A partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ into regions of equal measure and small diameter

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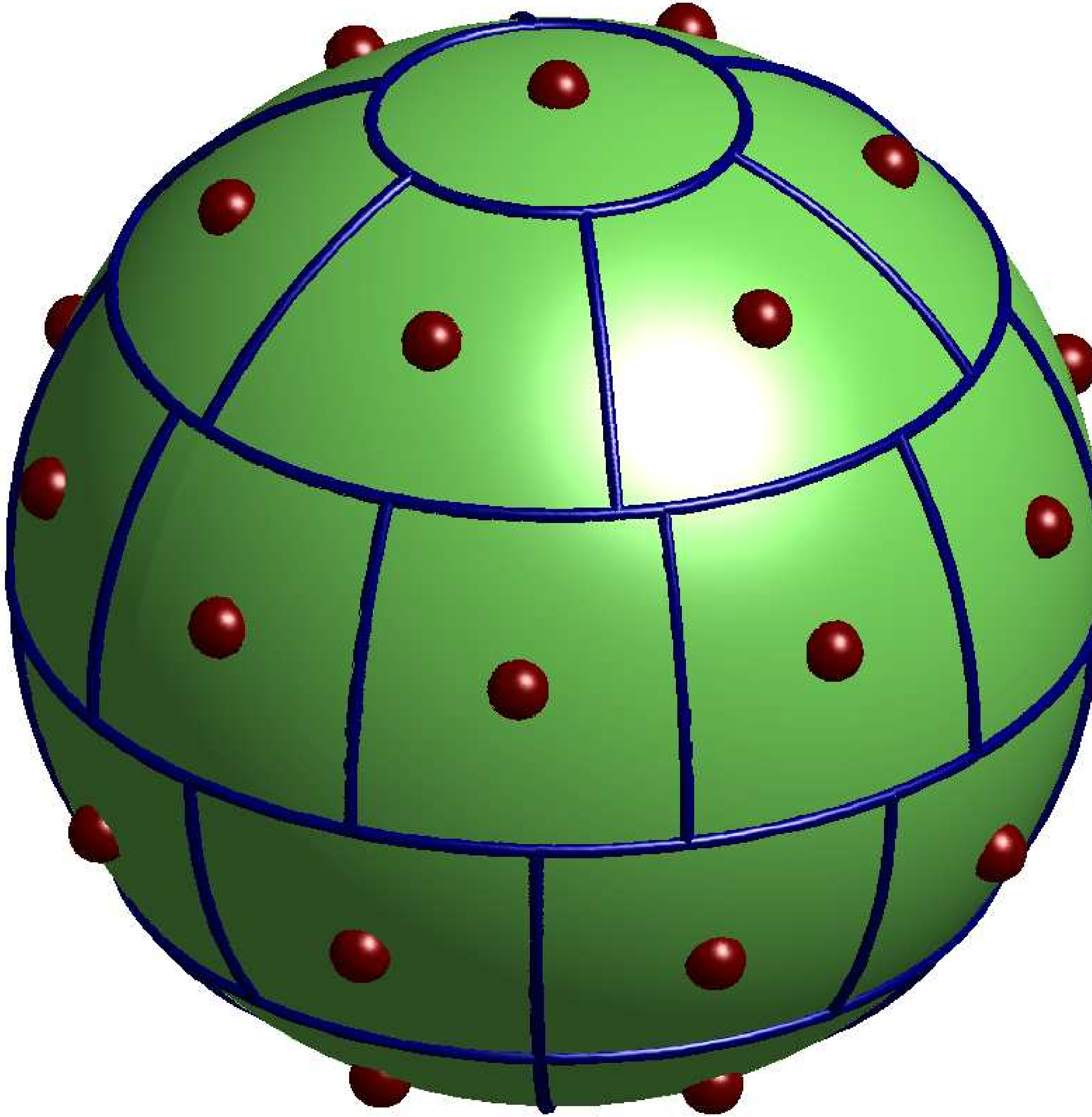
For presentation at Vanderbilt University, Nashville, November 2004.

RZ Partition of S^2
into 33 regions of equal area, showing the center point of each region.



