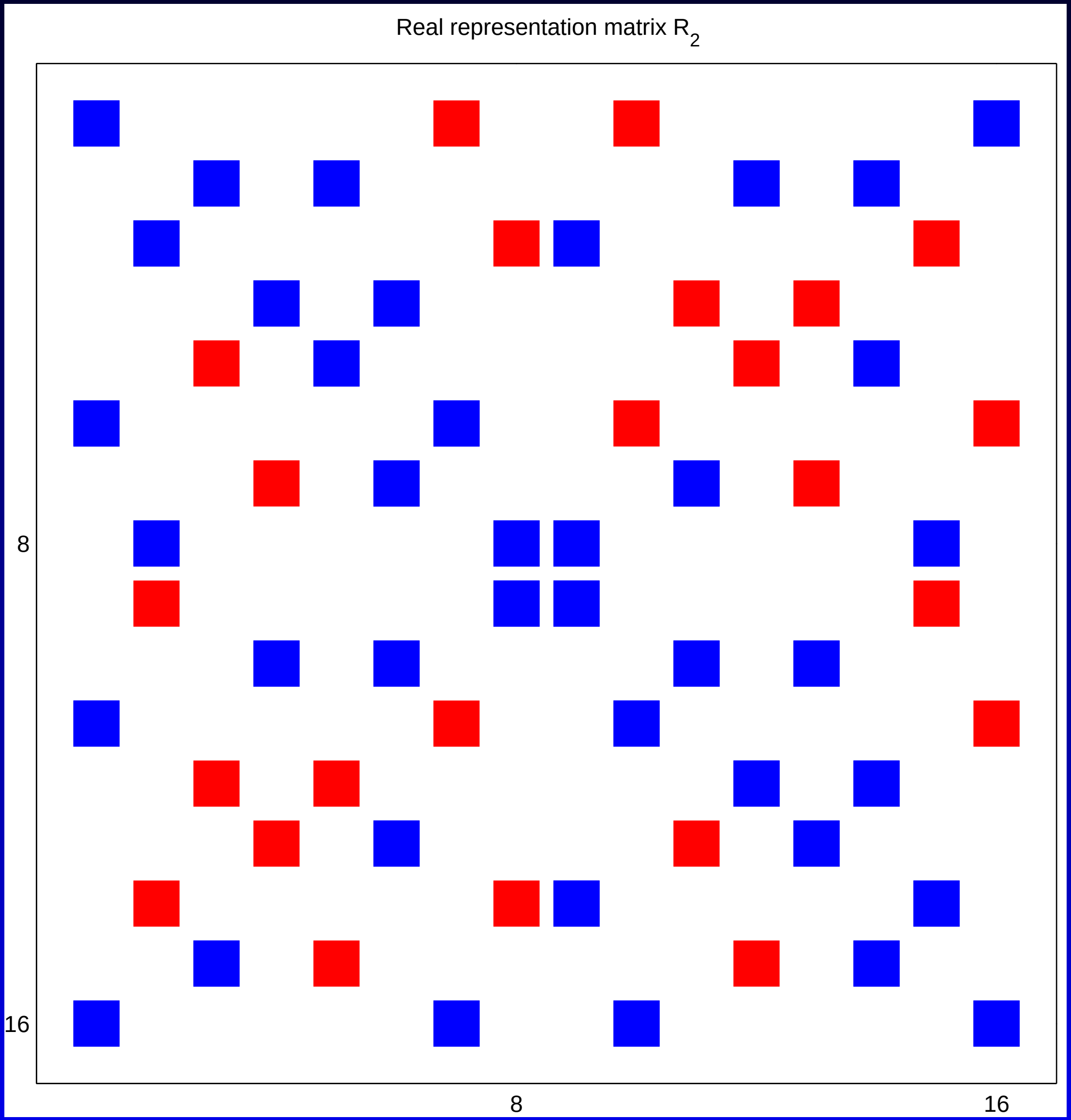
Proposition. *The normalized Frobenius inner product can be used to determine the coordinates* a_S *of the framed representation of* $a \in \mathbb{R}_{p,q}$ *from the real matrix representation* ρa *as follows:* $a_S = \rho a \bullet \rho v_S$

Proof. Use the corollary. Note that $\mathbb{R}_{p,q} \cong \mathbb{A} \subseteq \mathbb{R}_{m,m}$ for some $m \geqslant \max(p, q)$. The basis matrices of the real representation of $\mathbb{R}_{p,q}$ are $\pm$ the basis matrices for some $\mathbb{R}_{m,m}$, and are orthonormal with respect to $\bullet$. []

