The L_p-Minkowski Problem and the Minkowski Problem in Centroaffine Geometry

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Abstract

The L_p-Minkowski problem introduced by Lutwak is solved for $p \geqslant n+1$ in the smooth category. The relevant Monge-Ampère equation (1) is solved for all $p > 1$. The same equation for $p < 1$ is also studied and solved for $p \in (-n-1, 1)$. When $p = -n-1$ the equation is interpreted as a Minkowski problem in centroaffine geometry. A Kazdan-Warner type obstruction for this problem is obtained.

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§0. Introduction

Let f be a positive function defined on the unit sphere S^n in $\mathbb{R}^{n+1}$ and $p \in \mathbb{R}$. In this paper we study the following equation of Monge-Ampère type

$$\det(h_{ij} + h\delta_{ij}) = fh^{p-1} \tag{1}$$

on S^n. Here h_{ij} is the convariant differentiation of h with respect to an orthonormal frame on S^n. We look for a solution h which is the support function for some non-degenerate convex body. Recall that the relation between a convex body and its support function introduces a one-to-one correspondence between the set of all convex bodies, $\mathcal{K}$, in $\mathbb{R}^{n+1}$ and the set $\mathcal{S} = \big\{h \in C(S^n) : h$ is a convex function after being extended as a function of homogeneous degree one in$\mathbb{R}^{n+1}\big\}$. For any $p \geqslant 1$, given two convex bodies K and L with respective support functions h_K and h_L, and $\lambda, \mu > 0$, we can form a new convex body $\lambda \circ K +_p \mu \circ L$ whose support function is given by $(\lambda h_K^p + \mu h_L^p)^{\frac{1}{p}}$. For $p = 1$, this sum, which becomes $\lambda K + \mu L$, is called the Minkowski addition. It plays a central role in the theory of convex bodies. For $p > 1$, the addition was introduced by Firey[F] and further developed in Lutwak[L]. It has been shown that many basic notions and properties such as the mixed volumes, the quermassintegrals, Brunn-Minkowski inequality, have their natural counterparts for $p > 1$. In particular, the p-mixed volume, $V_p(K, L)$, is well-defined and is given by

$$\frac{n+1}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \circ L) - V(K)}{\varepsilon}$$

(here $V(K)$ is the volume of K). Let $\mathcal{K}_o$ be the collection of all convex bodies containing the origin in their interiors. For any $K \in \mathcal{K}_o$, there exists a Borel measure μ_p on S^n so that

$$V_p(K, L) = \frac{1}{n+1} \int_{S^n} h_L^p \, d\mu_p(K, \cdot)$$

for all $L \in \mathcal{K}_o$. The measure μ_p is called the p-area measure of K. When $p = 1$, it reduces to the ordinary area measure μ for K. It turns out that μ_p is related to μ by [L]:

$$h_K^{p-1} d\mu_p = d\mu.$$

Recall that the classical Minkowski problem is concerned with prescribing area measure (or Gauss curvature). It can be formulated as follows: Given a finite Borel measure m on S^n, find necessary and sufficient conditions on m so that it is the area function of a nondegenerate convex body. In the past the problem was also studied in the smooth category, that is, assuming the Radon-Nikodym derivative of m with respect to the spherical measure on S^n exists and is smooth, one looks for a solution of the Minkowski problem whose boundary is a smooth hypersurface. In terms of the support function this problem is equivalent to solving (1) for $p = 1$. It turns out that there are two necessary and sufficient conditions for the classical problem, namely, (i) $m(S^n) > m(C)$ where C is any great (n-1)-sphere, and (ii) for $j = 1, \cdots, n+1$,

$$\int_{S^n} x_j dm(x) = 0 \ . \tag{2}$$

Under (i) and (ii) the solution is unique up to translations. Furthermore, the boundary of the solution is smooth if f is smooth. For a full discussion on the Minkowski problem and its resolution, one may consult Pogorelov[P] and Cheng-Yau[CY].

Quite naturally, one may pose the same problem for p-area measure: *Given a finite Borel measure m on S^n, find necessary and sufficient conditions on m so that it is the p-area measure for some non-degenerate convex body in $\mathcal{K}_o$.* Let μ be the area measure of the solution of the problem. Then the L_p-Minkowski problem is equivalent to solving the equation

$$\mu(E, h) = h^{p-1} m(E), \tag{3}$$

for all Borel sets E in S^n. When $f = dm/dx$ is positive and the solution hypersurface has positive Gauss curvature, this equation reduces to (1). So (1) is the equation describing the L_p -Minkowski problem in the smooth category.

The L_p-Minkowski problem was first formulated and studied in Lutwak[L]. He showed that any even finite Borel measure is a p-area measure for a unique centrally symmetric convex body. The regularity of the convex body (when f is regular) was later established in Lutwak-Oliker[LO].

We observe that not every finite Borel measure is a p-area function. Let's call a measure "non-concentrating on hemisphere" if its measure on any (open) hemisphere is positive. Then the p-area measure of any hypersurface in $\mathcal{K}_o$ must be non-concentrating on hemisphere. For, let m vanish on some hemisphere H. Taking $E = H$ in the above equation, the right hand side vanishes and yet the left hand side is positive as $K \in \mathcal{K}_o$.

Now we state our main results. First we introduce some notations. Denote the class of all finite Borel measures on S^n which are non-concentrating on hemisphere by NCH. For an NCH measure m let f be its Radon-Nikodym derivative with respect to the spherical measure. We also let $\mathcal{K}_{cl}$ be the collection of all nondegenerate convex bodies which contain the origin in their interiors or on their boundaries.

Theorem A *Consider (1) and (3) for* $p > n + 1$.
(a) Let f *be a positive function in* $C^{\alpha}(S^n)$ *for some* $\alpha \in (0, 1)$. *Then (1) has a unique, positive solution in* $C^{2,\alpha}(S^n)$.
(b) Let $m \in$ *NCH. There exists a convex body in* $\mathcal{K}_{cl}$ *satisfying (3). It belongs to* $\mathcal{K}_o$ *when* f *is bounded from above.*

Next, we treat the case $p = n + 1$ as an eigenvalue problem.

Theorem B *(a) Let* f *be a positive function in* $C^{\alpha}(S^n)$ *for some* $\alpha \in (0, 1)$. *There exists a unique pair* $(h, \lambda), h > 0,$ *in* $C^{2,\alpha}(S^n)$ *and* $\lambda > 0$ *satisfying*

$$\det(h_{ij} + h\delta_{ij}) = \lambda f h^n \tag{4}$$

(b) Let $m \in$ *NCH. There exists a pair* $(K, \lambda), K \in \mathcal{K}_{cl}$ *and* $\lambda > 0$ *satisfying*

$$\mu(E, K) = \lambda h^n m(E), \tag{5}$$

for all Borel sets E. *Moreover,* $K \in \mathcal{K}_o$ *when* f *is bounded from above.*

In fact, λ *is characterized by*

$$\lambda = \sup_{S^n} \big\{ \big[\int f h^{n+1} dm \big]^{-1} : V(K) = 1, K \in \mathcal{K}_o \big\}. \tag{6}$$

When $p \in (1, n + 1)$ the situation is more delicate.

Theorem C *Consider (1) and (3) for* $1 < p < n + 1$.
(a) Let $f \in L^{\infty}(S^n)$, $f \geq f_0$ *for some positive constant* f_0. *Then (1) has a generalized non-negative solution in the sense of Aleksandrov.*
(b) Let $m \in$ *NCH. Then (3) has a solution in* $\mathcal{K}_{cl}$.

The regularity property in Theorems A and B follows from Proposition 1.2, which asserts that any positive solution h of (1) is smooth when f is smooth[C2]. As the solution is always positive when $p \geq n + 1$, we solve the smooth p-Minkowski problem in this case without any further necessary condition such as (2). This is not surprising because (2) originates from the translational invariance of the problem, which only holds for $p = 1$. The regularity in the case $1 < p < n+1$ will be treated in Theorem E below.

In [L] it is proved that there are at most one convex body in $\mathcal{K}_o$ satisfying (3). For (5) the following uniqueness result is also proved in the same work: Let (K_1, λ_1) and (K_2, λ_2) be two solutions of (5) where $K_i \in \mathcal{K}_o$, and $\lambda_i > 0, i = 1, 2$. Then $\lambda_1 = \lambda_2$ and K_2 is a dilate of K_1. Both results are consequences of a version of the Minkowski inequality for mixed p-Quermassintegrals.