The sphere $\mathbb{S}^d$

Definition 1. *For dimension* d, *the unit sphere* $\mathbb{S}^d$ *embedded in* $\mathbb{R}^{d+1}$ *is defined as*

$$\mathbb{S}^d := \left\{ x \in \mathbb{R}^{d+1} \,\middle|\, \sum_{k=1}^{d+1} x_k^2 = 1 \right\}.$$

Definition 2. Spherical polar coordinates *describe a point* p *of* $\mathbb{S}^d$ *using one longitude,* $p_1 \in [0, 2\pi]$, *and* $d-1$ *colatitudes,* $p_i \in [0, \pi]$, *for* $i \in \{2, \ldots, d\}$.

Equal-measure partitions

Definition 3. *Let* S *be a measurable set and* μ *a measure with* $0 < \mu(S) < \infty$.

An equal-measure partition *of* S *for* μ *is a nonempty finite set* P *of measurable subsets of* S, *such that for each* $Q, R \in P$ *with* $Q \neq R$,

$$\mu(Q) = \mu(R) = \mu(S)/|P|$$

and

$$\mu(Q \cap R) = 0.$$

Diameter bounded sets of partitions

Definition 4. *The* diameter *of a region* $\boldsymbol{R} \subset \mathbb{R}^{\boldsymbol{d}+\mathbf{1}}$ *is defined by*

$$\mathbf{diam}\,\boldsymbol{R} := \sup\{\boldsymbol{e}(\boldsymbol{x},\boldsymbol{y}) \mid \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{R}\},$$

where $\boldsymbol{e}(\boldsymbol{x},\boldsymbol{y})$ *is the* $\mathbb{R}^{\boldsymbol{d}+\mathbf{1}}$ *Euclidean distance* $\left\|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right\|$.

Definition 5. *A set* $\boldsymbol{Z}$ *of partitions of* $\boldsymbol{S} \in \mathbb{R}^{\boldsymbol{d}+\mathbf{1}}$ *is said to* have diameter bound $\boldsymbol{K} \in \mathbb{R}_{+}$ *if for all* $\boldsymbol{P} \in \boldsymbol{Z}$, *for each* $\boldsymbol{R} \in \boldsymbol{P}$, *for* $\boldsymbol{N} := |\boldsymbol{P}|$,

$$\mathbf{diam}\,\boldsymbol{R} \leqslant \boldsymbol{K}\boldsymbol{N}^{-\mathbf{1}/\boldsymbol{d}}.$$

$\boldsymbol{Z}$ *is said to be* diameter bounded *if there exists* $\boldsymbol{K} \in \mathbb{R}_{+}$ *such that* $\boldsymbol{Z}$ *has diameter bound* $\boldsymbol{K}$.

Key properties of the RZ partition of $\mathbb{S}^d$

The *recursive zonal* (*RZ*) partition of $\mathbb{S}^d$ into N regions is denoted as $RZ(d, N)$.
The set of partitions $RZ(d) := \{RZ(d, N) \mid N \in \mathbb{N}_+\}$.
The RZ partition satisfies the following theorems.

Theorem 1. *For dimension $d \geqslant 1$, let σ be the usual surface measure on $\mathbb{S}^d$ inherited from the Lebesgue measure on $\mathbb{R}$ via the usual embedding of $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$.*
Then for $N \geqslant 1$, $RZ(d, N)$ is an equal-measure partition for σ.

Theorem 2. *For $d \leqslant 8$, $RZ(d)$ is diameter-bounded in the sense of Definition 5.*

Precedents

The RZ partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Ed Saff, and on Ian Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Alexander (1972) uses the existence of a diameter-bounded set of equal-area partitions of $\mathbb{S}^2$ to analyse the maximum sum of distances between points. Alexander (1972) suggests a construction different from Zhou (1995).

Equal-area partitions of $\mathbb{S}^2$ used in the geosciences and astronomy do not have a proven bound on the diameter of regions.

Stolarsky's "Conjecture"

Stolasky (1973) asserts the existence of a diameter-bounded set of equal-measure partitions of $\mathbb{S}^d$ for all d, but offers no construction or existence proof.

Beck and Chen (1987) quotes Stolarsky. Bourgain and Lindenstrauss (1988) quotes Beck and Chen.

Wagner (1993) implies the existence of an RZ-like construction for $\mathbb{S}^d$. Bourgain and Lindenstrauss (1993) gives a partial construction.

Spherical zones, caps and collars

For $d > 1$, a *zone* can be described by

$$Z(a, b) := \{p \in \mathbb{S}^d \mid p_d \in [a, b]\},$$

where $0 \leqslant a < b \leqslant \pi$.