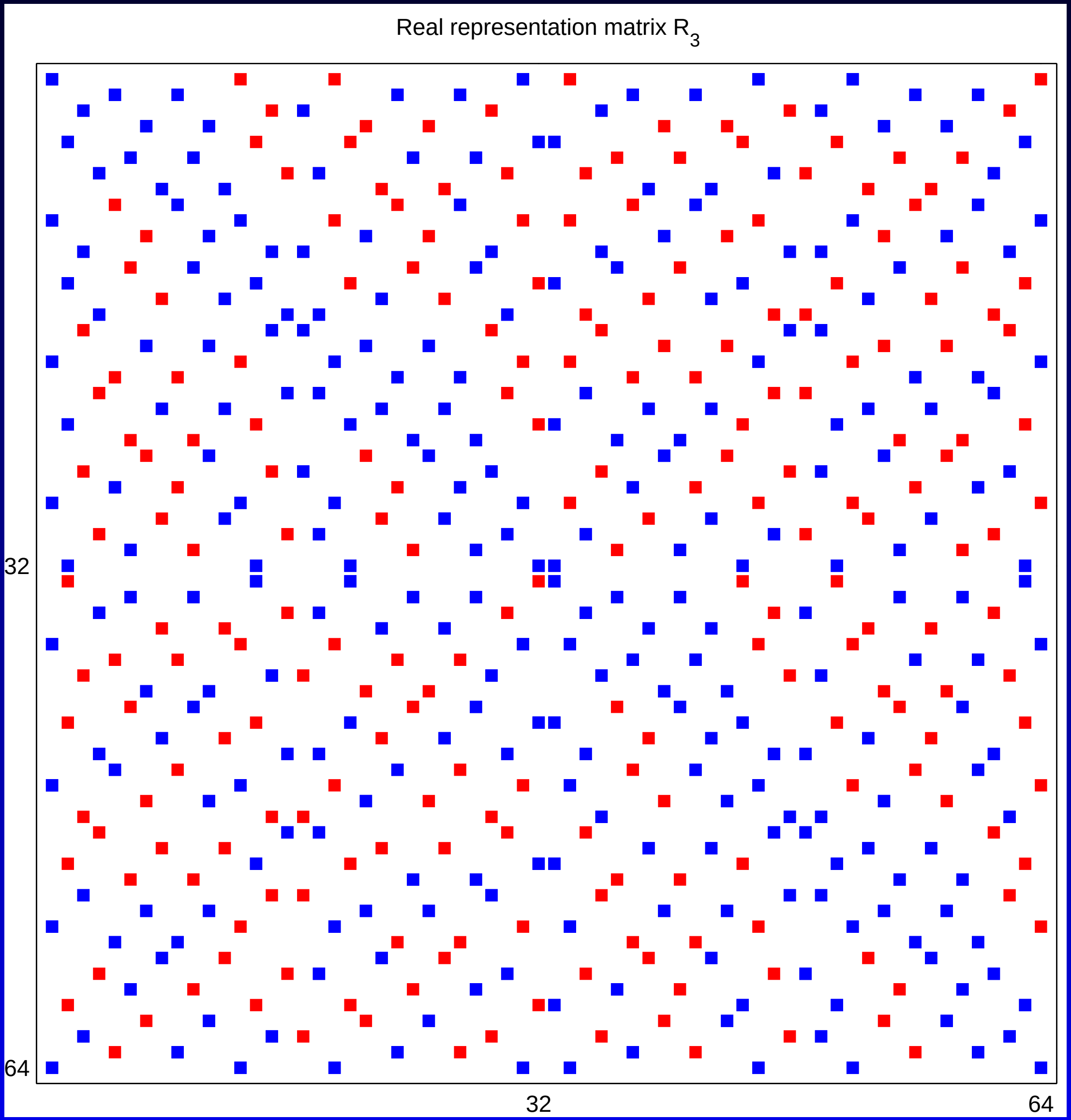
Real representation matrix R_1



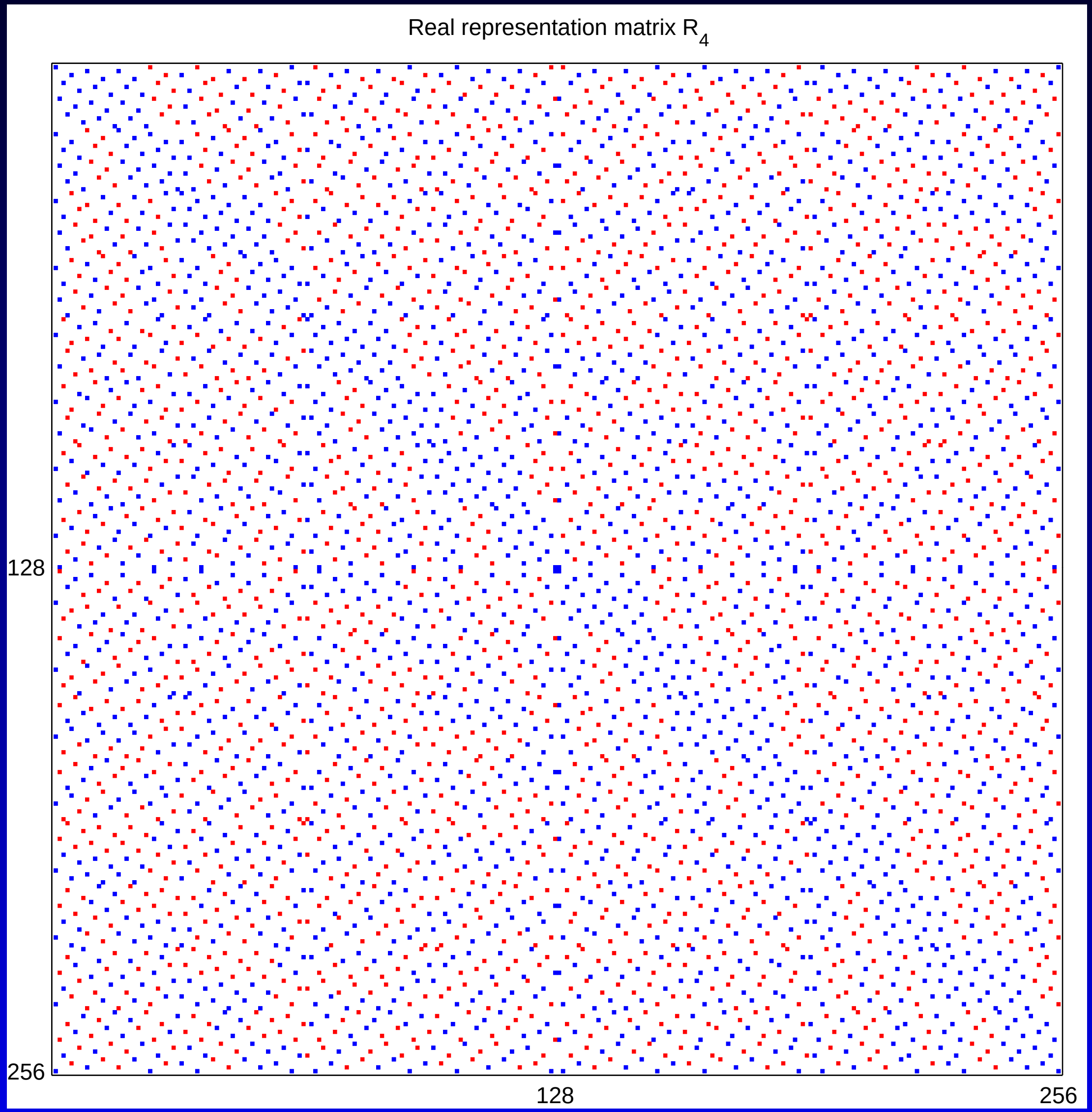
Real representation matrix R_2



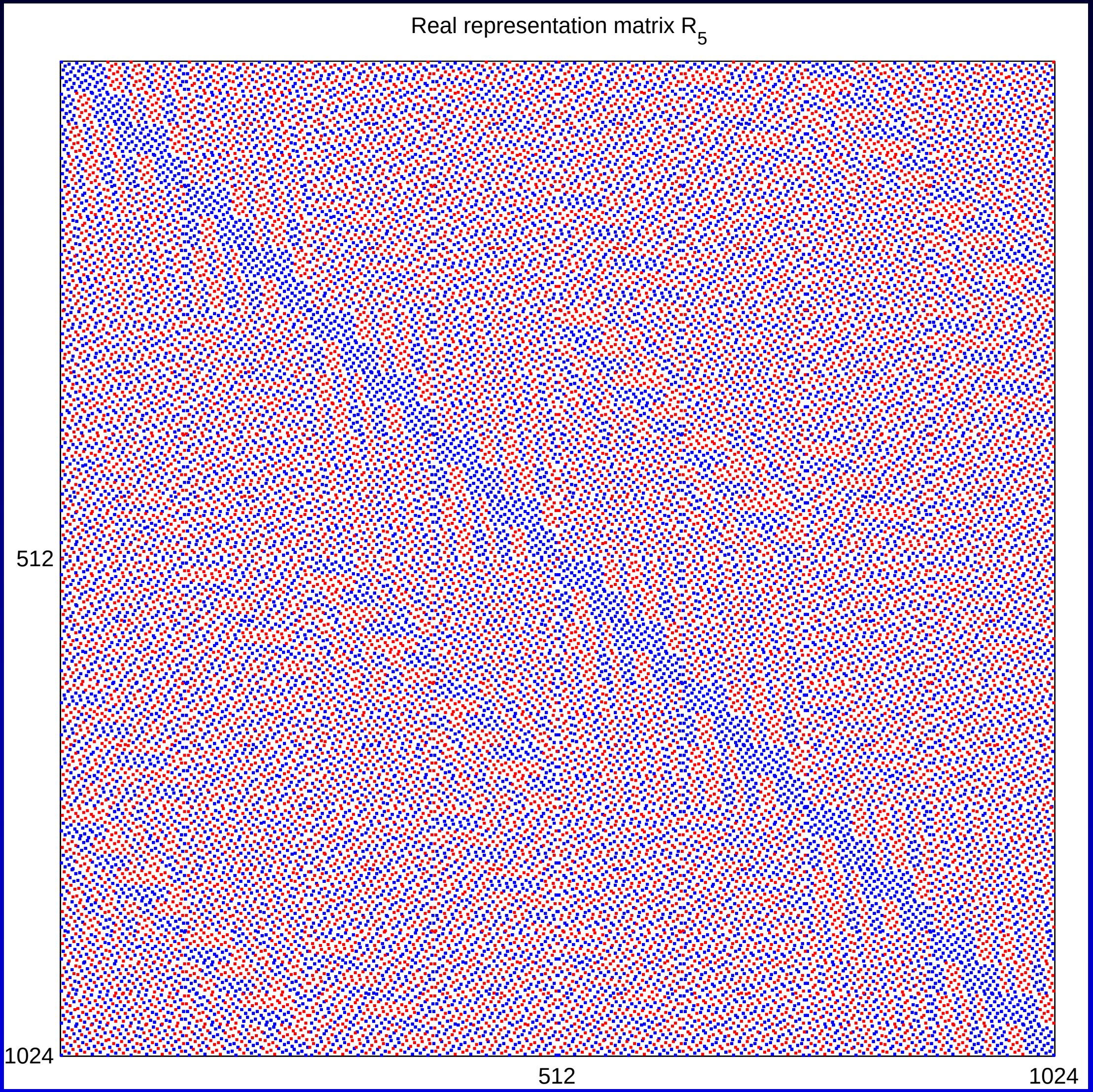
Real representation matrix R_3



Real representation matrix R_4

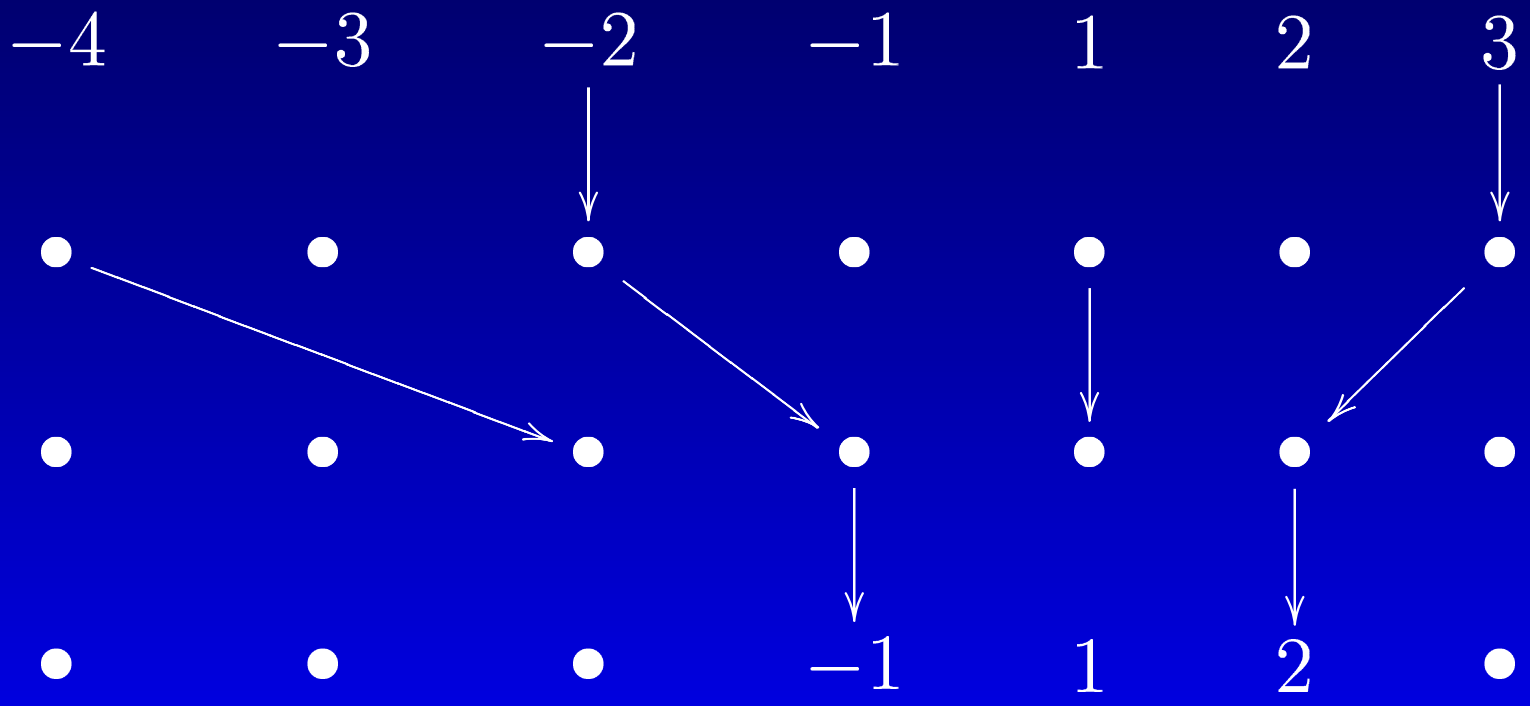


Real representation matrix R_5



Folding frames

- `fold` minimizes the size of matrices required
- eg.