The remaining cases in (1) $p < 1$ have not been studied in a systematic manner. Nevertheless, some significant special cases were discussed before. For example, when $p = 0$, $n = 2$ and $f(x)$=constant, (1) describes the ultimate shape of a worn stone in a model posed by Firey[F], who conjectured that the constant function is the unique solution. An affirmative answer is obtained relatively recently in Andrews[A2], where one may find a full discussion of the problem. Another important case is $p = -n-1$ and $f(x) \equiv 1$. It was Tzitséica who first studied this equations in 1908. He proved that all solutions are ellipsoids centered at the origin. The same equation was independently proposed again in the search for projective metric in a convex domain by Loewner-Nirenberg[LN]. In a different setting, it was studied over a bounded domain in a hemisphere with certain boundary condition. The problem was later solved by Cheng-Yau[CY]. We do not know any results when f is non-constant. From our work [CW3] on the Hessian equations, it is clear that $p = -n-1$ is the critical case for the Monge-Ampère operator on the sphere. We solve the subcritical case $p \in (-n-1, 1)$ in this paper.

Theorem D *Let $p \in (-n-1, 1)$, $f \in L^\infty(S^n)$, and $f \geq f_0$ for some constant $f_0 > 0$. Then there exists a generalized nonnegative solution of (1) in the sense of Aleksandrov. When $p \in (-n-1, -n+1]$ and $f \in C^\alpha(S^n)$ for some $\alpha \in (0, 1)$, the solution is positive and in $C^{2,\alpha}(S^n)$.*

Let us elaborate a little more on the regularity properties of the generalized solutions. When $-n+1 < p < n+1$, $p \neq 1$, even f is positive and smooth, the boundary of the solution convex body may touch the origin and hence the solution is not positive, see §6 for more. Examples can also be found in Andrews [A1] for $n = 1$ and Guan-Lin [GL] for all $n \geq 1$. In this case the Monge-Ampère equation (1) is either degenerate ($1 < p < n+1$) or singular ($-n+1 < p < 1$), and the solution is not C^2 in general. But we will prove the following regularity result.

Theorem E *Let h be a solution of equation (1) with $-n+1 < p < n+1$.*
(a) If $f \in L^\infty(S^n)$, $f \geq f_0$ for some constant $f_0 > 0$, then the solution is in $C^1(S^n)$ when $1 < p < n+1$ and the associated convex hypersurface is in C^1 when $-n+1 < p < 1$. Moreover $h \in C^{1,\gamma}(\{h > 0\})$ for some $\gamma \in (0,1)$. If furthermore $f \in C^\alpha$, then $h \in C^{2,\alpha}(\{h > 0\})$.
(b) If $f \in C^{0,1}(S^n)$, $f \geq f_0$, then the solution is in $C^{1,\alpha}(S^n)$ for some $\alpha \in (0,1)$ when $1 < p < n+1$ and the associated convex hypersurface is in $C^{1,\alpha}$ when $-n+1 < p < 1$.
(c) If $f \in C^{1,1}(S^2)$, $f \geq f_0$, and $p \in (\frac{n+1}{2}, n+1)$, then the solution is in $C^{1,1}(S^n)$.

The above theorem does not exclude the possibility that the solution is only Lipschitz when $-n+1 < p < 1$ and the associated convex hypersurface is Lipschitz when $1 < p < n+1$. By the function given in (6.4), the $C^{1,\alpha}$ estimate is optimal for $1 < p < \frac{1}{2}(n+1)$.

This paper does not go beyond $p < -n-1$. Yet the critical case $p = -n-1$ is very delicate and highly interesting because the equation becomes invariant under all projective transformations on the n-sphere. We shall make a preliminary study of it. First, using the concept of Klein geometry, we shall interpret it naturally as a Minkowski problem in centroaffine geometry. Next, we find a new necessary condition ("obstruction") for solving it.

Proposition F *Let h be a C^2-solution of (1) where $p = -n-1$. Then for any projective vector field ξ on S^n,*

$$\int_{S^n} (\nabla_\xi f) h^{-n-1} = 0 \ , \tag{7}$$

where $\nabla_\xi f$ is the derivative of f along ξ.

Incidentally, we point out that for positive p, (1) also describes self-similar solution for the expanding Gauss curvature flow, and, for negative p, self-similar solution for the contracting Gauss curvature flow. One may consult Andrews [A1] [A2] and Urbas[U1] [U2] for works in this direction.

The paper is organized as follows. After the preliminary Section 1, we prove Theorems A, B and C in Sections 2, 3 and 4 respectively. In Sections 2 and 4 we also present existence results on the general equation obtaining from (1) by replacing fh^{p-1} by some $f(x,h)$. See Propositions 2.1 and 4.1. In a board sense this equation may be regarded as a prescribed curvature problem. Without striving for full generality, our results illustrate how far our methods go. In Section 5 we study $p \in (-n-1, 1)$ and establish Theorem D. In Section 6 we prove Theorem E. Finally in Section 7 we give an introduction to the centroaffine Minkowski problem and prove Proposition F.

This paper was written over a number of years. The first draft [CW1] contains the proofs of Theorems A-C for positive measurable f and their extensions to more general right hand side $f(x,u)$, while Sections 5, 6 and 7 were completed relatively recently. In the meanwhile Guan and Lin [GL] independently obtained results similar to Theorem A and Theorem B without the variational characterization (6). Prior to us they also established Theorem E(c) by a different method.

We wish to point out further works on the L_p-Minkowski problem which have come into our knowledge after the completion of this paper. In [LYZ1] Lutwak, Yang and Zhang present another approach to the problem (still for even measures) and subsequently apply it in [LYZ2] to establish a sharp affine invariant L^p-Sobolev inequality. One may consult these papers for other related works. Concerning (1) for $p < 1$ a complete classification of all positive solutions when $n = 1$ and f is a constant has been carried out by Andrews [A4]. A surprising discovery is the existence of many non-circular solutions for $p < -7$.

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§1. Preliminaries

In this section we recall and collect some basic notions and results to be used in subsequent sections.

For a given convex body K in $\mathbb{R}^{n+1}$ we let X be its boundary. The convex body K is non-degenerate if its interior is non-empty and regular if X is a regular hypersurface. The support function of K (or X) is a continuous function defined on S^n given by $h(x) = \sup\{p \cdot x : p \in K\}$. It is convex after being extended as a function of homogeneous degree 1 in $\mathbb{R}^{n+1}$. It turns out that, conversely, any continuous, convex function h of homogeneous degree one determines a convex body $K = \{p \in \mathbb{R}^{n+1} : p \cdot x \leqslant h(x), \text{for all } x \in S^n\}$. So the collection of all convex bodies, $\mathcal{K}$, can be identified with the set

$$\begin{aligned}\mathcal{S} = \{h \in C(S^n) : \;& h \text{ is the restriction of a convex} \\ & \text{function of homogeneous degree one in } \mathbb{R}^{n+1}\}.\end{aligned}$$

$\mathcal{S}$ is regarded as a subspace of $C(S^n)$ in the sup-norm. Corresponding to all convex bodies containing the origin in their interior, $\mathcal{K}_o$, we have

$$\mathcal{S}^+ = \{h \in \mathcal{S} : h > 0\} \ .$$

We also set

$$\mathcal{S}^2 = \{h \in C^2(S^n) : (h_{ij} + \delta_{ij} h) > 0.\}$$

(where h_{ij} is the covariant differentiation of h with respect to an orthonormal frame on S^n), and

$$\mathcal{S}^{k,\alpha} = \mathcal{S} \cap C^{k,\alpha}(S^n) \ .$$

So elements in $\mathcal{S}^2$ determine convex hypersurfaces with positive Gauss curvature. Uniform convergence for support functions corresponds to convergence of convex bodies in the Hausdorff metric. The Blaschke selection theorem asserts that for any bounded sequence of convex bodies, one can select a convergent subsequence in the Hausdorff metric. Equivalently, it means that the support functions of the subsequence converge uniformly.

Any X, the boundary of some $K \in \mathcal{K}$, induces a Borel measure on S^n as follows: For any Borel set $E \in \mathcal{B}$,

$$\begin{aligned}\mu(E; X) = \mathcal{H}^n\{p \in X : \;& \text{There exists a supporting hyperplane} \\ & \text{passing } p \text{ whose unit outer normal lies in } E\},\end{aligned}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. Instead of $\mu(E, X)$ sometimes we write $\mu(E, h)$. The measure μ is called the area measure of X. When X belongs to $\mathcal{S}^2$,

$$
\begin{aligned}
d\mu &= K^{-1}dx \\
&= \det(h_{ij} + h\delta_{ij})dx \ .
\end{aligned}
$$

In view of this, $h \in \mathcal{S}$ is called a generalized solution of (1) if

$$\mu(E, h) = \int_E h^{p-1} f(x)dx, \quad \forall E \in \mathcal{B},$$

where h is the support function of X. More generally, for a given finite Borel measure m on S^n, a generalized solution to the L_p-Minkowski problem is a convex body K in $\mathcal{K}_{cl}$ whose area measure satisfies

$$\mu(E, h) = h^{p-1} m(E) \ , \quad \forall E \in \mathcal{B}. \tag{1.1}$$

Concerning (1.1) we have the following basic compactness result ([CY] or [P]).