$Z(0, b)$ is a North polar cap and $Z(a, \pi)$ is a South polar cap.
If $0 < a < b < \pi$, $Z(a, b)$ is a *collar*.

For $d > 1$, the measure of a spherical cap of spherical radius θ is

$$V(\theta) := \sigma\big(Z(0, \theta)\big) = \omega \int_0^\theta (\sin \xi)^{d-1} d\xi,$$

where $\omega = \sigma(\mathbb{S}^{d-1})$.

Outline of the RZ algorithm

The RZ algorithm is recursive in dimension $\boldsymbol{d}$.

Algorithm for $\boldsymbol{RZ(d, N)}$:

if $\boldsymbol{N = 1}$ **then**

There is a single region which is the whole sphere;

else if $\boldsymbol{d = 1}$ **then**

Divide the circle into $\boldsymbol{N}$ equal segments;

else

Divide the sphere into zones,

each the same measure as an integer number of regions:

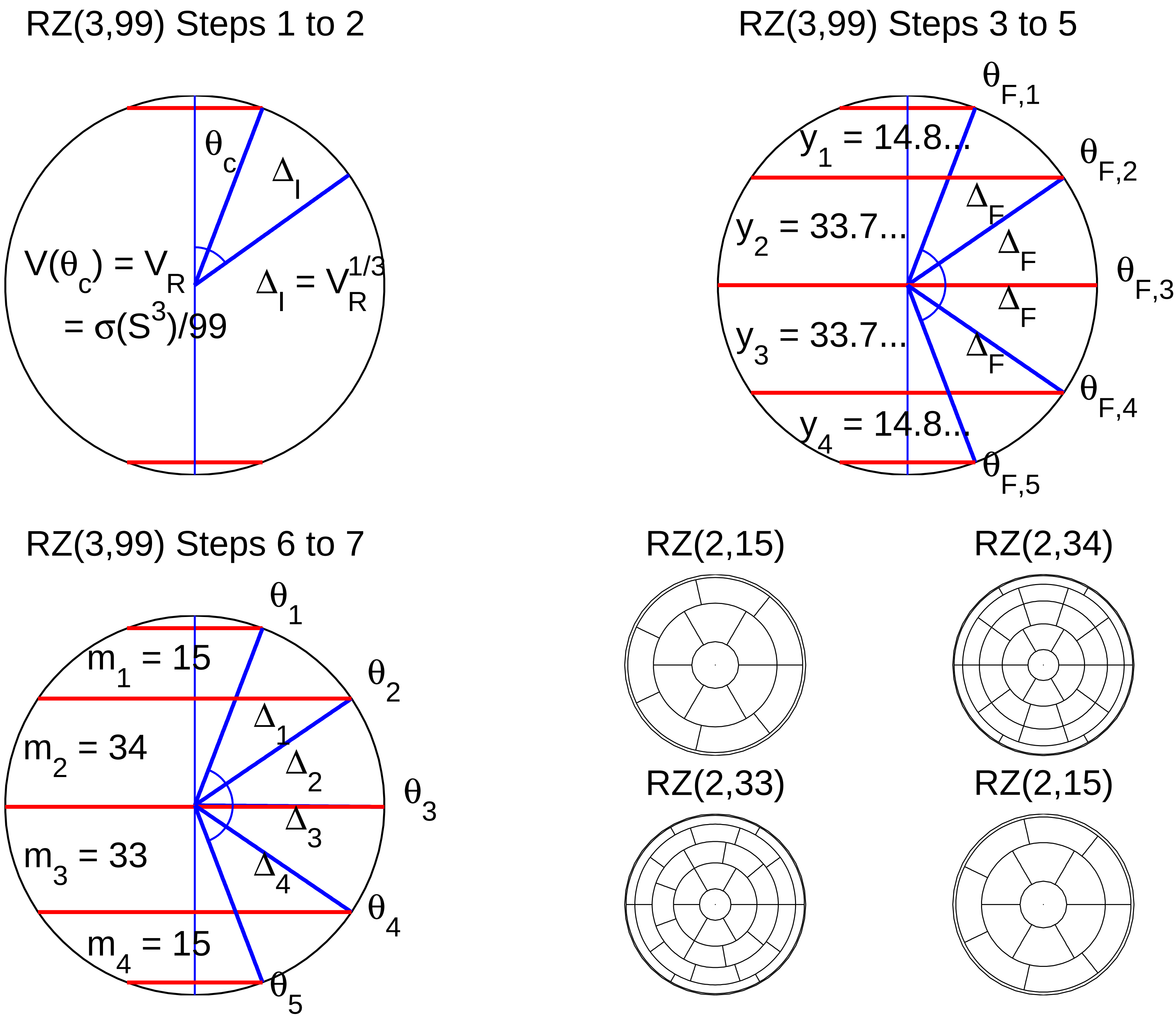
North and South polar spherical caps

and a number of spherical collars;

