4D-Kähler metrics admitting essential h-projective vector fields

Stefan Rosemann
(based on joint works with D. Calderbank, V. S. Matveev, T. Mettler)

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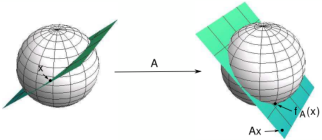

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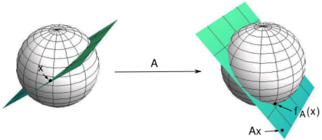

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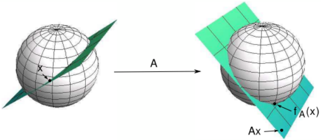

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- The "h-projective Lie-Problem" was mentioned explicitly in our application for the joint project Canberra-Jena.

Special cases: The compact setting is very rigid!

- Matveev, R$\sim$, 2012:
  The only closed connected Kähler $2n$-manifold with essential h-projective vector field is $(\mathbb{C}P(n), const \cdot g_{Fubini-Study}, J_{standard})$.

Special cases: The case of degree of mobility ≥ 3

- Classical (Mikes, Domashev 1978): Let (M, g, J) be Kähler $2n$-manifold. Then, the metrics $\bar{g}$, h-projectively equivalent to g, correspond to non-degenerate symmetric hermitian $(2,0)$-tensors A satisfying

$$(*) \quad \nabla_k A^{ij} = \delta_k^{(i}\Lambda^{j)} + J_k^{(i}J_l^{j)}\Lambda^l,$$

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$$(\hat{M} = \mathbb{R}_{>0} \times \mathbb{R} \times M, \hat{g} = dr^2 + r^2((dt - 2\tau)^2 + g), \hat{J})$$

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- The statement above does not remain to be true in dimensions > 4 (Matveev, R$\sim$, 2012).
- However, it remains to be true for arbitrary dimension (in any signature) if M is assumed to be closed (FKMR 2011).

- Consider again the equation

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- In 4D, a (non-trivial=non-parallel) solution A of $(*)$, can have either
    - Case 1: two non-constant eigenvalues ρ_1, ρ_2,
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  - Case 1: There are coordinates ρ_1, ρ_2, t_1, t_2 and functions F_1, F_2 of one variable such that (g, J) is given by

$$g = \tfrac{\rho_1 - \rho_2}{F_1(\rho_1)} d\rho_1^2 + \tfrac{\rho_2 - \rho_1}{F_2(\rho_2)} d\rho_2^2 + \tfrac{F_1(\rho_1)}{\rho_1 - \rho_2}(dt_1 + \rho_2 dt_2)^2 + \tfrac{F_2(\rho_2)}{\rho_2 - \rho_1}(dt_1 + \rho_1 dt_2)^2,$$
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The 4D-normal forms of Calderbank et al

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$$g = (c-\rho)h + \frac{\rho-c}{F(\rho)} d\rho^2 + \frac{F(\rho)}{\rho-c}\theta^2,$$
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    where $(h, i, \Omega = h(i.,.))$ is a $2D$ Kähler structure and θ is a 1-form on M satisfying $d\theta = -\Omega$.
- Case 1 is parameterized by arbitrary functions F_1, F_2.
- Case 2 is parameterized by an arbitrary function F and a $2D$ metric h.

Theorem:

Let (g, J) be 4D-Kähler structure of non-constant holomorphic section curvature with essential h-projective vector field v. Then, locally (g, J) and v are given by Case 1 or Case 2 below:

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where the functions F_i are given by one of the following subcases depending on the sign of $\frac{\beta^2}{4} - \alpha$ for certain constants α, β:

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  where the functions F_i are given by one of the following subcases depending on the sign of $\frac{\beta^2}{4} - \alpha$ for certain constants α, β:

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  Moreover, v takes the form (up to adding constant linear combinations of $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}$)

$$v = (\rho_1^2 + \beta\rho_1 + \alpha)\frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta\rho_2 + \alpha)\frac{\partial}{\partial \rho_2} + (-\beta t_1 - \alpha t_2)\frac{\partial}{\partial t_1} + (t_1 - 2\beta t_2)\frac{\partial}{\partial t_2}.$$

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Fixing a “backround metric g”, we can identify solutions σ of $(*)'$ with that of $(*)$ via

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$\Rightarrow$ We can choose $\bar{\sigma} = -\mathcal{L}_v \sigma$ as the second basis vector such that the matrix of $\mathcal{L}_v$ becomes

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The corresponding PDE on g, A and v

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Invariant distributions and the splitting of the PDE system

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- The Lie derivative $\mathcal{L}_v g$ splits into $\mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D} g_{D^\perp} + \mathcal{L}_{v_{D^\perp}} g_{D^\perp}$.