Proposition 1.1 *Let $\{h_j\}$ be a bounded sequence in $\mathcal{S}$, each h_j solving (1.1) for the measure m_j. Suppose that $\{m_j\}$ converges weakly to m. Then $\{h_j\}$ subconverges to a generalized solution of (1.1).*

The regularity property of the generalized solution is contained in the following proposition.

Proposition 1.2 *Let h be a generalized solution of (1.1). Let $Z = \{h = 0\}$. Suppose h is locally strictly convex away from Z. Then h is in $C^{1,\gamma}$ for any $\gamma \in (0, 1)$ and $C^{2,\alpha}$ away from Z when the Radon-Nikodym derivative of m with respect to the standard spherical measure on S^n, f, is in C and C^α for some $\alpha \in (0, 1)$ respectively. Moreover, it is in $C^{k+2,\alpha}(S^n \backslash Z)$ if $f \in C^{1,1}(S^n \backslash Z) \cap C^{k,\alpha}(S^n \backslash Z)$ for all $k \geqslant 1$.*

We remark that when $h > 0$ in the whole S^n, the local strict convexity is proved in [C1].

Proof For any $x_0 \in S^n \backslash Z$ denote the restriction of h on a supporting hyperplane through x_0 by u. Then u is a convex function in $\mathbb{R}^n$ which satisfies the standard Monge-Ampère equation

$$\det D^2 u = g(x) u^{p-1} \ , \tag{1.2}$$

where

$$g(x) = (1+|x|^2)^{-\frac{n+2}{2}-p+1} f\big(\frac{x, -1}{\sqrt{1+|x|^2}}\big)$$

(Here we have taken x_0 to be the south pole.) in the generalized sense of Aleksandrov. Since $u(x_0) > 0$, by assumption u is strictly convex near x_0. Hence for $\delta > 0$ small, u is positive in the domain $\{x : u(x) < u(x_0) + x \cdot y + \delta\}$. By Caffarelli's regularity theory [C2], u is in the Sobolev space $W^{2,p}$ for any $p > 1$ ($C^{2,\alpha}$ resp. for some $\alpha \in (0,1)$) whenever g is in C (C^α resp.). When f belongs to $C^{1,1} \cap C^{k,\alpha}$, by the Schauder estimates we infer furthermore that $u \in C^{k+2,\alpha}(S^n \backslash Z)$. □

To end this section we note a result on the equivalence between sup-norm and the L^p-norm on $\mathcal{S}^+$.

Proposition 1.3 *Suppose that m_j tends to minNCH weakly. There exists a positive $\rho = \rho(n, p, \mu)$ such that*

$$\rho h_{\max}^p \leqslant \int_{S^n} h^p d\mu_j \ , \tag{1.3}$$

for all large j and non-negative h in $\mathcal{S}$.

Proof By a compactness argument, there exists $\rho = \rho(n, p, m)$ such that

$$\int_{\mathcal{H}} (x \cdot \xi)^p dm \geqslant \rho > 0 \ ,$$

where $\mathcal{H} = \{x : x \cdot \xi > 0\}$ and ξ is any unit vector. Assume that $h_{\max}$ is attained at the north pole. By convexity we have $h(x) \geqslant h_{\max} x_{n+1}$ for $x_{n+1} > 0$. Therefore

$$\int_{S^n} h^p dm \geqslant \rho h_{\max}^p \ .$$

□

Let h be a solution of (1) or (3) with $m \in NCH$. By Proposition 1.3 we have the volume estimate

$$\begin{aligned} V(X) &= \frac{1}{n+1} \int_{S^n} h d\mu \\ &= \frac{1}{n+1} \int_{S^n} h^p dm \\ &\geqslant \frac{\rho}{n+1} h_{max}^p. \end{aligned} \tag{1.4}$$

§2. The Positive Case: $p > n + 1$

We shall give two proofs of Theorem A. The first one, which is based on the method of continuity, works for continuous $f's$. The second proof by gradient flow works for the general case. It also gives a variational characterization to the solution, which will be used in the next section.

Let's consider (1) where $f > 0$ in $C^\alpha(S^n)$ and h is a C^2-solution of (1). At $h(x_0) = h_{\max}$ ($h_{\min}$ resp.), $D^2h(x_0) \leqslant 0$ ($\geqslant 0$ resp.) Using this in the equation, we obtain

$$(\inf f)^{\frac{1}{p-n-1}} \leqslant h^{-1}(x) \leqslant (\sup f)^{\frac{1}{p-n-1}} \tag{2.1}$$

immediately. Let

$$\mathcal{I} = \{t \in [0,1] : (1) \text{ is solvable in } C^{2,\alpha} \text{ for } f_t = tf + (1-t)\} \ .$$

Clearly $0 \in \mathcal{I}$ and $\mathcal{I}$ is closed by (2.1) and Proposition 1.2. To show that $\mathcal{I}$ is open we look at the linearized problem

$$\mathcal{L}u = c_{ij}[h](u_{ij} + u\delta_{ij}) - (p-1)h^{p-2}u = g \in C^\alpha$$

where $c_{ij}[h]$ is the (i,j)-entry of the cofactor matrix of $(h_{ij} + h\delta_{ij})$. As is well-known, $c_{ij}[h]_j = 0$ and hence $\mathcal{L}$ is self-adjoint. To show its invertibility one needs to show ker$\mathcal{L} = 0$. But this follows from an inequality of Hilbert and Aleksandrov. We refer to Lutwak-Oliker[LO] for details. Since $\mathcal{I}$ is both open and closed, $\mathcal{I} = [0,1]$. In particular, (1) is solvable for $t = 1$.

To show uniqueness let's first note that $h \equiv 1$ is the unique solution (see (2.1)) when f is identically 1. Now, let h_1 and h_2 be two solutions of (1). We may connect each of them to 1 by line segments. Along the segments the linearized problem is invertible. As $h \equiv 1$ is the only solution to $f \equiv 1$, we conclude $h_1 \equiv h_2$.

Equation (1) has a natural variational structure. In fact, Minkowski solved the case $p = 0$ in the non-smooth category by a variational argument. See Pogorelov[P] and Schneider [S]. In [CW2] we use the gradient flow to furnish a variational proof of the Minkowski problem in the smooth category. Variational argument is also used for the L_p-Minkowski problem in [L] when the given measure m is even. For the present situation we shall use the gradient flow to solve (1) for more general nonlinearities. More specifically, let $f \in C^\alpha(S^n \times$

$(0,\infty))$, $\alpha \in (0,1)$, be a positive function, increasing in z and satisfying

$$\lim_{z\to\infty} \frac{f(x,z)}{z^n} = \infty \ , \tag{2.2}$$

$$\limsup_{z\to 0^+} \frac{f(x,z)}{z^n} < 1 \ , \tag{2.3}$$

uniformly on S^n.

Proposition 2.1 *Let f be given as above. Then the equation*

$$\det(h_{ij} + h\delta_{ij}) = f(x,h) \tag{2.4}$$

has a solution in $\mathcal{S}^+ \cap \mathcal{S}^{2,\alpha}$. It is unique if in addition $f(x,z)z^{-n}$ is increasing in z.

In the following proof we shall further assume f to be smooth so that the flow (2.5) is solvable. This additional regularity can be removed by an approximation argument easily. Let's consider the functional

$$\mathcal{I}(h) = \frac{1}{n+1}\int_{S^n} h \ \det(h_{ij} + h\delta_{ij}) - \int_{S^n} F(x,h),$$

where F is the primitive function of f satisfying $F(0) = 0$ on the space $C^\infty \cap \mathcal{S}^+$. We consider the Cauchy problem for the flow

$$\begin{cases} \dfrac{\partial h}{\partial t} = \log\det(h_{ij} + h\delta_{ij}) - \log f(x,h) \\ h(x,0) = h_0 \in \mathcal{S}^+ \cap C^\infty. \end{cases} \tag{2.5}$$

Along this flow,

$$\begin{aligned} \frac{d\mathcal{I}}{dt}(h) &= \int_{S^n} \big(\det(h_{ij} + h\delta_{ij}) - f(x,h))h_t \\ &= \int_{S^n} \big(\det(h_{ij} + h\delta_{ij}) - f(x,h)\big) \log\frac{\det(h_{ij} + h\delta_{ij})}{f(x,h)} \\ &\geqslant 0, \end{aligned} \tag{2.6}$$

and equality holds if and only if (2.4) holds. We shall show that (2.4) has a solution h' which satisfies

$$\mathcal{I}(h') = \max\{\mathcal{I}(h) : h \in \mathcal{S}^+\} \ .$$

We proceed in three steps.

