Projective parabolic geometries

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1. Introduction

What is special about the following manifolds?

- $\mathbb{R}P^n$, real projective space
- $\mathbb{C}P^m$, complex projective space
- $\mathbb{H}P^\ell$, quaternionic projective space
- $\mathbb{O}P^2$, the octonionic (Cayley) plane
- S^n, the conformal sphere

As riemannian manifolds, they are the rank 1 compact symmetric spaces, but the isometry group is a maximal compact subgroup of a larger symmetry group... making these spaces into generalized flag manifolds G/P, with G semisimple and P parabolic.

They are thus examples of symmetric R-spaces, but how are they characterized as R-spaces (generalized flag manifolds)? What is special about their curved analogues (parabolic geometries)?

1. Real projective structures

Let M be a (real) n-manifold. A (real) *affine connection* is a connection D on TM (e.g., $D = \nabla^g$ for a riemannian metric g).

Affine connections D and $\tilde{D}$ on TM are *projectively equivalent* iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^r \in C^\infty(M, \mathfrak{gl}(TM)),$$

where $$[\![X, \gamma]\!]^r(Y) := \tfrac{1}{2}\big(\gamma(X)Y + \gamma(Y)X\big).$$

We write $\tilde{D} = D + \gamma$ for short (instead of $\tilde{D} = D + [\![\cdot, \gamma]\!]^r$).

A *projective structure* on M^n ($n > 1$) is a projective class $\Pi^r = [D]$ of affine connections. Connections in Π^r have the same torsion, oftened assumed zero.

Basic questions.

1. When does there exist a metric connection in Π^r?
   Seek $g \in C^\infty(M, S^2T^*M)$ and $D \in \Pi^r$ with $Dg = 0$.
2. Is g unique?
   Seek geodesically equivalent metrics g_1 and g_2.

1. The real metrization equation

Key observation. The problem linearizes for the section $h = \mathrm{vol}_g^{2/(n+1)} \otimes g^{-1}$ of $\mathcal{L}^* \otimes S^2 TM$, where $\mathcal{L}^{\otimes(n+1)} = \wedge^n TM$.

Lemma. For $X \in TM, \gamma \in T^*M$: $[\![X, \gamma]\!] \cdot h = X \odot h(\gamma, \cdot)$.

Consequence. According to the decomposition

$$T^*M \otimes S^2 TM = \big(id \odot TM\big) \oplus \big(T^*M \otimes_0 S^2 TM\big),$$

tensored with $\mathcal{L}^*$, the second component $(Dh)_0$ of Dh is projectively invariant (independent of $D \in \Pi^r$).

Suppose that $(Dh)_0 = 0$. Then there is a section Z^D of $\mathcal{L}^* \otimes TM$ such that for all $X \in TM$,

$$D_X h = X \odot Z^D.$$

Conclude. If h is a nondegenerate solution, and $\nabla = D - h^{-1}(Z^D) \in \Pi^r$, then $\nabla h = 0$.

1. Complex projective structures

Let (M, J) be an almost complex manifold of real dimension $n = 2m$. A *complex affine connection* is a connection D on TM with $DJ = 0$ (e.g., $D = \nabla^g$ for a hermitian metric g).

Complex affine connections D and $\tilde{D}$ are *c-projectively equivalent* iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^c \in C^\infty(M, \mathfrak{gl}(TM, J)),$$
$$[\![X, \gamma]\!]^c(Y) := \tfrac{1}{2}\big(\gamma(X)Y + \gamma(Y)X - \gamma(JX)JY - \gamma(JY)JX\big).$$

We write $\tilde{D} = D + \gamma$ for short.

A *c-projective structure* on M^{2m} ($m > 1$) is an c-projective class $\Pi^c = [D]$ of complex affine connections. Connections in Π^c have the same torsion, often assumed type $(0, 2)$—and then given by the Nijenhuis tensor of J.

1. The hermitian metrization equation

Basic questions. When is there a connection $D \in \Pi^c$ preserving a hermitian metric g? If so, when is g unique?

Require $g(JX, JY) = g(X, Y)$ for all X, Y, and $Dg = 0$. If D is torsion-free then g is a Kähler metric.

Key fact. As in the real case, the problem linearizes for the section $h = \mathrm{vol}_g^{1/(m+1)} \otimes g^{-1}$ of $\mathcal{L}^* \otimes S^2 TM$, where $\mathcal{L}^{m+1} = \wedge^{2m} TM$.

The c-projectively invariant equation is:

$$D_X h = X \odot Z^D + JX \odot JZ^D.$$

Since h is hermitian, $\varphi := h(J\cdot, \cdot)$ is skew, i.e., a section of $\mathcal{L}^* \otimes \wedge^2 TM$. The equivalent equation on φ is

$$D_X \varphi = X \wedge K^D + JX \wedge JK^D$$

where $K^D = JZ^D$. Solutions are called *hamiltonian 2-vectors*.