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$$g = \begin{pmatrix} \frac{\rho_1-\rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2-\rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1)-F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2^2 F_1(\rho_1)-\rho_1^2 F_2(\rho_2)}{\rho_1-\rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1+\rho_2 & \rho_1\rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Consider the distributions

$$\begin{aligned} D &= \mathrm{span}\{\tfrac{\partial}{\partial \rho_1}, \tfrac{\partial}{\partial \rho_2}\} \overset{\text{invariantly}}{=} \mathrm{span}\{\mathrm{grad}\,\rho_1, \mathrm{grad}\,\rho_2\}, \\ D^{\perp} &= \mathrm{span}\{\tfrac{\partial}{\partial t_1}, \tfrac{\partial}{\partial t_2}\} \overset{\text{invariantly}}{=} \mathrm{span}\{J\mathrm{grad}\,\rho_1, J\mathrm{grad}\,\rho_2\}. \end{aligned}$$

- D is an h-projectively invariant distribution $\Rightarrow D^{\perp} = JD$ is h-projectively invariant.

  $\Rightarrow$ If f is h-projective transformation we have $f_* D = D$, $f_* D^{\perp} = D^{\perp}$.

  $\Rightarrow$ The h-projective vector field v looks like

$$v = \underbrace{(\rho_1^2 + \beta\rho_1 + \alpha)\frac{\partial}{\partial \rho_1} + (\rho_2^2 + \beta\rho_2 + \alpha)\frac{\partial}{\partial \rho_2}}_{v_D} + \underbrace{v^3(t_1, t_2)\frac{\partial}{\partial t_1} + v^4(t_1, t_2)\frac{\partial}{\partial t_2}}_{v_{D^{\perp}}}.$$

- The Lie derivative $\mathcal{L}_v g$ splits into $\mathcal{L}_v g = \mathcal{L}_{v_D} g_D + \mathcal{L}_{v_D} g_{D^{\perp}} + \mathcal{L}_{v_{D^{\perp}}} g_{D^{\perp}}$.

  $\Rightarrow$ The PDE $\mathcal{L}_v g = gA + \frac{1}{2}\mathrm{trace}(A) g$ splits into an upper-left block (containing $\rho_1, \rho_2, F_1, F_2, F_1', F_2'$) and a lower-right block.

Invariant distributions and the splitting of the PDE system

- Normal forms of g and A in Case 1: In certain coordinates ρ_1, ρ_2, t_1, t_2, we have

$$g = \begin{pmatrix} \frac{\rho_1-\rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2-\rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1)-F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2^2 F_1(\rho_1)-\rho_1^2 F_2(\rho_2)}{\rho_1-\rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1+\rho_2 & \rho_1\rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Consider the distributions

$$\begin{aligned} D &= \mathrm{span}\{\tfrac{\partial}{\partial\rho_1}, \tfrac{\partial}{\partial\rho_2}\} \overset{\text{invariantly}}{=} \mathrm{span}\{\mathrm{grad}\,\rho_1, \mathrm{grad}\,\rho_2\}, \\ D^\perp &= \mathrm{span}\{\tfrac{\partial}{\partial t_1}, \tfrac{\partial}{\partial t_2}\} \overset{\text{invariantly}}{=} \mathrm{span}\{J\mathrm{grad}\,\rho_1, J\mathrm{grad}\,\rho_2\}. \end{aligned}$$

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$$v = \underbrace{(\rho_1^2 + \beta\rho_1 + \alpha)\frac{\partial}{\partial\rho_1} + (\rho_2^2 + \beta\rho_2 + \alpha)\frac{\partial}{\partial\rho_2}}_{v_D} + \underbrace{v^3(t_1,t_2)\frac{\partial}{\partial t_1} + v^4(t_1,t_2)\frac{\partial}{\partial t_2}}_{v_{D^\perp}}.$$

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  $\Rightarrow$ The PDE $\mathcal{L}_v g = gA + \frac{1}{2}\mathrm{trace}(A) g$ splits into an upper-left block (containing $\rho_1, \rho_2, F_1, F_2, F_1', F_2'$) and a lower-right block.

- Upper-left block gives ODE's on F_1 and F_2 respectively.

- Normal forms of g and A in Case 1: In certain coordinates ρ_1, ρ_2, t_1, t_2, we have

$$g = \begin{pmatrix} \frac{\rho_1-\rho_2}{F_1(\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{\rho_2-\rho_1}{F_2(\rho_2)} & 0 & 0 \\ 0 & 0 & \frac{F_1(\rho_1)-F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} \\ 0 & 0 & \frac{\rho_2 F_1(\rho_1)-\rho_1 F_2(\rho_2)}{\rho_1-\rho_2} & \frac{\rho_2^2 F_1(\rho_1)-\rho_1^2 F_2(\rho_2)}{\rho_1-\rho_2} \end{pmatrix}, \quad A = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1+\rho_2 & \rho_1\rho_2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Consider the distributions

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  $\Rightarrow$ The PDE $\mathcal{L}_v g = gA + \frac{1}{2}\mathrm{trace}(A) g$ splits into an upper-left block (containing $\rho_1, \rho_2, F_1, F_2, F_1', F_2'$) and a lower-right block.
- Upper-left block gives ODE's on F_1 and F_2 respectively. Inserting solutions for F_1, F_2 into lower-right block gives equations on $v^3(t_1,t_2), v^4(t_1,t_2)$ which can be solved.

Thanks for listening!