STEP 1 $\mathcal{I}$ **is bounded from above.** Let $h \in \mathcal{S}$ and X its associated hypersurface. Suppose that the maximum of h is attained at the north pole x_0. By convexity $h(x) \geqslant h(x_0)x_{n+1}$. Hence,

$$\begin{aligned}\int_{S^n} f(x,h) &\geqslant \int_{\{x_{n+1}>\frac{1}{2}\}} f\big(x, \frac{1}{2}h(x_0)\big) \\ &\geqslant Mh(x_0)^n \ ,\end{aligned}$$

where $M \to \infty$ as $h(x_0) \to \infty$ by (2.2). On the other hand,

$$\int_{S^n} \det(h_{ij} + h\delta_{ij}) \leqslant \omega_n h^n(x_0) \ ,$$

as the right hand side is the area of X. Therefore, $\mathcal{I}(h)$ becomes negative outside the set $\{h : h_{\max} < h_0\}$ for some large constant h_0.

STEP 2 A priori estimate for the solution of the flow. According to the C^2-estimate in [CW2] and $\tilde{C}^{2,\alpha}$-and higher regularity results of Krylov (here $\tilde{C}^{k,\alpha}$ is the parabolic Hölder space), the estimates

$$\|h\|_{\tilde{C}^{k+2,\alpha}(S^n\times[0,\infty))} \leqslant C \ , \ k \geqslant 2 \ ,$$

follow from the uniform estimates

$$0 < C_1 \leqslant h(x,t) \leq C_2 \ , \ (x,t) \in S^n \times [0,\infty) \ . \tag{2.7}$$

So it suffices to prove (2.7). Let $h(x_0,t_0) = \min h$ in $S^n \times [0,T]$. When $t_0 > 0$, we've

$$\begin{aligned}0 \geqslant h_t &= \log \frac{\det(h_{ij} + h\delta_{ij})}{f(x,h)} \\ &\geqslant \log \frac{h^n}{f}\end{aligned}$$

at this point. It follows from (2.3) that the first inequality in (2.7) holds. Similarly, using (2.2) one gets the other estimate.

STEP 3 Existence of a maximiser for $\mathcal{I}$. The *a priori* estimates in Step 2 enable us to solve (2.5) using a maximizing sequence $\{h_j^0\}$ as initial data to obtain a family of $\{h_j\}$ in $\mathcal{S}^+ \cap C^\infty(S^n \times [0,\infty))$. By (2.6) and Step 2, for each j, one may extract a sequence $h_j(x,t_i)$ which converges smoothly to a solution of (2.4) as $t_i \to \infty$. In this way we obtain a sequence of solutions $\{h_j^*\}$ which is again maximizing. Applying the maximum principle to (2.4), just like the way we derived (2.1), we obtain uniform two-sided bounds on h_j^*. Hence it contains a subsequence converging to a maximizer of $\mathcal{I}$ among all smooth functions in $\mathcal{S}^+$.

Observe that the first term in the functional $\mathcal{I}$ is simply the volume enclosed by the hypersurface determined by h, and hence it is continuous on $\mathcal{S}$. So this maximizer is in fact a maximizer in $\mathcal{S}^+$.

To complete the proof of Proposition 2.1 we prove the uniqueness of solution when f/z^n is increasing. Let h_1 and h_2 be two solutions of (2.4). Suppose $G(x_0) = G_{\max}$, where $G = h_1/h_2$. Then at x_0,

$$0 = \nabla G = \frac{(\nabla h_1)h_2 - h_1 \nabla h_2}{h_2^2} \ ,$$

and

$$0 \geqslant \{G_{ij}\} = \frac{h_2(D^2 h_1) - h_1(D^2 h_2)}{h_2^2} \ ,$$

i.e.,

$$\left\{\frac{D^2 h_1}{h_1}\right\} \leqslant \left\{\frac{D^2 h_2}{h_2}\right\} \, .$$

Hence

$$\begin{aligned} f(x_0, h_1(x_0)) &= h_1^n(x_0) \det\big(\frac{D^2 h_1}{h_1} + I\big) \\ &\leqslant h_1^n(x_0) \det\big(\frac{D^2 h_2}{h_2} + I\big) \\ &= \frac{h_1^n(x_0)}{h_2^n(x_0)} f(x_0, h_2(x_0)) \, . \end{aligned}$$

Since f/z^n is increasing, $h_1(x_0) \leqslant h_2(x_0)$, i.e., $h_1(x) \leqslant h_2(x)$. Similarly one can show that $h_1 \geqslant h_2$.

We remark that the solution may not be unique if f/z^n is not increasing. A simple example is $f = \alpha z^{n+1} + \beta$, $\alpha, \beta > 0$. One can easily show that it admits two spherical solutions.

Returning to the proof of Part (b) in Theorem A, let m be a finite Borel measure not concentrating on hemisphere.

Let $\{m_j\}$, $dm_j = f_j(x)dx$, f_j positive and smooth, be a sequence of measures converging weakly to m and let $\{h_j\}$ be the solution of (1) for $f = f_j$. By Proposition 2.1, each h_j can be taken to be the maximizer of the corresponding functional $\mathcal{I}_j$.

The area of the hypersurface determined by h_j, A_j, satisfies

$$\begin{aligned} \omega_n h^n_{j\max} &\geqslant A_j \\ &= \int_{S^n} f_j(x) h_j^{p-1} dx \qquad (2.8) \\ &\geqslant \rho h^{p-1}_{j\max} \end{aligned}$$

by Proposition 1.3. As here $p-1>n$, a uniform bound on $h_{j\,\max}$ comes out. On the other hand, by the variational characterization of the solution,

$$
\begin{aligned}
\mathcal{I}_j(h_j) &\geqslant \sup_{R>0}\ \mathcal{I}_j(R) \\
&= \Big(\frac{1}{n+1}-\frac{1}{p}\Big)\frac{\omega_n^{\frac{p}{n+1}}}{|m_j(S^n)|^{\frac{n+1}{p-n-1}}} \\
&\geqslant \frac{1}{2}\Big(\frac{1}{n+1}-\frac{1}{p}\Big)\frac{\omega_n^{\frac{p}{n+1}}}{|m(S^n)|^{\frac{n+1}{p-n-1}}}
\end{aligned}
$$

for sufficiently large j. By Proposition 2.1, $\{h_j\}$ subconverges to a solution in $\overline{\mathcal{S}}^+$ of (1) which determines a non-degenerate convex body.

To show that h is in fact positive when f is bounded from above we claim

$$
d_j = \inf_x h_j(x) \geqslant \delta > 0 \ .
$$

For, if not, we may suppose $d_j \to 0$ and the infimum is attained at the south pole. Then $u_j = (1+|x|^2)^{\frac{1}{2}} h(x_1, \cdots, x_n, -1)$ satisfies

$$
\det D^2 u_j = u^{p-1} g_j(x) \ , \quad x \in \mathbb{R}^n
$$

where

$$
g_j(x) = (1+|x|^2)^{-\frac{n}{2}-p} f_j\Big(\frac{x,-1}{\sqrt{1+|x|^2}}\Big) \ ,
$$

and $u_j(0) = \inf_x u_j \to 0$ as $j \to \infty$. Since all X_j's have bounded diameters and their inradii are positively bounded from below, there exists $R>0$ such that $u_j(x) \geqslant 1$ for $|x| \geqslant R$. Letting $j \to \infty$, we conclude that $\{u_j\}$ subconverges to a convex function u such that $u(x) \geqslant 1$ for $|x| \geqslant R$ and $u(0) = \inf u$. It satisfies, in the generalized sense of Aleksandrov,

$$
\det D^2 u \leqslant \Lambda u^{p-1} \ ,
$$

for some positive Λ.

Consider the zero set $Z = \{u = 0\}$ and $Z_\delta = \{u < \delta\}$. If $|Z| = 0$, then $|Z_\delta| \to 0$ as $\delta \to 0$. By comparing the normal image of u over $Z_\delta, N_u(Z_\delta)$, with the normal image of the cone whose base is $Z_\delta \times \{\delta\}$ and whose vertex is $(0,0)$, we have

$$
|N_u(Z_\delta)| \geqslant \frac{c\delta^n}{|Z_\delta|}
$$

On the other hand, we have

$$
\begin{aligned}
|N_u(Z_\delta)| &= \int_{Z_\delta} \det D^2 u dx \\
&\leqslant \Lambda \int_{Z_\delta} u^{p-1} dx \\
&\leqslant \Lambda \delta^{p-1} |Z_\delta| \ .
\end{aligned}
$$

Therefore,

$$\delta^{n-p+1} \leqslant C|Z_\delta|^2 \ .$$

Letting $\delta \to 0$ we have a contradiction.