  `{-2,3}.fold({-4,-2,1,3})=={-1,-2}`



Isomorphisms

$$\sharp T^{+} = \sharp S^{+}, \sharp T^{-} = \sharp S^{-} \Rightarrow \mathbb{R}_T \equiv \mathbb{R}_S$$

- *Proof.* The following diagram commutes

$$\begin{array}{ccccc}
T & \xrightarrow{\gamma_T} & R^T & \xrightarrow{\varphi_T} & R_{\sharp T^{+},\sharp T^{-}} \\
f \downarrow & & \nu_{TS} \downarrow & & \| \\
S & \xrightarrow[\gamma_S]{} & R^S & \xrightarrow[\varphi_S]{} & R_{\sharp S^{+},\sharp S^{-}}
\end{array}$$

The map ν_{TS} is a quadratic space isomorphism []

Division and inverses

(Golub and van Loan 1996, Higham 1996)

- Left and right division are defined via the inverse

$$B/A := BA^{-1} \quad A\backslash B := A^{-1}B$$

- The matrix representation can be used to compute the inverse. Not all multivectors have inverses.
- ***GluCat*** 0.0.6 implements the inverse as a special case of division
- Division uses LU decomposition followed by successive LU solves, with iterative refinement

`elliptic()` $= \mathfrak{i}$

(Braden 1985; Lam and Smith 1989; Porteous 1995; Bergdolt 1996)

- Define $\mathfrak{i}$ to be the unit volume element $\mathfrak{i} := vS$, of the *real–complex* algebra, $\mathbb{R}_S$, such that:

$$\mathfrak{i}^2 = -1, \mathfrak{i}x = x\mathfrak{i} \qquad \forall x \in \mathbb{R}_S$$

