Almost H-projective structures and their description as parabolic geometries

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Kioloa, March 2013

1. Almost complex manifolds

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Theorem (Newlander-Nirenberg 1957)

(M, J) is a complex manifold $\iff N \equiv 0$.

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Corollary

There exists a complex torsion-free connection on $(M, J) \iff N \equiv 0$.

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Two affine connections ∇ and $\hat{\nabla}$ on an almost complex manifold (M, J) are H-projectively equivalent $:\iff$ there exists a real 1-form $\Upsilon \in \Omega^1(M)$ such that

$$\nabla_X Y = \hat{\nabla}_X Y + \underbrace{\Upsilon(X)Y + \Upsilon(Y)X - \Upsilon(JX)JY - \Upsilon(JY)JX}_{:=\upsilon_\Upsilon(X)(Y)}$$

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- $\upsilon_\Upsilon \in \Omega^1(M, \mathfrak{gl}(TM, J))$
- Hence, if $\hat{\nabla}$ is a complex connection, then any *H*-projectively equivalent connection ∇ is complex too.
- Since $\upsilon_\Upsilon(X)(Y) = \upsilon_\Upsilon(Y)(X)$, *H*-projectively equivalent connections have the same torsion.

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Remark

- A smooth curve $c : I \to M$ is J-planar with respect to a complex connection $\nabla :\iff \nabla_{\dot{c}}\dot{c} \in \text{span}\{\dot{c}, J\dot{c}\}$.

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- A smooth curve $c : I \to M$ is J-planar with respect to a complex connection $\nabla :\iff \nabla_{\dot{c}}\dot{c} \in \text{span}\{\dot{c}, J\dot{c}\}$.
- Two complex connections are H-projectively equivalent $\iff$ they have the same J-planar curves.

3. Parabolic geometries

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A Cartan geometry of type (G, P) on a manifold M is given by a principal P-bundle $\mathcal{G} \to M$ together with a Cartan connection, i.e. a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that:

(1) ω is P-equivariant: $(r^p)^*\omega = Ad(p)^{-1} \circ \omega$, $\forall p \in P$

(2) ω reproduces generators of fundamental vector fields

(3) $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

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Curvature

The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ is given by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

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Natural vector bundles

- Any P-module $\mathbb{V}$ gives rise to a vector bundle

$$V := \mathcal{G} \times_P \mathbb{V} := \mathcal{G} \times \mathbb{V}/\sim, \quad \text{where } (u, v) \sim (u \cdot p, p^{-1} \cdot v) \ \forall p \in P.$$

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- Any P- module homomorphism $\mathbb{V} \to \mathbb{W}$ induces a vector bundle homomorphism $V \to W$.

The Cartan connection induces an isomorphism as follows:

$$\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM$$

$$[u, X + \mathfrak{p}] \mapsto T_u p \omega^{-1}(X).$$

Consequently, all tensor bundles over M are associated vector bundles.

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$$\Lambda^2 T^*M \otimes \mathcal{A}M \cong \mathcal{G} \times_P \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g},$$

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In this picture K corresponds to the following P-equivariant function

$$\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) = K(\omega^{-1}(X)(u), \omega^{-1}(X)(u)).$$

Prolongation procedures of Tanaka (1979), Morimoto (1993), and Čap-Schichl (2000)

Normalising the curvature of a regular parabolic geometry induces an equivalence of categories between regular normal parabolic geometries and certain underlying geometric structures, which admit descriptions in more conventional geometric terms.

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- Consider the complex for computing the homology $H_*(\mathfrak{p}_+, \mathfrak{g})$ of the nilradical $\mathfrak{p}_+$ of the parabolic subalgebra $\mathfrak{p}$ with values in $\mathfrak{g}$

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- A parabolic geometry is normal $:\Longleftrightarrow \partial^*\kappa = 0$.

- The curvature κ of a normal parabolic geometry therefore gives rise to a P-equivariant function, called the harmonic curvature,

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- For a regular normal parabolic geometry, it can be shown that

$$\kappa = 0 \iff \kappa_h = 0.$$

4. Almost H-projective structures as parabolic geometries

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- $$\mathfrak{gl}(n+1,\mathbb{C}) \cong \{A \in \mathfrak{gl}(2(n+1),\mathbb{R}) : A\mathbb{J} = \mathbb{J}A\} =$$

$$= \left\{ \begin{pmatrix} A_{1,1} & \dots & A_{1,n+1} \\ \vdots & \ddots & \vdots \\ A_{n+1,1} & \dots & A_{n+1,n+1} \end{pmatrix} : A_{i,j} = \begin{pmatrix} a_{i,j} & -b_{i,j} \\ b_{i,j} & a_{i,j} \end{pmatrix} \right\}.$$

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- Then

$$\mathfrak{sl}(n+1,\mathbb{C}) = \left\{ \begin{pmatrix} -tr_{\mathbb{C}}(A) & Z \\ X & A \end{pmatrix} : A \in \mathfrak{gl}(n,\mathbb{C}), X \in \mathbb{C}^n, Z \in \mathbb{C}^{n^*} \right\}$$

- Hence, $\mathfrak{sl}(n+1,\mathbb{C})$ admits a |1|-grading as follows:

$$\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0 \cong \mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{g}_{-1} \cong \mathbb{C}^n$ resp. $\mathfrak{g}_1 \cong \mathbb{C}^{n^*}$ as $\mathfrak{g}_0$-modules.