1. Quaternionic structures

Let (M, Q) be a quaternionic manifold of real dimension $n = 4\ell$ (thus $Q \subset \mathfrak{gl}(TM)$, with fibres isomorphic to $\mathfrak{sp}(1)$, spanned by imaginary quaternions J_1, J_2, J_3). A *quaternionic affine connection* is a connection on TM preserving Q (e.g., $D = \nabla^g$ for a quaternion Kähler metric g on M).

Fact. For any two quaternionic connections D and $\tilde{D}$ with the same torsion, $\exists\, \gamma \in \Omega^1(M)$ with

$$\begin{aligned} \tilde{D}_X - D_X &= [\![X, \gamma]\!]^q \in C^\infty(M, \mathfrak{gl}(TM, Q)), \\ [\![X, \gamma]\!]^q(Y) &:= \tfrac{1}{2}\Big(\gamma(X)Y + \gamma(Y)X \\ &\qquad\qquad - \sum_i \big(\gamma(J_iX)J_iY + \gamma(J_iY)J_iX\big)\Big). \end{aligned}$$

An equivalence class of quaternionic connections may be denoted $\Pi^q = [D]$. There is a canonically determined class whose torsion is the intrinsic torsion of the quaternionic structure.

1. Quaternion Kähler metrics

On (M, Q), compatible (quaternion Kähler) metrics are again given by a linear equation, for Q-hermitian sections h of $\mathcal{L}^* \otimes S^2 TM$ (with $\mathcal{L}^{\ell+1} = \wedge^{4\ell} TM$):

$$D_X h = X \odot Z^D + \sum_i J_i X \odot J_i Z^D$$

for some (hence any) $D \in \Pi^q$ (with Z^D a trace of Dh).

Aside. In the torsion-free case, this has an interpretation in terms of the twistor space Z of M, which is the unit sphere bundle in Q.

- Z is a complex $2\ell + 1$ manifold with real structure, containing real "twistor lines" (rational curves $\mathbb{C}P^1 \subset Z$ with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2\ell}$): M is the space of such twistor lines.
- Under Penrose transform, h corresponds to a holomorphic section π of $\wedge^2 TZ \otimes K_Z^{1/(\ell+1)}$. The standard twistor theory of quaternionic Kähler metrics g uses the section θ of $T^*Z \otimes K_Z^{-1/(\ell+1)}$ dual to $\pi^{\wedge \ell}$, and the inverse of π on $\ker \theta$. If g is hyperkähler, this defines a symplectic foliation of Z over $\mathbb{C}P^1$; if not, θ is a contact structure on Z.

2. A common framework: abelian parabolic geometries

Real, complex and quaternionic projective geometries all have a natural metrization equation, suggesting a common framework. In each case, the geometry is specified by two pieces of data:

- A principal P^0-subbundle F^0 of the frame bundle $GL(M)$ (where P^0 is $GL(n, \mathbb{R})$, $GL(m, \mathbb{C})$ or $GL(\ell, \mathbb{H}) \cdot \mathrm{Sp}(1)$).
- An equivalence class Π of connections on F^0, which form an affine space modelled on $\Omega^1(M)$, using an algebraic bracket $[\![\cdot, \cdot]\!]\colon TM \times T^*M \to \mathfrak{p}^0_M := F^0 \times_{P^0} \mathfrak{p}^0 \subseteq \mathfrak{gl}(TM)$.

These are characteristic features of *abelian parabolic geometries*, i.e., Cartan geometries modelled on generalized flag varieties (or R-spaces) in cominuscule (or $|1|$-graded) representations.

The model spaces for the projective geometries are the projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^m$ and $\mathbb{H}P^\ell$. Another abelian parabolic geometry is conformal geometry, modelled on the conformal n-sphere S^n.

Goal. Develop metrization theory for a class of abelian parabolic geometries: "projective parabolic geometries".

2. Parabolic subalgebras and generalized flag manifolds

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie subalgebra whose Killing perp $\mathfrak{p}^\perp$ is a nilpotent subalgebra of $\mathfrak{p}$.

Consequences: $\mathfrak{p}^\perp$ is the nilpotent radical (or nilpotency ideal) of $\mathfrak{p}$, and $\mathfrak{p}^0 := \mathfrak{p}/\mathfrak{p}^\perp$ is a reductive Lie algebra, with $\mathfrak{p} \cong \mathfrak{p}^0 \ltimes \mathfrak{p}^\perp$. Also, the Killing form of $\mathfrak{g}$ induces a duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}^\perp$.

A *generalized flag manifold* (or *R-space*) X is an adjoint orbit of parabolic subalgebras. Since parabolic subalgebras are self-normalizing, the stabilizer of a point $\mathfrak{p} \in X$ is the Lie subgroup P of G with Lie algebra $\mathfrak{p}$. Thus $X \cong G/P$.

Generalized flag manifolds arise as highest weight orbits in projectivized representations $P(V)$ of G. The parabolic subalgebras $\mathfrak{p}$ are the infinitesimal stabilizers of highest weight spaces, and their nilpotent radicals $\mathfrak{p}^\perp$ are abelian iff V is cominuscule.