If $|Z| > 0$, we take $x_0 \in \partial Z_\delta$ so that $(x - x_0) \cdot e_n \geqslant 0$ for all $x \in Z_\delta$. Let $Z_{\delta,\varepsilon} = \{x : u(x) \leqslant u(x_0) - \varepsilon(x - x_0) \cdot e_n\}$. One can select $\delta \to 0$ and $\varepsilon(\delta) \to 0$ such that

$$\inf \left\{ u(x) - \big(u(x_0) - \varepsilon(x - x_0) \cdot e_n\big) : x \in Z_{\delta,\varepsilon} \right\} \leqslant -\frac{1}{2}\delta$$

and $|Z_{\delta,\varepsilon}| \to 0$. Similarly as above, we can derive

$$\delta^{n-p+1} \leqslant C|Z_{\delta,\varepsilon}|^2$$

and the same contradiction holds.

§3. An Eigenvalue Problem

In this section we prove Theorem B. We shall first prove it for a positive, Hölder continuous f. Let $\overline{h}_\varepsilon$, $\varepsilon \in (0, 1)$, be the unique solution of (1) for $p = n + 1 + \varepsilon$ and let $\overline{X}_\varepsilon$ be the associated hypersurface. We dilate $\overline{X}_\varepsilon$ to a hypersurface X_ε whose enclosed volume is the same as the unit ball. Its support function, $h_\varepsilon = \Big[\dfrac{(n+1)}{\omega_n} V(\overline{X}_\varepsilon)\Big]^{-\frac{1}{n+1}} \overline{h}_\varepsilon$, satisfies

$$\det(h_{\varepsilon ij} + h_\varepsilon \delta_{ij}) = \lambda_\varepsilon h_\varepsilon^{n+\varepsilon} f(x), \tag{3.1}$$

where $\lambda_\varepsilon = V(\overline{X}_\varepsilon)^{\varepsilon/n+1}$. Here and below we also denote by $V(X)$ or $V(h)$ the volume of the associate convex body K. Consider $h_\varepsilon(x_0) = h_{\varepsilon \max}$. At x_0 we have

$$\lambda_\varepsilon h_\varepsilon^{n+\varepsilon}(x_0) f(x_0) \leqslant h_\varepsilon^n(x_0) \ ,$$

so

$$\begin{aligned}\lambda_\varepsilon &\leqslant \frac{1}{f(x_0)} h_{\varepsilon\max}^{-\varepsilon}(x_0)\\ &\leqslant \frac{1}{f(x_0)}\ ,\end{aligned}$$

since $V(X_\varepsilon) = \omega_n/(n+1)$. Similarly one gets a lower bound for λ_ε. We have

$$\frac{1}{\sup f} \leqslant \lambda_\varepsilon \leqslant \frac{1}{\inf f}\ . \tag{3.2}$$

It follows that

$$\begin{aligned}\frac{\omega_n}{n+1} &= V(X_\varepsilon)\\ &= \frac{\lambda_\varepsilon}{n+1}\int h_\varepsilon^{n+1} f \\ &\geqslant c h_{\varepsilon\max}^{n+1}\ ,\end{aligned} \tag{3.3}$$

by Proposition 1.3. Hence $\{h_\varepsilon\}$ is uniformly bounded in $\mathcal{S}^+$. By passing to a subsequence we may assume that $\{(h_\varepsilon, \lambda_\varepsilon)\}$ converges uniformly to (h_0, λ_0) as $\varepsilon \downarrow 0$. One can check that the proof of positivity of the support function in the last section still works for h_ε. Hence $h_0 \in \mathcal{S}^+$.

By the variational characterization of h_ε we have

$$\mathcal{I}_\varepsilon(h_\varepsilon) = \max\left\{\mathcal{I}_\varepsilon(h) : h \in \mathcal{S}^+\right\} \geqslant 0\ .$$

Letting $\varepsilon \downarrow 0$, h_0 satisfies

$$\mathcal{I}_0(h_0) = \max\left\{\mathcal{I}_0(h) : h \in \mathcal{S}^+\right\} \geqslant 0$$

But $\mathcal{I}_0(th_0) = t^{n+1}\mathcal{I}_0(h_0)$. So $\mathcal{I}_0(h_0) = 0$, i.e.,

$$\sup\{\int_{S^n}[h\det(h_{ij} + h\delta_{ij}) - \lambda_0 \int_{S^n} fh^{n+1}] = 0. \tag{3.4}$$

We have proved Part (a) of Theorem B. Now Part (b) can be established by an approximation argument. Note that (6) follows from (3.4).

As pointed out in the introduction, the uniqueness of the solution pair has already proved in [L]. When f is positive and continuous, an analytic proof can be given as follows. First, suppose that (λ_1, h_1) and (λ_2, h_2) are two solutions of (1) for $p = n+1$. By multiplying h_2 with a suitable constant, we may assume X_2 is contained inside X_1, with some point touching X_1. At this point, say, x_0, we have

$$\begin{aligned}\lambda_1 f(x_0) h_1^n(x_0) &= \det(h_{1ij} + h_1\delta_{ij})\\ &\geqslant \det(h_{2ij} + h_2\delta_{ij})\\ &= \lambda_2 f(x_0) h_2^n(x_0)\ .\end{aligned}$$

Hence $\lambda_1 \geqslant \lambda_2$. By symmetry we get $\lambda_1 = \lambda_2$.

Next, assume h_1 and h_2 are two different solutions of (1) with the same λ_0. By multiplying h_2 with a suitable constant we may assume the set $E = \{x \in S^n : h_1(x) > h_2(x)\}$ is an open set in the south hemisphere. We can always do this when h_2 is not a constant multiple of h_1. Now, let u_1 and u_2 be the restriction of h_1 and h_2 respectively on the tangent hyperplane of S^n at the south pole. Then both u_1 and u_2 satisfy

$$\begin{cases} \det D^2 u_i = g(x) u_i^n \text{ in } \Omega \\ u_1 = u_2 \text{ on } \partial\Omega \\ u_1 > u_2 \text{ in } \Omega \end{cases}$$

But this is impossible by the comparison principle.

§4. The Positive Case: $1 < p < n + 1$

In this section we prove Theorem C as well as its generalization. First of all, we consider an approximation problem to (1) : For any $\varepsilon \in (0, 1)$,

$$\det(h_{ij} + h\delta_{ij}) = f(x)\phi_\varepsilon(h), \tag{4.1}$$

where f is positive, in C^α for some $\alpha \in (0, 1)$ and ϕ_ε is a smooth, strictly increasing function satisfying $\phi_\varepsilon \geq \frac{\varepsilon}{2}$ and $\phi_\varepsilon(z) = z^{p-1} + \varepsilon$ when $z \geq 0$. The assumptions that $f \in C^\alpha$ and $\phi_\varepsilon \geq \frac{\varepsilon}{2}$ imply that the solution is $C^{2,\alpha}$ by [C2]. By the estimates (4.2), (4.5), and the volume estimate thereafter, the Hölder continuity of f can be removed by approximation.

We shall show that the sup-norm of any classical solution of this equation admits positive bounds from both sides independent of ε. Indeed, at $h(x_0) = h_{\max} > 0$, we have

$$\begin{aligned} h_{\max}^n &\geqslant \det(h_{ij}(x_0) + h(x_0)\delta_{ij}) \\ &= f(x_0)(\varepsilon + h_{\max}^{p-1}) \ . \end{aligned}$$

Hence

$$h_{\max} \geqslant (\inf f)^{\frac{1}{n+1-p}} \equiv C_1 \tag{4.2}$$

for all ε. On the other hand, the area and enclosed volume of X, the hypersurface associated to h, satisfy

$$\begin{aligned} A(X) &= \int_{S^n} \det(h_{ij} + h\delta_{ij}) \\ &\leqslant \varepsilon \|f\|_{L^1} + \int_{\{h \geqslant 0\}} f h^{p-1} \ , \end{aligned}$$

and

$$\begin{aligned} V(X) &= \frac{1}{n+1} \int_{S^n} h \ \det(h_{ij} + h\delta_{ij}) \\ &\geqslant \frac{1}{n+1} \Big[\varepsilon \int_{\{h<0\}} fh + \int_{\{h \geqslant 0\}} (\varepsilon + h^{p-1}) hf \Big] \end{aligned}$$

respectively. By the isopermetric inequality,

$$\Big(\varepsilon \|f\|_{L^1} + \int_{\{h \geqslant 0\}} f h^{p-1} \Big)^{\frac{n+1}{n}} \geqslant \omega_n^{\frac{1}{n}} \Big[\varepsilon \int_{\{h<0\}} fh + \int_{\{h \geqslant 0\}} (\varepsilon + h^{p-1}) hf \Big] \tag{4.3}$$