- ***GluCat*** 0.0.6 provides `elliptic()`, which returns $\mathfrak{i}$

Complex representations of $\mathbb{R}_{p,q}$

(Porteous 1995; Lounesto 1997)

p \ q →	0	1	2	3	4	5	6	7	8
↓ 0	1	**1**	2	4	4	**4**	8	16	16
1	2	2	**2**	4	8	8	**8**	16	32
2	2	4	4	**4**	8	16	16	**16**	32
3	**2**	4	8	8	**8**	16	32	32	**32**
4	4	**4**	8	16	16	**16**	32	64	64
5	8	8	**8**	16	32	32	**32**	64	128
6	8	16	16	**16**	32	64	64	**64**	128
7	**8**	16	32	32	**32**	64	128	128	**128**
8	16	**16**	32	64	64	**64**	128	256	256

- The table lists N for $\mathbb{C}(N)$
- $\mathbb{R}_{p,q}$ chooses generators from $\mathbb{R}_{P,Q}$ with **red** N

Functions of multivectors

(Golub and van Loan 1996; Rinehart 1955)

- Consider the real Clifford algebra $\mathbb{R}_{p,q}$ to be a subalgebra of the real–complex Clifford algebra $\mathbb{R}_{P,Q}$ isomorphic to some complex matrix algebra $\mathbb{C}(N)$. Try to define a function of $x \in \mathbb{R}_{p,q}$ in terms of the corresponding function in $\mathbb{C}$.

$$f(x) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \boldsymbol{\Omega}} f(z)(z - x)^{-1} dz$$

- Use norm equivalence between $\mathbb{R}_{p,q}$ and $\mathbb{C}(N)$

Functions of matrices

(Golub and van Loan 1996; Rinehart 1955)

- In the matrix case, we have:

$$f(X) := \frac{1}{2\pi i} \int_{\partial\Omega} f(z)(zI - X)^{-1} dz$$

with Ω an open subset of $\mathbb{C}$ with smooth boundary $\partial\Omega$ and

$$\sigma(X) \subset \Omega$$

where $\sigma(X)$ is the spectrum of X.

Taylor series of matrices

(Golub and van Loan 1996; Rinehart 1955)

- The *Taylor series* for $f(X)$ allows us to define transcendental functions of matrices.

  Theorem. *If* $f(z)$ *has a power series representation*

  $$f(z) = \sum_{k=0}^{\infty} c_k z^k \quad \textit{in an open disk}$$

  containing $\sigma(X)$, *then* $\quad f(X) = \sum_{k=0}^{\infty} c_k X^k$

Square root

(Golub and van Loan 1996; Gerald and Wheatley 1999; Cheng, Higham, Kenney and Laub 1999)

GluCat 0.0.6 uses the following algorithm:

- Scale by dividing by the norm
- If the real part of the scalar part is negative, negate the argument and multiply the result by `i`
- If `abs(val-1) < 1`, use diagonal Padé approximation
- Otherwise use product form of Denman-Beavers square root iteration

Transcendental functions

(Golub and van Loan 1996; Gerald and Wheatley 1999; Abramowicz and Stegun 1965)

- `exp()` uses the scaling and squaring Padé approximation.
- `cos`, `cosh`, `acos`, etc. are based on `exp` and `log`.
- Approximated using expressions which may involve the use of `i`.
- Advantage of using `i` is that scaling is simplified.
- Disadvantage is that it is possible to “poke out” of the subalgebra.

Cosine and sine

Proposition. *Within a real–complex Clifford algebra, using the real framed norm, the definitions of the functions* $\cos$ *and* $\sin$ *in terms of* $\exp$ *agree with their definitions as formal power series.*

Proof. For $\cos$:

$$\begin{aligned}\frac{\exp(\mathrm{i}x)+\exp(-\mathrm{i}x)}{2} &= \sum_{k=0}^{\infty}\frac{\mathrm{i}^k x^k}{2(k)!}+\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k x^k}{2(k)!}\\ &= \sum_{m=0}^{\infty}\frac{\mathrm{i}^{2m}x^{2m}}{(2m)!}\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!}\\ &= cos(x)\end{aligned}$$

[]

Logarithm

(Cheng, Higham, Kenney and Laub 1999)

GluCat 0.0.6 uses inverse scaling and squaring using the incomplete Denman–Beavers square root cascade and the diagonal Padé approximation

GluCat

(Lounesto et al. 1987; Lounesto 1992; Raja 1996; Bangerth et al.; Karmesin et al.; Siek et al.)

- Generic library of universal Clifford algebra templates
- C++ template library for use with other libraries such as *deal.II* and *POOMA*
- For details, see `http://glucat.sf.net` and the thesis, “Practical computation with Clifford algebras”