2. Classification of cominuscule representations

Irreducible cominuscule representations V of semisimple Lie algebras $\mathfrak{g}$ are real forms of the following:

$A_{n,k}$ $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ acting on $V = \bigwedge^k \mathbb{C}^{n+1}$;

BD_n $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ acting on $V = \mathbb{C}^n$;

D_n $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ acting on a half-spin representation V;

C_n $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ acting on $V = \bigwedge_0^n \mathbb{C}^{2n}$;

E_6 a 27-dimensional irreducible reresentation V of $\mathfrak{g} = \mathfrak{e}_6(\mathbb{C})$;

E_7 a 56-dimensional irreducible reresentation V of $\mathfrak{g} = \mathfrak{e}_7(\mathbb{C})$.

The corresponding generalized flag manifolds are: (A) grassmannians, (BD) conformal quadrics, (D) maximal isotropic grassmannians, (C) lagrangian grassmannians, and (E) two exceptional manifolds of dimensions 16 and 27.

2. Cartan geometries

A Cartan geometry on M, modelled on G/P, where $\dim M = \dim G/P$, is a principal G-bundle $F_G \to M$ with

- a principal G-connection and
- a reduction $F_P \subseteq F_G$ of structure group to $P \subseteq G$

satisfying the *Cartan condition*: the induced 1-form on M with values in the associated bundle $\mathfrak{g}_M/\mathfrak{p}_M := F_P \times_P \mathfrak{g}/\mathfrak{p}$ is a bundle isomorphism. Thus M inherits the first order geometry of G/P.

In the parabolic case, the duality between $\mathfrak{p}^\perp$ and $\mathfrak{g}/\mathfrak{p}$ implies that T^*M is isomorphic to the associated bundle $\mathfrak{p}_M^\perp$ of nilpotent ideals in $\mathfrak{p}_M$. Hence $\mathfrak{p}_M \cong \mathfrak{p}_M^0 \ltimes T^*M$, where $\mathfrak{p}_M^0 \subset \mathfrak{gl}(TM)$.

If also $\mathfrak{p}^\perp$ is abelian, then there is an algebraic bracket

$$[\![,]\!]\colon TM \times T^*M \to \mathfrak{p}_M^0 \subseteq \mathfrak{gl}(TM)$$

induced by the Lie bracket on $\mathfrak{g}_M$ (the “adjoint tractor bundle”).

2. The bundle of Weyl structures

The principal G-connection on F_G induces a covariant derivative on any bundle associated to a G-representation V. In particular, with $V = \mathfrak{g}$, there is a connection on $\mathfrak{g}_M$.

A *Weyl structure* is a Lie algebra splitting $\mathfrak{g}_M \cong TM \oplus \mathfrak{p}^0_M \oplus T^*M$. It induces a connection on TM by restriction and projection.

Fact. Weyl structures are sections of an affine bundle $\mathcal{W}$ modelled on T^*M. This is an algebraic fact about filtered isomorphisms of $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{p} \oplus \mathfrak{p}^0 \oplus \mathfrak{p}^\perp$.

Weyl structures thus form an affine space Π modelled on $\Omega^1(M)$.

In terms of connections, we can describe $\mathcal{W}$ using jets. An affine connection may be viewed as a splitting of the 1-jet sequence

$$0 \to T^*M \otimes TM \to J^1(TM) \to TM \to 0.$$

It is thus a section of an affine bundle modelled on $\mathrm{Hom}(TM, T^*M \otimes TM)$. $\mathcal{W}$ is a affine subbundle of this bundle of splittings, modelled on T^*M.

2. Computing with Weyl connections

A function Φ on Π is an *invariant* if it is constant, i.e., $\forall\, D \in \Pi$, $\gamma \in \Omega^1(M)$, $\delta_\gamma \Phi(D) := \frac{d}{dt}\Phi(D + t\gamma)|_{t=0}$ is zero.

For a section s of a vector bundle E associated to the frame bundle, $\delta_\gamma D_X s = [\![X, \gamma]\!] \cdot s$ (the natural action of $\mathfrak{p}^0_M$ on E).

Variation of the second derivative:

$$\delta_\gamma D^2_{X,Y} s = [\![X, \gamma]\!] \cdot D_Y s + [\![Y, \gamma]\!] \cdot D_X s - D_{[\![X,\gamma]\!] \cdot Y} s + [\![Y, D_X \gamma]\!] \cdot s.$$

Hence the curvature $R^D \in \Omega^2(M, \mathfrak{p}^0_M)$ of D, given by $D^2_{X,Y} s - D^2_{Y,X} s = R^D_{X,Y} \cdot s$, satisfies

$$\delta_\gamma R^D_{X,Y} = -[\![Id \wedge D\gamma]\!]_{X,Y} := -[\![X, D_Y \gamma]\!] + [\![Y, D_X \gamma]\!].$$

Can write: $R^D = W + [\![Id \wedge r^D]\!]$, where W is invariant ($\delta_\gamma W = 0$), and the *normalized Ricci tensor* $r^D \in \Omega^1(M, T^*M)$ satisfies $\delta_\gamma r^D = -D\gamma$.