By Hölder and Young's inequalities,

$$\begin{aligned} \Big(\int_{\{h \geqslant 0\}} f h^{p-1} \Big)^{\frac{n+1}{n}} &\leqslant \Big[\Big(\int_{\{h \geqslant 0\}} f \Big)^{\frac{1}{p}} \Big(\int_{\{h \geqslant 0\}} f h^p \Big)^{\frac{p-1}{p}} \Big]^{\frac{n+1}{n}} \\ &\leqslant \delta \int_{\{h \geqslant 0\}} h^p f + C_\delta \Big(\int_{\{h \geqslant 0\}} f \Big)^{\frac{n+1}{n+1-p}} \end{aligned}$$

By choosing δ small, the first term on the right hand side can be absorbed to the right hand side of (4.3). Hence we have

$$C\big(1 + \|f\|_{L^1}\big)^{\frac{n+1}{n+1-p}} \geqslant \int_{\{h \geqslant 0\}} f h^p + \varepsilon \int_{\{h<0\}} hf \ , \tag{4.4}$$

for some constant C depending only on p and n. Using $p > 1$ and Proposition 1.3 we conclude that

$$h_{\max} \leqslant C_2 \ , \tag{4.5}$$

where C_2 depends on p, n and $\|f\|_{L^1}$. From (4.2) and (4.5), we have from Proposition 1.3 that

$$V(X) \geq C_0 > 0.$$

Now, we use a degree-theoretic argument to solve (4.1).

Lemma 4.1 *For each $h \in C(S^n)$, there exists a unique vector $\xi = \xi_h$, depending continuously on h, such that*

$$\int_{S^n} f(x) \phi_\varepsilon(h + \xi \cdot x) x_i = 0 \ , \ i = 1, \cdots, n+1 \ . \tag{4.6}$$

Proof Let

$$F(\xi) = \int_{S^n} f(x)\Phi(h + \xi \cdot x)dx, \tag{4.7}$$

where Φ is a primitive function of ϕ. By our assumption on ϕ it is clear that F tends to infinity uniformly in x as $|\xi| \to \infty$. Hence the minimum of F is attained and it satisfies (4.6). By the strictly increasing of ϕ we know that F is strictly convex, so there are no other critical points. $\square$

Given $h \in C(S^n)$, by (4.6) we can solve

$$\det(u_{ij} + u\delta_{ij}) = f(x)\phi_\varepsilon(h + \xi_h \cdot x) \tag{4.8}$$

to obtain a generalized solution u in $\mathcal{S}$ which is unique up to translation. We can fix it by requiring

$$\int f(x)\phi_\varepsilon(u)x_j = 0 \ , \quad j = 1, \cdots, n+1 \ .$$

By doing this we have defined a map T from $C(S^n)$ to $\mathcal{S}$ given by $u = Th$. Since the inclusion $\mathcal{S} \hookrightarrow C(S^n)$ is compact by the Blaschke selection theorem, T is a compact map. By replacing the f in (4.7) by $f_\lambda = (1-\lambda) + \lambda f$, we obtain in the same way a continuous family of compact mappings $T_\lambda, \lambda \in [0,1]$. Any fixed point h of T_λ is a generalized solution of (4.1) with $f_\lambda \phi_\varepsilon$ on its right. By the regularity results in [C1] and [C2] h belongs to $C^{2,\beta}$ for some β. Therefore, it satisfies (4.2) and (4.5) where C_1 and C_2 can be chosen to be independent of λ. So the Leray-Schauder degree $\deg(id - T_\lambda, M, 0)$, where $M = \{h \in C(S^n) : C_1 < h_{\max} < C_2\}$, are well-defined and all equal.

We shall show that $\deg(id - T_0, M, 0) = 1$. To prove this we consider another homotopy family of compact mapping S_μ, by solving

$$\det(u_{ij} + u\delta_{ij}) = (1-\mu) + \mu\phi_\varepsilon(h + \xi_h \cdot x),$$

and requiring the solution to satisfy the above integrability condition.

When $\mu = 0$, we define $S_\mu h \equiv 1$.

Lemma 4.2 *Let h be a fixed point of S_μ, $\mu \in (0,1]$. Then h is positive.*

Proof Suppose $h_{\min}$ is attained at the south pole. For any $x \in S^n$, let $\tilde{x} = x - 2e_{n+1}$, the reflection of x with respect to the hyperplane $x_{n+1} = 0$.

When $h_{\min} \leqslant 0$, it is clear that $h(\tilde{x}) > h(x)$. Since ϕ is strictly increasing, $\phi(h(\tilde{x})) > \phi(h(x))$. Hence

$$\int_{S^n_+} \big(1 - \mu + \mu\phi(h)\big) x_{n+1} > \int_{S^n_-} \big(1 - \mu + \mu\phi(h)\big) |x_{n+1}| \ ,$$

contradicting our definition of S_μ. Hence h must be positive. $\square$

By this lemma, the integrals over $\{h < 0\}$ in (4.3) are vacuous. Hence the constant C_2 in (4.5) can be chosen to be independent of μ and λ. On the other hand, it is readily seen that C_1 can be also chosen independent of μ and λ. As a result,

$$\begin{aligned}\deg(id - T_1, M, 0) &= \deg(id - T_0, M, 0)\\ &= \deg(id - S_1, M, 0)\\ &= \deg(id - S_0, M, 0)\\ &= 1 \ .\end{aligned}$$

We have shown that (4.1) admits a solution h_ε. Using the uniform bounds (4.2) and (4.5), we can extract a sequence $\{h_{\varepsilon_j}\}$ which converges uniformly to a generalized solution h of (1).

The volume estimate

$$\begin{aligned}V(X_{\varepsilon_j}) &\geqslant \frac{1}{n+1} \int_{\{h_{\varepsilon_j} \geqslant 0\}} h_{\varepsilon_j}(\varepsilon_j + h^{p-1}_{\varepsilon_j}) + \frac{\varepsilon_j}{n+1} \int_{\{h_{\varepsilon_j} < 0\}} h_{\varepsilon_j}\\ &\geqslant C\big(h^p_{\varepsilon_j \max} - \varepsilon_j h_{\varepsilon_j \max}\big)\end{aligned}$$

implies that the convex hypersurface determined by h is non-degenerate. Moreover, we've

$$\begin{aligned}\omega_n &\geqslant \int_{X_{\varepsilon_j} \cap \{h_{\varepsilon_j} \leqslant \delta\}} K_{\varepsilon_j} ds\\ &= \int_{X_{\varepsilon_j} \cap \{h_{\varepsilon_j} \leqslant \delta\}} \phi^{-1}_{\varepsilon_j} f^{-1}(x) ds\\ &\geqslant C(\varepsilon_j + \delta^{p-1})^{-1} \mathcal{H}^n\{X_{\varepsilon_j} : h_{\varepsilon_j} \leqslant \delta\} \ .\end{aligned}$$

So,

$$\begin{aligned}\mathcal{H}^n\big\{X : h \leqslant \delta\big\} &\leqslant \overline{\lim_{j\to\infty}} \mathcal{H}^n\big\{X_{\varepsilon_j} : h_{\varepsilon_j} \leqslant \delta\big\}\\ &\leqslant C\delta^{p-1} \ .\end{aligned}$$

Letting δ tend to 0^+, we conclude that h is non-negative and

$$\mathcal{H}^n\big\{X : h = 0\big\} = 0. \tag{4.9}$$

For a finite Borel measure m not concentrating on hemisphere, we take $dm_j = f_j(x)dx$, $\{m_j\}$ converges weakly to m, and let h_j be the corresponding non-negative solutions. By (4.4), h_j are uniformly bounded by a constant depending on the L^1- norm of f_j, which in terms are controlled by the total measure of m. On the other hand, by integrating the equation and using Proposition 1.3, $\{h_{j\,\max}\}$ is uniformly bounded below by a positive constant. Hence $\{h_j\}$ subconverges to a non-trivial solution of (3) for m. From the volume formula we see that it determines a non-degenerate convex body. The proof of Theorem C is complete.

Next we consider the more general equation (2.4). We shall take f to be a non-negative function in $C^\alpha(S^n \times \mathbb{R})$ increasing in z and satisfying either

$$f(x,z) > 0,\ \ (x,z) \in S^n \times \mathbb{R}, \tag{4.10}$$

or

$$\liminf_{z \to 0^+} \frac{f(x,z)}{z^n} > 1,\ \ f(x,0) = 0. \tag{4.11}$$

It also satisfies

$$\lim_{z \to \infty} \frac{f(x,z)}{z^n} = 0 \tag{4.12}$$

and

$$\lim_{z \to \infty} f(x,z) = \infty\ , \tag{4.13}$$

uniformly in S^n.