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- Set $G := PSL(n+1,\mathbb{C})$ and let P be the stabiliser in G of the complex line generated by the first standard basis vector of $\mathbb{R}^{2(n+1)}$.

- Therefore, the Levi subgroup G_0 of P consists of equivalence classes of matrices of the form

$$\begin{pmatrix} \det_{\mathbb{C}}(C)^{-1} & 0 \\ 0 & C \end{pmatrix} \qquad \text{where } C \in GL(n,\mathbb{C})$$

It follows that the adjoint action of G_0 on $\mathfrak{g}$ induces an isomorphism

$$G_0 \cong GL(\mathfrak{g}_{-1},\mathbb{J}) \cong GL(n,\mathbb{C}).$$

Theorem (Yoshimatsu (1978), Hrdina (2009))

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$$\{\text{Almost } H\text{-projective structures } (J, [\nabla]) \text{ on } M\}$$

$$\updownarrow 1:1$$

$$\{\text{Normal (real) parabolic geometries of type } (PSL(n+1, \mathbb{C}), P) \text{ on } M\}.$$

Theorem (Yoshimatsu (1978), Hrdina (2009))

Suppose that M is a manifold with $\dim_{\mathbb{R}}(M) = 2n > 2$. Then there is an equivalence of categories between

$$\{\text{Almost } H\text{-projective structures } (J, [\nabla]) \text{ on } M\}$$

$$\updownarrow 1:1$$

$$\{\text{Normal (real) parabolic geometries of type } (PSL(n+1, \mathbb{C}), P) \text{ on } M\}.$$

Given an almost H-projective manifold $(M, J, [\nabla])$, then J defines reduction of structure group

$$\begin{array}{ccc} \mathcal{G}_0 & \xrightarrow{\ \phi\ } & \mathcal{F}M \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ M & \xrightarrow{\ id\ } & M \end{array}$$

corresponding to the inclusion $G_0 \cong GL(\mathfrak{g}_{-1}, \mathbb{J}) \hookrightarrow GL(\mathfrak{g}_{-1}) \cong GL(2n, \mathbb{R})$

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- Let $\pi : \mathcal{G} \to \mathcal{G}_0$ be the natural projection. The tautological 1-form $\tau \in \Omega^1(\mathcal{G}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ given by

$$\tau(\gamma^\nabla(u))(\eta) = (\theta + \gamma^\nabla(u))(T\pi\eta)$$

can be extended to a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ (which is unique up to isomorphism).

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- Hence, one has a algebraic Hodge structure

$$\Lambda^i \mathfrak{g}_-^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \ker(\Box)}^{\ker(\partial^*)} \oplus \mathrm{im}(\partial),$$

where $\Box := \partial\partial^* + \partial^*\partial$.

- In particular, as G_0-module $H^i(\mathfrak{g}_-, \mathfrak{g}) \cong H_i(\mathfrak{p}_+, \mathfrak{g})$ is isomorphic to G_0-submodule $\ker(\Box)$ in $\Lambda^i \mathfrak{g}_-^* \otimes \mathfrak{g}$.

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- Hence, $H^2_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}_{-1}, \mathfrak{g}^{\mathbb{C}}) \cong H^2_{\mathbb{C}}(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1}, \mathfrak{g} \oplus \mathfrak{g})$ as $\mathfrak{g}^{\mathbb{C}}_0 \cong \mathfrak{g}_0 \oplus \mathfrak{g}_0$-module.

H^2 (for $n = 5$)

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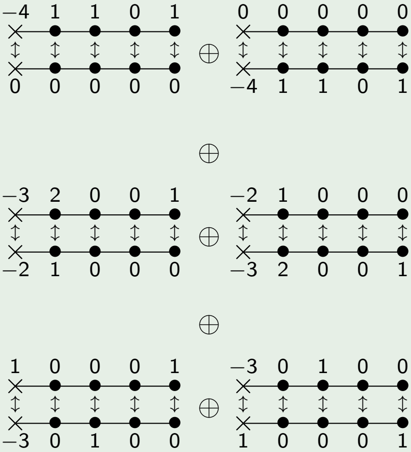

Hence, κ_h has values in three irreducible G_0-modules. Correspondingly, we shall write $\kappa_h = W^{2,0} + W^{1,1} + T$.

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4. If $T = 0$ and $[\nabla]$ is *Kählerisable*, then $W^{(2,0)} = 0$.

- The Hodge decomposition

$$\Lambda^i T^*M \otimes \mathrm{gr}(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \Lambda^i \mathfrak{g}_{-1}^* \otimes \mathfrak{g} = \mathrm{im}(\partial) \oplus \ker(\Box) \oplus \mathrm{im}(\partial^*)$$

implies that for any $\nabla \in [\nabla]$:

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- $R^{\nabla}_{ab}{}^a{}_d = (\partial \mathsf{P}^\nabla)_{ab}{}^a{}_d = \frac{2n+1}{2} \mathsf{P}^\nabla_{bd} - \frac{1}{2} \mathsf{P}^\nabla_{db} + J_{(b}{}^e J_{d)}{}^f \mathsf{P}^\nabla_{fe}$

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- The components $W^{2,0}$ and $W^{1,1}$ are independent of the choice of the connection in $[\nabla]$. These are the two harmonic curvature components.