2. Recovering the Cartan connection

Abelian parabolic geometries are thus equipped with the data we see in projective geometries:

- A principal P^0-subbundle F^0 of the frame bundle $GL(M)$ (the fibre $GL(M)_x$ consists of linear isomorphisms $\mathfrak{g}/\mathfrak{p} \to T_xM$).
- An affine bundle $\mathcal{W}$ of Weyl structures, modelled on T^*M, whose sections form an affine space Π of connections on F^0.

Conversely, these data suffice to construct a normal Cartan connection on the pullback F_P of $\mathcal{W}$ to F^0, which is an affine bundle modelled on $\mathfrak{p}^\perp$, hence also a principal $\exp \mathfrak{p}^\perp$-bundle over F^0. As a bundle over M, F_P is then a principal P-bundle.
The Cartan connection θ on F_P is then the sum of three pieces.

- The solder form of F^0 pulls back to $\theta_1 \in \Omega^1(F_P, \mathfrak{g}/\mathfrak{p})$.
- The tautological Weyl structure defines $\theta_0 \in \Omega^1(F_P, \mathfrak{p}^0)$.
- The normalized Ricci tensor r^D is a 1-form on $\mathcal{W}$ with values in T^*M, and so pulls back to $\theta_{-1} \in \Omega^1(F_P, \mathfrak{p}^\perp)$.

2. Linear representations of the Cartan connection

Let M be a parabolic geometry modelled on G/P and let V be a representation of G. The $\mathfrak{p}^\perp$ action induces a filtration

$$V \supset \mathfrak{p}^\perp V \supset \cdots \supset (\mathfrak{p}^\perp)^k V \supset \cdots \supset V^0 \supset \{0\},$$

where the *socle* V^0 is the kernel of the $\mathfrak{p}^\perp$ action. The *top* of the filtration is the quotient $H_0(V) := V/\mathfrak{p}^\perp V$. Note $V^0 \cong H_0(V^*)^*$.

Similarly, $V_M := F_P \times_P V$ is a filtered bundle. In particular, V_M has a natural subbundle V_M^0 and quotient $H_0(V_M)$. A Weyl structure splits this filtration.

Since $V_M \cong F_G \times_G V$, it has an induced connection. This gives rise to an invariant linear differential equation of finite type on $C^\infty(H_0(V_M))$. Solutions correspond to parallel sections of V_M with respect to a "prolongation connection", which agrees with the induced connection modulo curvature terms.

For $M \cong G/P$, $V_M \cong M \times V$ is trivialized by parallel sections. However, V_M^0 is not constant in this trivialization. In particular, if V_M^0 is a line bundle, then $x \mapsto (V_M^0)_x$ maps M into $P(V)$.

3. Projective parabolic geometries and prolongation

We wish to identify projective parabolic geometries as a subclass of abelian parabolic geometries.

Working principle. Projective parabolic geometries should have a "nice" metrization problem: compatible metrics are nondegenerate solutions of an invariant linear differential equation of finite type.

Consider again the projective case, and the equation $(Dh)_0 = 0$, i.e., $Dh = X \odot Z^D$.
Question: what equation is satisfied by Z^D?
Differentiate and skew symmetrize (this is called "prolongation"):

$$R^D_{X,Y} \cdot h = X \odot D_Y Z^D - Y \odot D_X Z^D.$$

This determines DZ^D modulo its trace, i.e., we may introduce a section λ^D of $\mathcal{L}^*$ with $DZ^D = \lambda^D id +$ curvature terms.
A further prolongation determines $D\lambda^D$ completely.

3. The prolongation connection and parallel sections

Conclude. Compatible metrics correspond to parallel sections (h, Z^D, λ^D) of $W_M := (\mathcal{L}^* \otimes S^2 TM) \oplus (\mathcal{L}^* \otimes TM) \oplus \mathcal{L}^*$ for the prolongation connection

$$\mathcal{D}_X \begin{bmatrix} h \\ Z^D \\ \lambda^D \end{bmatrix} = \begin{bmatrix} D_X h - X \odot Z^D \\ D_X Z^D - \lambda^D X + h(r_X^D) + \cdots \\ D_X \lambda^D + r_X^D(Z_D) + \cdots \end{bmatrix}$$

where the dots denote projective curvature terms. W_M has rank $\frac{1}{2}n(n+1) + n + 1 = \frac{1}{2}(n+1)(n+2)$, which provides an upper bound on the space of parallel sections.

In the projectively flat case, we have a linear representation of the (flat) Cartan connection. On $\mathbb{R}P^n$, the solution space is $W := S^2\mathbb{R}^{n+1}$, i.e., the compatible metrics on $\mathbb{R}P^n$ are those induced by nondegenerate inner products on $\mathbb{R}^{n+1}$.

The embedding of $\mathbb{R}P^n$ in $P(W)$ induced by $\mathcal{L}^* \subset \mathbb{R}P^n \times W$ is not minimal: it is the Veronese embedding $[x] \mapsto [x \otimes x]$.