Proposition 4.1 *(a) Assume that (4.10), (4.12) and (4.13) hold. Then (2.4) has a solution in $\mathcal{S}^{2,\alpha}$.*
(b) Assume that (4.11), (4.12) and (4.13) hold. Then (2.4) has a non-negative generalized solution in $\mathcal{S}$.

We shall discuss the proof of (b) only. Part (a) can be proved by a similar way.

Instead of f let's consider (2.4) with $f_\varepsilon = f + \varepsilon$. Let $h = h_\varepsilon$ be a classical solution of (2.4) for $f = f_\varepsilon$. From the above degree argument, it suffices to derive two-sided uniform bounds for $h_{\max} = \sup h_\varepsilon$. A lower positive bound for $h_{\max}$ can be obtained in the same way as in (4.2). In the following we give an upper bound for $h_{\max}$.

Let E be the minimum ellipsoid of X, the hypersurface determined by h. Without loss of generality we may assume

$$E = \{x : \Sigma |x_i - x_i^0|^2 / a_i^2 = 1\} ,$$

where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n+1}$. Recall that by John's lemma[C1]

$$\frac{1}{n+1}(E - x^0) \subseteq K - x^0 \subseteq E - x^0 .$$

Let Ω be the projection of X onto the hyperplane $x_1 = 0$ and let z be the center of the minimum ellipsoid of Ω. Let $\Omega_t = \{z + t(x - z) : x \in \Omega\}$. The subset of X, $F = \{x \in X : x \text{ projects into } \Omega_{1/2n}\}$ consists of two disjoint pieces, F_1 and F_2, one of which satisfies $h \leqslant Ca_1$. Taking it to be F_1, say, we note that its n-dimensional Hausdorff measure $\geqslant Ca_2 \cdots a_{n+1}$. Using $f_\varepsilon(x, h) \leqslant f_\varepsilon(x, Ca_1) \leqslant C(1 + a_1^n)$, we have

$$\begin{aligned} \int_{F_1} K ds &= \int_{F_1} f_\varepsilon^{-1}(x, h) ds \\ &\geqslant C \frac{a_2 \cdots a_{n+1}}{1 + a_1^n} . \end{aligned} \tag{4.14}$$

To obtain an upper bound on the left hand side, we represent F_1 as a graph $x_1 = u(y)$, $y = (y_2, \cdots, y_{n+1}), \in \Omega$. By convexity $|\nabla u| \leqslant C$ on $\Omega_{1/2n}$. So

$$\begin{aligned} \int_{F_1} K ds &= \int_{\Omega_{1/2n}} \frac{\det D^2 u}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}} (1 + |\nabla u|^2)^{\frac{1}{2}} dy \\ &\leqslant C \int_{\Omega_{1/2n}} \det D^2 u dy \\ &\leqslant C |N_u(\Omega_{1/2n})| , \end{aligned}$$

where $N_u(\Omega_{1/2n})$ is the normal image of u over $\Omega_{1/2n}$. By Lemma 4.3 below, we have

$$\begin{aligned} \int_{F_1} K ds &\leqslant C a_1^n / |\Omega_{\frac{1}{2n}}| \\ &\leqslant \frac{C a_1^n}{a_2 \cdots a_{n+1}} . \end{aligned}$$

Putting this estimate into (4.14), we get

$$(1 + a_1^n) a_1^{n+2} \geq C(a_1 ... a_{n+1})^2 \ \ (\approx CV(X)^2). \tag{4.15}$$

Using (4.13) in

$$V(X) \geqslant \frac{1}{n+1} \int_{\{h \geqslant 0\}} h f_\varepsilon + \frac{\varepsilon}{n+1} \int_{\{h < 0\}} h$$

we have

$$V(X)h_{\max}^{-1} \to \infty,$$

as $h_{\max} \to \infty$. In view of (4.15), it implies $Ca_1 \geqslant a_{n+1}$ for large a_{n+1}. Denoting the maximal width of X by

$$\omega(X) = \sup_x \frac{1}{2}\big(h(x) + h(-x)\big),$$

we have

$$V(X) \geqslant C\omega(X)^{n+1} \; . \tag{4.16}$$

Lemma 4.3 *Let Ω be a convex domain whose center of its minimum ellipsoid is the origin. Let u be a convex function whose boundary value are non-positive and let $\Phi = \big\{(x, \varphi(x)) : x \in \Omega\big\}$ be the convex cone with vertex at $(0, u(0))$ and $\varphi(x) = 0$ on $\partial\Omega$. Then for any $0 < t < 1$, there exists a constant $C = C(n, t)$ such that*

$$N_u(\Omega_t) \subseteq N_{C\varphi}(\Omega),$$

where $\Omega_t = \{tx : x \in \Omega\}$.

Proof For any $t > 0$, there is some C such that $u \leqslant C\varphi = 0$ on $\partial\Omega$ and $u \geqslant C\varphi$ on Ω_t. Letting $\Omega' = \{u < C\varphi\}$, we have

$$\begin{aligned} N_u(\Omega_t) &\subseteq N_u(\Omega') \\ &\subseteq N_{C\varphi}(\Omega') \\ &\subseteq N_{C\varphi}(\Omega) \; . \end{aligned}$$

□

Returning to the proof, now we relate the maximal width to $h_{\max}$.

Lemma 4.4 *There exists a constant C depending on f and n, independent of ε, such that*

$$\omega(X) \geqslant \frac{1}{4} h_{\max}$$

if $h_{\max} \geqslant C$.

Proof We have

$$\begin{aligned} \omega_n &\geqslant \int_{\{h<0\}} K ds \\ &\geqslant \varepsilon^{-1} \mathcal{H}^n \{X : h < 0\} \; . \end{aligned}$$

Hence $\mathcal{H}^n\{X : h < 0\} \leqslant \omega_n$ for all $\varepsilon \in (0,1)$. If $\omega_X \leqslant \dfrac{1}{4}h_{\max}$, by an elementary geometric argument,

$$\begin{aligned} V(X) &\leqslant C_n \mathcal{H}^n\{X : h < 0\} h_{\max} \\ &\leqslant C_n' h_{\max} \ . \end{aligned}$$

However, this is impossible for large $h_{\max}$ because $V(X)h_{\max}^{-1}$ becomes large with $h_{\max}$. $\square$

So, by (4.16) we have

$$V(X) \geqslant C h_{\max}^{n+1} \ , \tag{4.17}$$

when $h_{\max}$ is large. However, on the other hand, by (4.12)

$$\begin{aligned} V(X) &\leqslant C h_{\max} \sup f_\varepsilon(x, h_{\max}) \\ &= o(h_{\max}^{n+1}) \end{aligned}$$

as $h_{\max} \to \infty$, contradiction holds. We have derived an upper bound on h_ε independent of ε.

§5. The Subcritical Case $q \in (0, n+2)$

In this section we prove Theorem D. Letting $q = -(p-1) \in (0, n+2)$, we consider the functional

$$\mathcal{J}(h) = \begin{cases} \dfrac{1}{q-1} \int_{S^n} f h^{1-q}, & q \neq 1 \\ -\int_{S^n} f \log h \ , & q = 1, \end{cases}$$

for $h \in \mathcal{S}^+$. We shall use the Blaschke-Santalo inequality

$$\sup_{h \in \mathcal{S}} \inf_{\xi \in K} V(h) \int_{S^n} \frac{1}{(h - \xi \cdot x)^{n+1}} \leq \frac{\omega_n^2}{n+1},$$

where $K = K_h$ is the convex body determined by h and the infimum is taken over all ξ satisfying $h - \xi \cdot x \in \mathcal{S}^+$. Note that the left hand side of this inequality is invariant under all affine transformations. It is known that equality in this inequality if and only if K is a centered ellipsoid.