Partition each spherical collar into regions of equal measure,

using the RZ algorithm for dimension $\boldsymbol{d - 1}$;

endif.


RZ(3,99) Steps 1 to 2
θc
ΔI
V(θc) = VR
= σ(S3)/99
ΔI = VR1/3
RZ(3,99) Steps 3 to 5
θF,1
θF,2
θF,3
θF,4
θF,5
y1 = 14.8...
y2 = 33.7...
y3 = 33.7...
y4 = 14.8...
ΔF
ΔF
ΔF
ΔF
RZ(3,99) Steps 6 to 7
θ1
θ2
θ3
θ4
θ5
m1 = 15
m2 = 34
m3 = 33
m4 = 15
Δ1
Δ2
Δ3
Δ4
RZ(2,15)
RZ(2,34)
RZ(2,33)
RZ(2,15)

Rounding the number of regions per collar

Similarly to Zhou (1995), given the sequence y_i for n collars, with

$$\sum_{i=1}^{n} y_i = N - 2,$$

define the sequences a and m by:
$a_0 := 0$, and for $i \in \{1, \ldots, n\}$,

$$m_i := \mathrm{round}(y_i + a_{i-1}), \quad a_i := \sum_{j=1}^{i} (y_j - m_j).$$

Then m_i is the required number of regions in collar i, and we can show that $a_i \in [-1/2, 1/2)$ and $a_n = 0$.

Geometry of regions

Each region $\boldsymbol{R}$ in collar $\boldsymbol{i}$ of $\boldsymbol{RZ(d,N)}$ is of the form

$$\boldsymbol{R = R_{d-1} \times [\theta_i, \theta_{i+1}]},$$

in spherical polar coordinates, where
$\boldsymbol{R_{d-1} = [t_1, b_1] \times \ldots \times [t_{d-1}, b_{d-1}]}$, with $\boldsymbol{t, b \in \mathbb{S}^{d-1}}$.

We can show that

$$\textbf{diam}\, \boldsymbol{R} \leqslant \sqrt{\boldsymbol{\Delta_i^2 + w_i^2 (\textbf{diam}\, R_{d-1})^2}},$$

where $\boldsymbol{\Delta_i := \theta_{i+1} - \theta_i}$ and $\boldsymbol{w_i := \max_{\xi \in [\theta_i, \theta_{i+1}]} \sin \xi}$.