3. Representations and compatible metrics

Suppose $\mathfrak{p}$ is a parabolic in $\mathfrak{g}$ with $\mathfrak{p}^\perp$ abelian, and choose an splitting of $\mathfrak{g}$ as a graded Lie algebra $\mathfrak{g}/\mathfrak{p} \oplus \mathfrak{p}^0 \oplus \mathfrak{p}^\perp$.

Suppose W is a representation of $\mathfrak{g}$ with graded decomposition

$$W = (L^* \otimes B) \oplus (L^* \otimes \mathfrak{g}/\mathfrak{p}) \oplus L^*$$

for representations L and B of $\mathfrak{p}^0$, with $\dim L = 1$.

On a parabolic geometry M of this type, W induces a bundle W_M, which splits using a Weyl structure as

$$W_M = (\mathcal{L}^* \otimes \mathcal{B}) \oplus (\mathcal{L}^* \otimes TM) \oplus \mathcal{L}^*.$$

Defining $h(\alpha, \beta) = \alpha \cdot (\beta \cdot h)$, $\mathcal{B}$ maps to a subbundle of $S^2 TM$. Also $\mathcal{L}^* \otimes TM$ is a subbundle of $T^*M \otimes \mathcal{L}^* \otimes \mathcal{B}$.

Proposition. The equation $(Dh)_0 = 0$, i.e., $D_X h = X \cdot Z^D$ for a section Z^D of $\mathcal{L}^* \otimes TM$, is invariant (i.e., independent of $D \in \Pi$).

Proof. Since $X \cdot h = 0$, $[\![X, \gamma]\!] \cdot h = X \cdot (\gamma \cdot h)$.

3. Characterizing projective parabolic geometries

Idea. An abelian parabolic geometry, modelled on G/P, is a projective parabolic geometry if G has a representation W of the previous form. *This is quite restrictive.*

- The 1-dimensional summand $L^* \subset W$ implies that G/P is highest weight orbit in $P(W)$.
- The natural filtration of W has height 3.

The irreducible W are real forms of the following:

- $G = SL_n(\mathbb{C})$ acting on $W = S^2\mathbb{C}^n$;
- $G = SL_p(\mathbb{C}) \times SL_q(\mathbb{C})$ acting on $W = \mathbb{C}^p \otimes \mathbb{C}^q$;
- $G = SL_{2\ell}(\mathbb{C})$ acting on $W = \wedge^2\mathbb{C}^{2\ell}$;
- $G = SO_{n+1}(\mathbb{C})$ acting on $W = \mathbb{C}^{n+1}$;
- $G = SO_{10}(\mathbb{C})$ acting on $W = \mathbb{C}^{16}$ (half-spin representation);
- $G = E_6(\mathbb{C})$ acting on $W = \mathbb{C}^{27}$.

These are the isotropy representations of symmetric R-spaces!

3. Symmetric R-spaces revisited

Proposition. Suppose W is a representation of $\mathfrak{g}$ with $\wedge^2 W$ irreducible. Let $\mathfrak{q}^0 = \mathfrak{g} \oplus \mathbb{C}$ be a 1-dimensional central extension. Then the weight of the centre can be chosen so that $\mathfrak{q} := \mathfrak{q}^0 \ltimes W$ is a parabolic subalgebra with abelian nilpotent radical W in $\mathfrak{h} \cong W^* \oplus \mathfrak{q}^0 \oplus W$.

Equivalently (W, W^*) is a Jordan pair. Thus if we choose $y \in W$, $x \cdot z = [x, [y, z]]$ makes W^* into a Jordan algebra.
If $F = (\mathrm{ad}\, y)^2 \colon W^* \to W$ is invertible, then $F^{-1}(y)$ is an identity element for the multiplication.

Proposition. W admits elements with $(\mathrm{ad}\, y)^2$ invertible if and only if H/Q is a *self-dual* symmetric R-space, i.e., $\mathfrak{h}$ admits an inner involution interchanging W and W^*.
(Or equivalently, an $\mathfrak{sl}_2$ subalgebra containing a grading element.)

3. Definition via selfdual symmetric R-spaces

Defn. A *projective parabolic geometry* is a Cartan geometry with model the generalized flag manifold G/P in the projective isotropy representation $P(W)$ of a selfdual symmetric R-space H/Q.

Equivalently, the model is a space of primitive idempotents in a semisimple Jordan algebra (W, y) (with $y \in W$ generic).

Thus $\mathfrak{h} \cong W \oplus (\mathfrak{g} \oplus \mathbb{R}) \oplus W^*$ provides M with a distinguished bundle W_M, which is a subbundle of a Lie algebra bundle $\mathfrak{h}_M$ containing $\mathfrak{g}_M$, and splits as $(\mathcal{L}^* \otimes \mathcal{B}) \oplus (\mathcal{L}^* \otimes TM) \oplus \mathcal{L}^*$.

The classification of selfdual symmetric R-spaces is well known. They are real forms of the following.

Type	$\mathfrak{h}$	$\mathfrak{q}^0$	W
C	$\mathfrak{sp}_{2n+2}(\mathbb{C})$	$\mathfrak{gl}_{n+1}(\mathbb{C})$	$S^2\mathbb{C}^{n+1}$
A	$\mathfrak{sl}_{2n+2}(\mathbb{C})$	$\mathfrak{s}(\mathfrak{gl}_{n+1}(\mathbb{C}) \oplus \mathfrak{gl}_{n+1}(\mathbb{C}))$	$\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$
D	$\mathfrak{so}_{4n+4}(\mathbb{C})$	$\mathfrak{gl}_{2n+2}(\mathbb{C})$	$\wedge^2\mathbb{C}^{2n+2}$
BD	$\mathfrak{so}_{n+4}(\mathbb{C})$	$\mathfrak{co}_{n+2}(\mathbb{C})$	$\mathbb{C}^{n+2}$
E	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{ce}_6(\mathbb{C})$	$\mathbb{C}^{27}$

3. Projective parabolic geometries of riemannian type

Projective parabolic geometries of *riemannian type* are those where the model admits compatible positive definite metrics. Such models depend on two parameters r, d (with $\dim M = n = rd$) and the Jordan algebra $W \cong J_{d+1}(\mathbb{R}^r)$ is "formally real".

Type	$\mathfrak{h}$	$G = (Q^0)^{ss}$	Jordan alg.	G/P	r, d
C	$\mathfrak{sp}_{2n+2}(\mathbb{R})$	$\mathrm{PGL}_{n+1}(\mathbb{R})$	$J_{n+1}(\mathbb{R})$	$\mathbb{R}P^n$	$1, n$
A	$\mathfrak{su}(m+1, m+1)$	$\mathrm{PGL}_{m+1}(\mathbb{C})$	$J_{m+1}(\mathbb{C})$	$\mathbb{C}P^m$	$2, m$
D	$\mathfrak{so}_{2\ell+2}(\mathbb{H})$	$\mathrm{PGL}_{\ell+1}(\mathbb{H})$	$J_{\ell+1}(\mathbb{H})$	$\mathbb{H}P^\ell$	$4, \ell$
BD	$\mathfrak{so}(n+2, 2)$	$\mathrm{SO}(n+1, 1)$	$J_2(\mathbb{R}^n)$	S^n	$n, 1$
E	$\mathfrak{e}_{7(-25)}$	$\mathrm{E}_{6(-26)}$	$J_3(\mathbb{O})$	$\mathbb{O}\mathrm{P}^2$	$8, 2$

Note that W_M has rank $\frac{1}{2}(d+1)(n+2)$, and $\mathcal{L}^{(n+r)/2} = \wedge^n TM$. Also, n-dimensional conformal geometry appears as "projective geometry of dimension 1 over $\mathbb{R}^n$", with model $S^n = \mathbb{R}^n P^1$.

3. The projective hessian equation

The linear differential equation on sections h of the quotient $\mathcal{L}^* \otimes \mathcal{B}$ of W_M is the *projective metric equation*.

The dual bundle $W_M^* \cong \mathcal{L} \oplus (\mathcal{L} \otimes T^*M) \oplus (\mathcal{L} \otimes \mathcal{B}^*)$ has natural quotient $\mathcal{L}$, on which the induced equation is second order.

Proposition. On sections ℓ of $\mathcal{L}$, the equation

$$D^2\ell + [\![r^D, \ell]\!] \in \mathcal{L} \otimes \mathcal{B}^*$$

is projectively invariant (independent of $D \in \Pi$).

Proof. $\delta_\gamma(D^2\ell + [\![r^D, \ell]\!])(X, Y) =$
$\Big[\!\!\Big[[\![X, \gamma]\!], [\![Y, D\ell]\!] \Big]\!\!\Big] + \Big[\!\!\Big[[\![Y, \gamma]\!], [\![X, D\ell]\!] \Big]\!\!\Big] - \Big[\!\!\Big[[\![[\![X, \gamma]\!], Y]\!], D\ell \Big]\!\!\Big]$. By two applications of the Jacobi identity, this is $-\Big[\!\!\Big[X, \Big[\!\!\Big[Y, [\![D\ell, \gamma]\!] \Big]\!\!\Big] \Big]\!\!\Big]$; thus $\delta_\gamma(D^2\ell + [\![r^D, \ell]\!]) = [\![\gamma, D\ell]\!]$, which is a section of $\mathcal{L} \otimes \mathcal{B}^*$.

Remarks. In the real case ($r = 1$), $\mathcal{B}^* = S^2T^*M$: this equation is vacuous and there is an invariant third order equation on sections of $\mathcal{L}$.

In the complex case ($r = 2$), solutions have J-invariant hessian and are hamiltonians for Killing fields of compatible Kähler metrics.

When $d = 1$, r^D is not well-defined: the choice of a hessian equation is a "Möbius structure", and is part of the data defining a c-projective curve.

3. Projective Killing tensors

Any generalized flag variety G/P in $P(W)$ is an intersection of quadrics, determined by the complement of the Cartan square $\odot^2 W^*$ in $S^2 W^*$ (homogeneous quadratic polynomials on W).