The inductive step

Assuming that $RZ(d-1)$ has diameter bound κ, define

$$P_i := w_i \, \kappa \, m_i^{\frac{-1}{d-1}}.$$

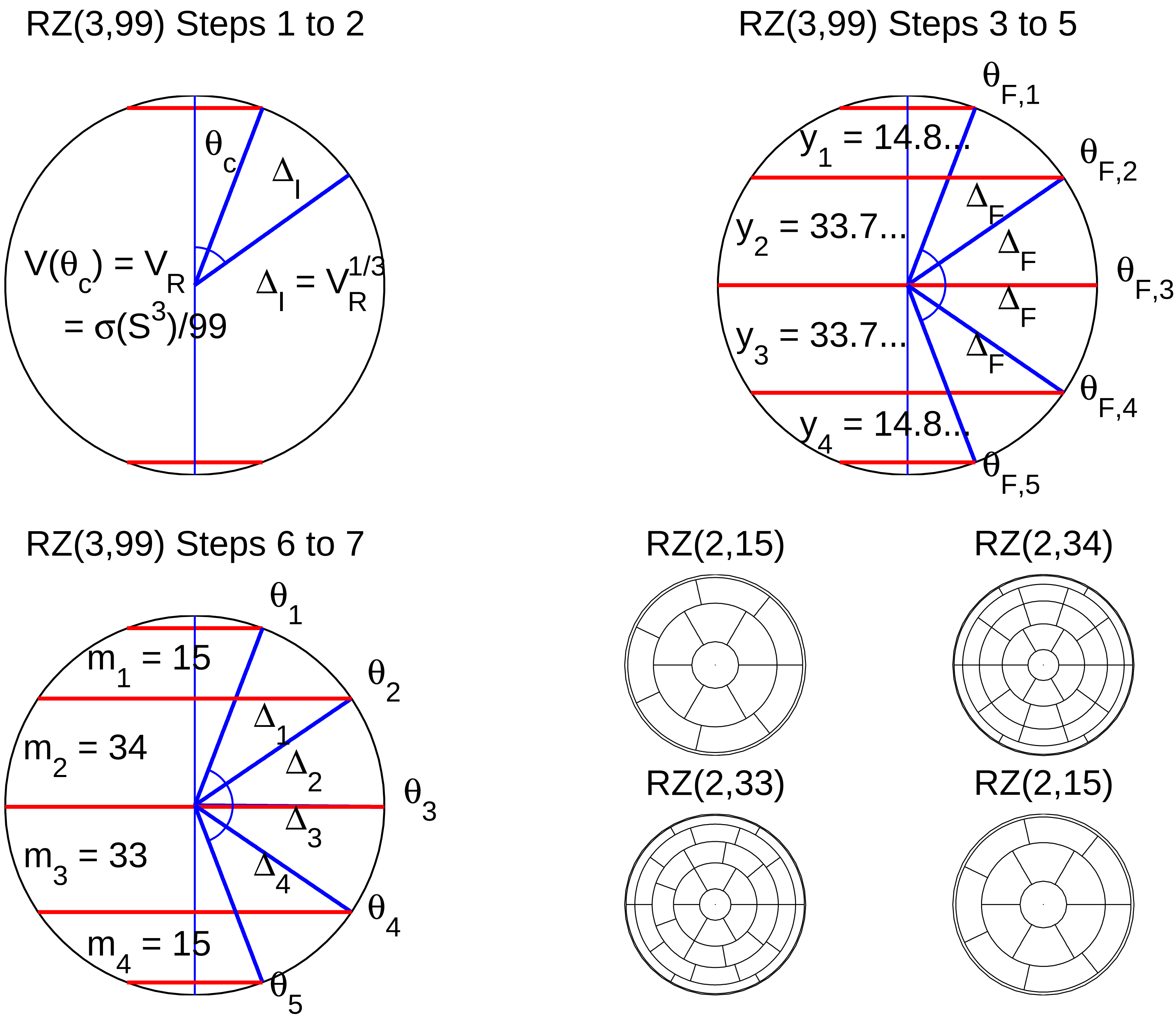
Then we can show that

$$\operatorname{diam} R \leqslant \sqrt{\left(\max_{i \in \{1,\ldots,n\}} \Delta_i\right)^2 + \left(\max_{i \in \{1,\ldots,n\}} P_i\right)^2}.$$

Continuous analogs

Define $\Theta := V^{-1}$,

$$
\begin{aligned}
Y(\theta) &:= \frac{V(\theta + \Delta_F) - V(\theta)}{V_R},\\
T(\tau, \theta) &:= \Theta\big(V(\theta) - \tau V_R\big),\\
B(\beta, \theta) &:= \Theta\big(V(\theta + \Delta_F) + \beta V_R\big),\\
M(\tau, \beta, \theta) &:= Y(\theta) + \tau + \beta,\\
\Delta(\tau, \beta, \theta) &:= B(\beta, \theta) - T(\tau, \theta),\\
W(\tau, \beta, \theta) &:= \max_{\xi \in [T(\tau,\theta), B(\beta,\theta)]} \sin \xi,\\
P(\tau, \beta, \theta) &:= \kappa W(\tau, \beta, \theta) M(\tau, \beta, \theta)^{\frac{-1}{d-1}}.
\end{aligned}
$$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RZ(2,33)
RZ(2,15)