For projective parabolic geometries, this complement is an irreducible representation $U \subseteq S^2 W^*$, with quotient $\mathcal{L}^2 \otimes \mathcal{B}^*$.

Proposition. For sections k of $\mathcal{L}^2 \otimes \mathcal{B}^*$, the equation

$$(D_X k)(Y, Z) + (D_Y k)(Z, X) + (D_Z k)(X, Y) = 0$$

is projectively invariant (independent of $D \in \Pi$).

Proof. Just check that the symmetrization of $([\![X, \gamma]\!] \cdot k)(Y, Z)$ vanishes.

Solutions will be called *projective Killing* 2*-tensors*. They give quadratic integrals of the geodesic flow with respect to any $D \in \Pi$: if $D_X X = 0$ and then $D_X(k(X, X)) = (D_X k)(X, X) = 0$.

3. Metrics and integrability

For a section h of $\mathcal{L}^* \otimes B$, its volume form vol_h is a section of $\mathcal{L}^{-n/2} \otimes \bigwedge^n TM \cong \mathcal{L}^{r/2}$. As an $\mathcal{L}^*$-valued metric on T^*M, h is nondegenerate iff vol_h is nonvanishing, in which case there is an inverse metric h^{-1} on TM, which is a section of $\mathcal{L} \otimes B^*$.

Let $h^* = \mathrm{vol}_h^{2/r}\, h^{-1}$, which is a section of $\mathcal{L}^2 \otimes B^*$.

Proposition. h^* is a polynomial in h of degree $d-1$.

Proof. It is easy to check that h^* has homogeneity $d-1$. In fact it is essentially the "adjugate" of "cofactor matrix" of h.

Proposition. Let h be a solution of the projective metric equation. Then h^* is a projective Killing 2-tensor.

[This reflects the well-known fact that the geodesic flow of a metric is hamiltonian, and the energy, given by the metric, is conserved.]

Proof. If h is nondegenerate, we can choose $D \in \Pi$ with $Dh = 0$. Then $Dh^* = 0$, so h^* is a projective Killing 2-tensor. In general we approximate h by nondegenerate pointwise solutions.

3. Mobility and pencils of compatible metrics

Definition. The *mobility* of a projective parabolic geometry is the dimension of the span of the set of nondegenerate solutions to the projective metric equation.

A *pencil of compatible metrics* is a 2-dimensional linear space $\mathbb{V}$ of solutions of the projective metric equation.

Thus the mobility is at most $\dim W = \frac{1}{2}(d+1)(n+2)$. It is ≥ 1 if there is a compatible metric, and ≥ 2 if there is a pencil of compatible metrics.

The "pencil" here is the projective line $\mathrm{P}(\mathbb{V})$ of solutions (in $\mathbb{V}$) up to constant rescaling: in these terms, the 2-dimensional family of solutions may be viewed as a section $\mathbf{h}$ of $\mathcal{O}(1) \otimes (\mathcal{L}^* \otimes \mathcal{B})$ over $\mathrm{P}(\mathbb{V}) \times M$—instead of $\mathbb{V}^* \otimes (\mathcal{L}^* \otimes \mathcal{B})$ over M.

Such a pencil $\mathbf{h}$ yields a degree $d-1$ polynomial of projective Killing 2-tensors $\mathbf{h}^*$ (a section of $\mathcal{O}(d-1) \otimes \mathcal{L}^2 \otimes \mathcal{B}^*$ over $\mathrm{P}(\mathbb{V}) \times M$—or equivalently $S^{d-1}\mathbb{V}^* \otimes \mathcal{L}^2 \otimes \mathcal{B}$ over M).

These determine a family of quadratic integrals of the geodesic flow for any compatible metric.

3. Pencils of compatible metrics: classification

A further contraction of $\mathbf{h}^*$ with $\mathbf{h}$ yields a degree d polynomial of volume forms. For torsion-free geometries, such volume forms solve the projective hessian equation. For complex and quaternionic geometries, this implies the existence of a degree $d-1$ polynomial of Killing vector fields.

These results have already yielded classifications of compatible pencils in the real and complex cases (the quaternionic case has yet to be worked out).

In the complex case, we use the classification of hamiltonian 2-forms, and write $\mathrm{vol}\,\mathbf{h} = \pi_c\tilde{\pi}$, where π_c has distinct constant roots $\eta_i : 1 \le i \le N$ and $\tilde{\pi}$ has nonconstant roots $\xi_j : 1 \le j \le \ell$. We have compatible metrics $\mathbf{h}(v)^{-1} : v \in V$ of the form

$$\sum_{i=1}^{N} \frac{\tilde{\pi}(\eta_i)}{\varepsilon({\eta_i}^\natural, v)} g_i + \sum_{j=1}^{\ell} \frac{\Delta_j \Theta_j(\xi_j)}{\varepsilon({\xi_j}^\natural, v)} \left(\left(\frac{d\xi_j}{\Theta_j(\xi_j)} \right)^2 + \left(J \frac{d\xi_j}{\Theta_j(\xi_j)} \right)^2 \right),$$

where $0 \neq \varepsilon \in \wedge^2 \mathbb{V}^*$, $\Delta_j = \prod_k \varepsilon({\xi_j}^\natural, {\xi_k}^\natural)$, the g_is are Kähler metrics and the Θ_js are functions of one variable.

3. Tautological bundles and projective coordinates

In the real case, the embedding of $\mathbb{R}P^n$ in $P(W)$ is not minimal: since $W = S^2\mathbb{R}^{n+1}$, it is the Veronese embedding $[x] \mapsto [x \otimes x]$. The minimal embedding identifies $\mathbb{R}P^n$ with $P(\mathbb{R}^{n+1})$, which gives rise to a rank $n+1$ bundle $V_M^{\mathbb{R}}$, whose quotient is $\mathcal{L}^{1/2}$. This is the tautological bundle $\mathcal{O}_{\mathbb{R}}(1)$, on whose sections μ the equation

$$D^2\mu + r^D\mu = 0$$

is projectively invariant. The solutions are "affine sections of $\mathcal{O}(1)$". If the projective structure on M is flat, they provide the homogeneous coordinates for the development of M into $\mathbb{R}P^n$.

Similarly, in the complex and quaternionic cases, there are tautological complex and quaternionic line bundles $\mathcal{O}_{\mathbb{C}}(1)$ and $\mathcal{O}_{\mathbb{H}}(1)$, equipped with differential equations whose solutions are the coordinates needed to develop M into $\mathbb{C}P^m$ or $\mathbb{H}P^\ell$.

3. Projective geometries and Cartan holonomy

Basic fact 1. $\mathbb{R}P^{2m+1}$ is an $S^1 \cong \mathbb{R}P^1$ bundle over $\mathbb{C}P^m$ (the Hopf fibration), given by a choice of complex structure on the fundamental representation $\mathbb{R}^{2m+2}$ of $GL(2m+2, \mathbb{R})$ (yielding the fundamental representation $\mathbb{C}^{m+1}$ of $GL(m+1, \mathbb{C})$).

For any c-projective manifold M^{2m}, the real projectivization of $\mathcal{O}_{\mathbb{C}}(1)$ is an $\mathbb{R}P^1$ bundle N^{2m+1} with a real projective structure on it, and the projective Cartan connection preserves a complex structure in its fundamental representation.

Conversely, a projective structure on a $(2m+1)$-manifold whose Cartan connection has such a holonomy reduction is locally a circle bundle over an c-projective manifold (S. Armstrong).

Basic fact 2. $\mathbb{C}P^{2\ell+1}$ is a $S^2 \cong \mathbb{C}P^1$ bundle over $\mathbb{H}P^\ell$. The generalization is the twistor space $Z^{2\ell+1} \to M^{4\ell}$ of a quaternionic manifold $M^{4\ell}$, which is a 2-sphere bundle in Q, or the complex projectivization of $\mathcal{O}_{\mathbb{H}}(1)$. Twistor spaces are (locally) c-projective manifolds with reduced holonomy Cartan connection.

3. Projective submanifolds

Observation 1. Let (M^{2m}, J) be a c-projective manifold. A submanifold N is *totally real* if $J(TN) \cap TN = 0$. If N is maximal ($\dim N = m$), then $TM \cong TN \oplus J(TN)$ along N.

By projecting c-projective connections onto TN, N inherits a projective structure: for $X, Y \in TN$, the projection of $[\![X, \gamma]\!]^c(Y)$ is $[\![X, i^*\gamma]\!]^r(Y)$, where $i \colon M \to N$ is the inclusion.

Observation 2. Let $(M^{4\ell}, Q)$ be a quaternionic manifold. A submanifold N is *totally complex* if TN is invariant under some $J \in Q$ (along N), but $I(TN) \cap TN = 0$ for any $I \in Q$ anticommuting with J. If N is a maximal ($\dim N = 2\ell$), then $I(TN)$ is an I-independent complement to TN.

This induces a c-projective structure on N: for $X, Y \in TN$, the projection of $[\![X, \gamma]\!]^q(Y)$ is $[\![X, i^*\gamma]\!]^c(Y)$.

Furthermore, using $\pm J$, N lifts to a pair of complex submanifolds in the twistor space Z. When the blow-up of Z along these submanifolds is a $\mathbb{C}P^1$ bundle over N, this generalizes the Feix–Kaledin construction of hyperkähler metrics (Borowka–C).

Outlook

1. Projective parabolic geometries seem to provide a framework with the right level of generality for coherent nontrivial results. Much remains to be investigated.

2. In particular, quaternionic geometries are (still!) in their infancy, and much can be learned by analogy with the complex case. It should also be amusing to study the octonionic case.

3. It is natural to seek generalizations, for instance to nonabelian nilpotent radical. The metrization problem may again be used as motivation: one approach seeks subriemannian metrics on the horizontal distribution in TM which are horizontally parallel for some Weyl structure.

4. Thanks for your attention!