Properties of continuous analogs

For each collar $i \in \{1, \ldots, n\}$, if we define $\theta_{F,i} := \theta_c + (i-1)\Delta_F$, then we can show that

$$
\begin{aligned}
Y(\theta_{F,i}) &= y_i, \\
T(-a_{i-1}, \theta_{F,i}) &= \theta_i, \\
B(a_i, \theta_{F,i}) &= \theta_{i+1}, \\
M(-a_{i-1}, a_i, \theta_{F,i}) &= m_i, \\
\Delta(-a_{i-1}, a_i, \theta_{F,i}) &= \Delta_i, \\
W(-a_{i-1}, a_i, \theta_{F,i}) &= w_i, \\
P(-a_{i-1}, a_i, \theta_{F,i}) &= P_i.
\end{aligned}
$$

Feasible domains

Define the *feasible domain* $\mathbb{D} := \mathbb{D}_t \cup \mathbb{D}_m \cup \mathbb{D}_b$, where

$$\begin{aligned}
\mathbb{D}_t &:= \{(0, \beta, \theta_c) \mid \beta \in [-1/2, 1/2]\}, \\
\mathbb{D}_m &:= \{(\tau, \beta, \theta) \mid \tau \in [-1/2, 1/2], \beta \in [-1/2, 1/2], \\
&\quad \theta \in [\theta_{F,2}, \pi - \theta_c - 2\Delta_F]\}, \\
\mathbb{D}_b &:= \{(\tau, 0, \pi - \theta_c - \Delta_F) \mid \tau \in [-1/2, 1/2]\}.
\end{aligned}$$

Assuming that $RZ(d-1)$ has diameter bound κ, then for $N > 2$, for R in collar i of $RZ(d, N)$, we can show

$$\operatorname{diam} R \leqslant \sqrt{\left(\max_{\mathbb{D}} \Delta\right)^2 + \left(\max_{\mathbb{D}} P\right)^2}.$$

Properties and estimates of V

- V is smooth on $[0, \pi]$ and is monotonic increasing in $(0, \pi)$.
- DV is positive and monotonic increasing in $(0, \pi/2)$.
- $DV(\theta) = DV(\pi - \theta)$.
- For $\theta, h \geqslant 0$ and $\theta + h \in [0, \pi/2]$,

$$V(\theta + h) - V(\theta) \in [hDV(\theta), hDV(\theta + h)].$$

- For $\theta \in (0, \pi/2)$, $V(\theta) \in \left[L_V\, \theta^d, H_V\, \theta^d\right]$, where

$$L_V := \frac{\omega}{d}\left(\frac{2}{\pi}\right)^{d-1} \quad \text{and} \quad H_V := \frac{\omega}{d}.$$

Cap, Δ, P bounds

We can use properties and estimates of V to show that:

- There is a constant $K_c > 0$ such that for $N > 1$, the diameter of each polar cap of $RZ(d, N)$ is bounded by $K_c N^{-1/d}$.
- For $1 < d \leqslant 8$, if $RZ(d-1)$ is diameter bounded, then there are constants $K_\Delta > 0, K_P > 0$, $N_\Delta, N_P \in \mathbb{N}$ such that for $RZ(d, N)$ with $N > \max(N_\Delta, N_P)$,

$$\max_{\mathbb{D}} \Delta \leqslant K_\Delta N^{-1/d},$$

$$\max_{\mathbb{D}} P \leqslant K_P N^{-1/d}.$$

Outline of proof of Theorem 2

Assume that $N > 2$ and $d > 1$.
Define $N_H := \max(N_\Delta, N_P)$.

Then if $d \leqslant 8$, if $RZ(d-1)$ has diameter bound κ, and if $N > N_H$, we have $\operatorname{maxdiam}(d, N) \leqslant K_H N^{-1/d}$, where $K_H := \max\left(K_c, \sqrt{K_\Delta^2 + K_P^2}\right)$.

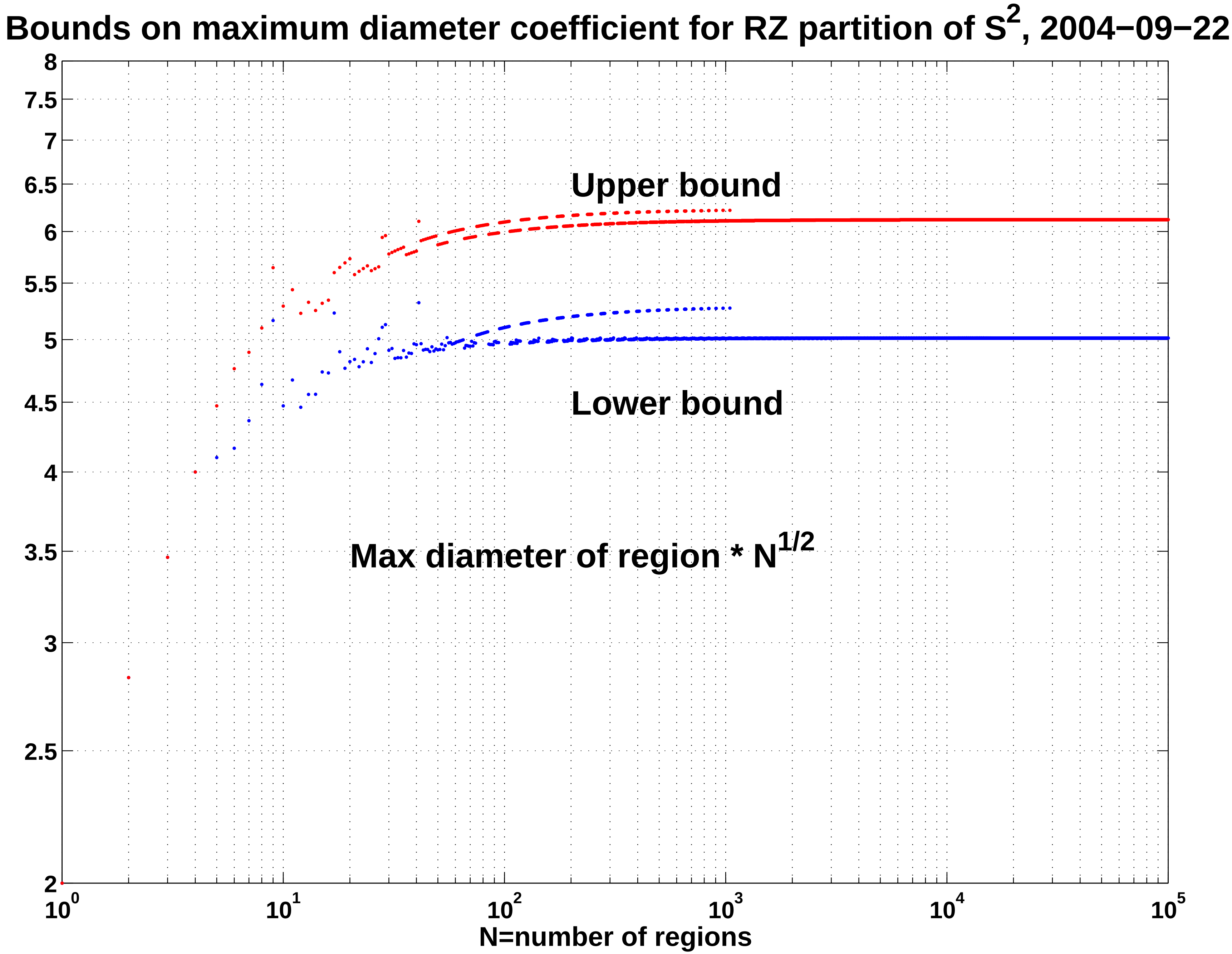
The diameter of any region is bounded by 2.
Therefore for $N \leqslant N_H$, $\operatorname{maxdiam}(d, N) \leqslant K_L N^{-1/d}$, where $K_L := 2N_H^{1/d}$.

$RZ(1, N)$ consists of N equal segments, so $RZ(1)$ has diameter bound 2π. The result follows by induction.

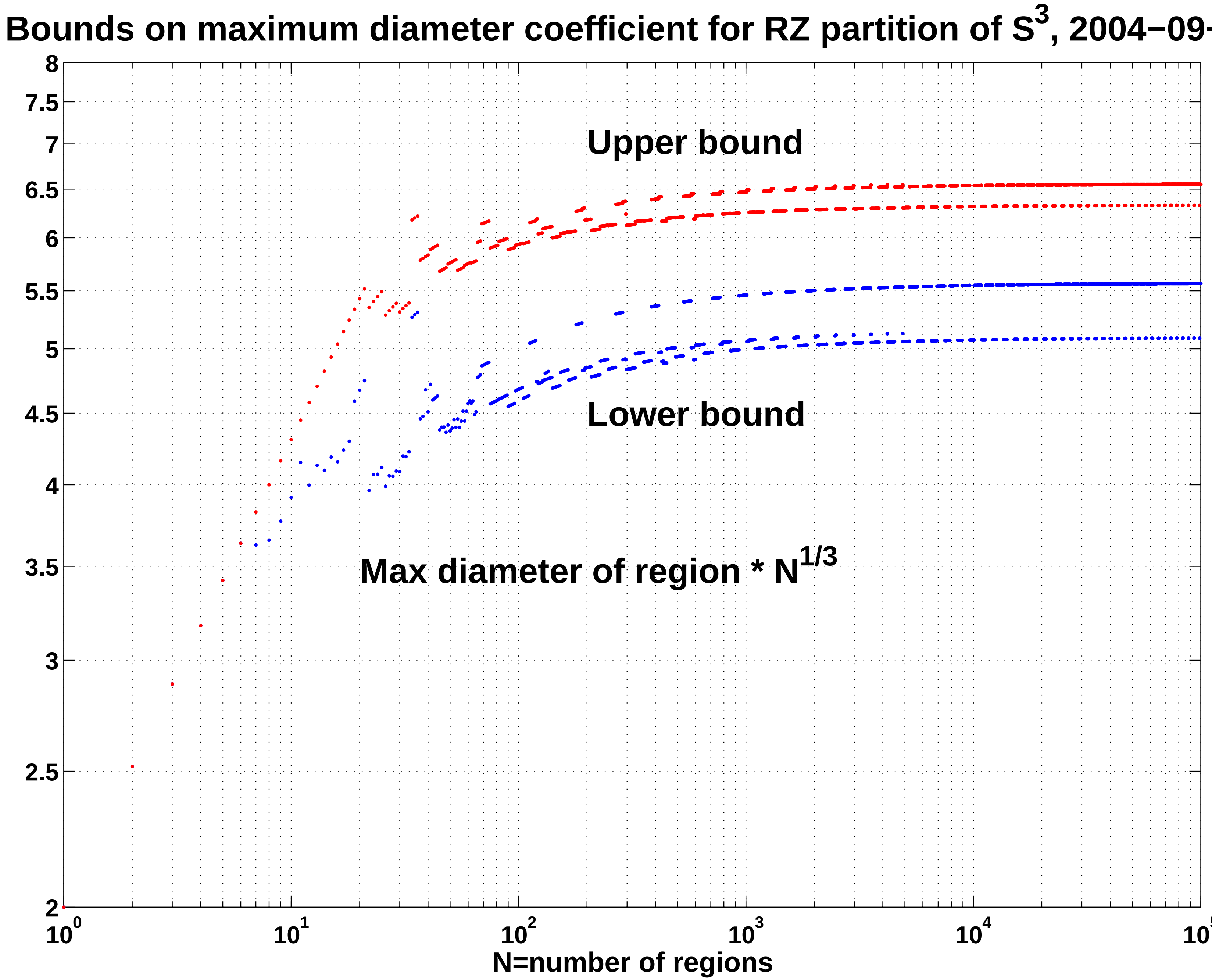
Numerical results - constants

d	K_d
2	18.4
3	59.9
4	205
5	724
6	2.63×10^3
7	9.76×10^3
8	3.57×10^4

Zhou obtains $K_2 \leqslant 7$ for his (1995) algorithm.



Bounds on maximum diameter coefficient for RZ partition of S2, 2004−09−22
Upper bound
Lower bound
Max diameter of region * N1/2
N=number of regions
8
7.5
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
100
101
102
103
104
105


Bounds on maximum diameter coefficient for RZ partition of S^3, 2004−09−22
Upper bound
Lower bound
Max diameter of region * $N^{1/3}$
N=number of regions
8
7.5
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
10^0
10^1
10^2
10^3
10^4
10^5

Stereographic projection of $\mathbb{S}^3$ to $\mathbb{R}^3$

In Cartesian coordinates, the stereographic projection $\mathbb{R}^4 \to \mathbb{R}^3 \cup \infty$ is

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)/(1 - x_4), \quad \text{if} \quad x_4 \neq 1,$$
$$(x_1, x_2, x_3, 1) \mapsto \infty.$$

When restricted to $\mathbb{S}^3$,

- The north pole projects to ∞.
- The south polar cap projects to a ball.
- Collars project to differences between balls.
- Spheres project to generalized spheres.

Illustration of RZ partition of $\mathbb{S}^3$