To find a solution of (1) in this case, we consider the maximization problem

$$\sup_{h \in \mathcal{S}^+} \left\{ \inf_{\xi \in K_h} \mathcal{J}(h - \xi \cdot x) : V(h) = 1 \right\} \tag{5.1}$$

To verify that a maximizer h satisfies the corresponding Euler equation, one will need its positivity. However, when $0 < q < n$, a maximizer of (5.1) may fail to be positive. Therefore, we consider instead an approximation problem first. For $\varepsilon > 0$ small, let $\varphi = \varphi_\varepsilon$ be a positive, convex, monotone decreasing function on $(0, \infty)$ such that

$$\varphi(z) \begin{cases} = \dfrac{1}{q-1} z^{1-q} (q \neq 1), \quad -\log z (q = 1), \text{ for } z \geqslant 1 \\ \geq \frac{1}{n} z^{-n}, \quad 0 < z < \varepsilon \\ \leq \frac{1}{q'-1} z^{1-q'}, \quad 0 < z < \dfrac{1}{4}, \end{cases}$$

where $\varepsilon \in (0, 1/4)$ and $q' \in [n+1, n+2)$ is a fixed constant. For $q \in [n+1, n+2)$, we can take

$$\varphi(z) = \frac{1}{q-1} z^{1-q} \ ,$$

for all $z > 0$. Let

$$\tilde{\mathcal{J}}(h) = \int_{S^n} f \varphi(h) \ .$$

For any $h \in \mathcal{S}^+$, it is clear that $\tilde{\mathcal{J}}(h - \xi \cdot x) \to \infty$ as $\xi \in K_h$ and $\xi \to \partial K_h$. By the convexity of φ, we conclude that there exists a unique $\xi \in K^0$ which attains $\inf\{\tilde{\mathcal{J}}(h - \xi \cdot x) : \xi \in K\}$. Let

$$c = \sup_{h \in \mathcal{S}^+} \left\{ \inf_{\xi \in K} \tilde{\mathcal{J}}(h - \xi \cdot x) : V(h) = 1 \right\} . \tag{5.2}$$

We estimate the critical value c as follows. First, taking $h \equiv 1$, we've

$$\begin{aligned} c &\geqslant \inf_{\xi \in B_1} \tilde{\mathcal{J}}(1 - \xi \cdot x) \\ &\geqslant \int_{S^n} f \varphi(2) \\ &\geqslant \begin{cases} C_1 \varphi(2) > 0 \ , & q \in (1, n+2) \\ -C_1' \ , & q \in (0, 1] \ , \end{cases} \end{aligned} \tag{5.3}$$

since $|\xi \cdot x| \leqslant 1$ and φ is monotone decreasing. On the other hand, using $\varphi(z) \leqslant C + z^{-n-1}$ for $z > 0$ and the Blaschke-Santalo inequality,

$$\inf_{\xi \in K} \tilde{\mathcal{J}}(h - \xi \cdot x) \leq C + \inf_{\xi \in K} \int \frac{f}{(h - \xi \cdot x)^{n+1}} \leq C$$

for all $h \in \mathcal{S}^+$, $V(h) = 1$. Hence

$$c \leqslant C_2 \tag{5.4}$$

where the constants in (5.3) and (5.4) are independent of ε.

Lemma 5.1 *The maximization problem (5.2) has a solution.*

Proof Let $\{h_j\} \subseteq \mathcal{S}^+$, $V(h_j) = 1$ be a maximizing sequence. We claim

$$d_j \leqslant C \tag{5.5}$$

for some constant $C > 0$ independent of ε, where d_j is the diameter of X_j, the convex hypersurface determined by h_j. Observe that $\inf\{\tilde{\mathcal{J}}(h - \xi \cdot x)$: $\xi \in K_h\}$ is invariant under any translation of X_h, we may assume the minimium ellipsoid of X_j, E_j, is centered at the origin. Then $\frac{1}{n+1} E_j \subseteq X_j \subseteq E_j$ and $\frac{1}{n+1} h_{E_j} \leqslant h_j \leqslant h_{E_j}$. Suppose on the contrary that $d_j \to \infty$ as $j \to \infty$. We set $S^n = S_1 \cup S_2 \cup S_3$ where

$$\begin{aligned}
S_1 &= S^n \cap \{h_{E_j} < \delta\} \\
S_2 &= S^n \cap \{\delta < h_{E_j} < 1/\delta\}\ , \text{ and} \\
S_3 &= S^n \cap \{h_{E_j} > 1/\delta\}\ ,
\end{aligned}$$

where $\delta \in (0, 1/4)$ is a fixed small constant. Then

$$\begin{aligned}
\inf_{\xi \in K_j} \tilde{\mathcal{J}}(h_j - \xi \cdot x) &\leqslant \tilde{\mathcal{J}}(h_j) \\
&\leqslant \tilde{\mathcal{J}}(\frac{1}{n+1} h_{E_j}) \\
&= \int_{S^n} f \varphi(\frac{h_{E_j}}{n+1})
\end{aligned}$$

by the monotonicity of φ. As $d_j \to \infty$, we have, for any fixed δ,

$$\begin{aligned}
\int_{S_1} f \varphi\left(\frac{h_{E_j}}{n+1}\right) &\leqslant \int_{S_1} \frac{C}{h_{E_j}^{q'-1}} \\
&\leqslant C \left(\int \frac{1}{h_{E_j}^{n+1}} \right)^{\frac{q'-1}{n+1}} |S_1|^{\frac{n+2-q'}{n+1}} \\
&\leqslant C |S_1|^{\frac{n+2-q'}{n+1}} \\
&\longrightarrow 0,
\end{aligned}$$

by the Blaschke-Santalo inequality. Noting that we also have $|S_2| \to 0$ as $d_j \to \infty$, we have

$$\begin{aligned}
\int_{S^n} f \varphi\left(\frac{h_{E_j}}{n+1}\right) &= o(1) + \int_{S_3} f \varphi\left(\frac{h_{E_j}}{n+1}\right) \\
&\leqslant o(1) + C \varphi\left(\frac{1}{(n+1)\delta}\right) .
\end{aligned}$$

In other words,

$$\inf_{\xi \in K_j} \tilde{\mathcal{J}}(h_j - \xi \cdot x) \leqslant o(1) + C\varphi\left(\frac{1}{(n+1)\delta}\right) ,$$

when C is independent of j, δ and ε. Sending $j \to \infty$ we obtain $c \leqslant C\varphi(\delta^{-1})$. As $\varphi(\delta^{-1})$ tends to 0 $(1 < q < n+2)$ or to $-\infty$ $(0 < q \leqslant 1)$, this inequality is in conflict with (5.3). Hence (5.5) holds.

Now, by passing to a converging subsequence we conclude that $\{h_j\}$ converges to a maximizer h of (5.2), whose diameter satisfies the bound (5.5). By a translation we may suppose that $\inf_{\xi \in K_h} \tilde{\mathcal{I}}(h - \xi \cdot x)$ is attained at $\xi = 0$. Our assumption of φ implies that $h > 0$ on S^n. $\square$

Next, we consider the variation of the volume functional. Since the hypersurface detemined by the maximizer may not be strictly convex, one must be cautious about the variation. For any $h \in \mathcal{S}^+$ and any $\eta \in C^\infty(S^n)$, let

$$K_t = \{p : p \cdot x \leqslant (h + t\eta)(x) \ , \ x \in S^n\}$$

$X_t = \partial K_t$,and h_t the support function of X_t. Note that $h_0 = h$, $X_0 = X$ and $K_0 = K$.

Lemma 5.2 *Suppose that X is C^1 at p. Then*

$$\lim_{t \to 0^+} \frac{h_t(x_0) - h(x_0)}{t} = \eta(x_0) ,$$

where x_0 is the unit outer normal of X at p.

Proof Choose a coordinate system so that p is the origin and $X \subseteq \{x_{n+1} \geqslant 0\}$. Then $\{x_{n+1} = 0\}$ is the tangent plane at p and x_0 is the south pole.

Since X at C^1 at p, $h(x) > 0$ for all $x \neq x_0$. Therefore, for any $x \neq x_0$

$$h_t(x) > h_t(x_0) \tag{5.6}$$

for sufficiently small t. By the definition of h_t and (5.6), there exists $x_t \in S^n$ such that $h_t(x_0) = (h + t\eta)(x_t)$ with $x_t \to x_0$ as $t \to 0^+$. Hence

$$\varliminf_{t \to 0^+} \frac{h_t(x_0) - h(x_0)}{t} \geqslant \eta(x_0) .$$

On the other hand, by definition we have $h_t \leqslant h + t\eta$. So

$$\varlimsup_{t \to 0^+} \frac{h_t(x_0) - h(x_0)}{t} \leqslant \eta(x_0) .$$

$\square$

Corallary 5.3 *We have*

$$\lim_{t\to 0^+} \frac{1}{t}\left(V(h_t) - V(h)\right) = \int_{S^n} \eta d\mu \ ,$$

where μ *is the area measure of* X.

Proof Choose an interior point of X as the origin and represent X as a radial graph. Then X is C^1 a.e.. So,

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t}\left(V(h_t) - V(h)\right) &= \int_X \eta \\ &= \int_{S^n} \eta d\mu. \end{aligned}$$

□

Corallary 5.4 *Let* $h = h_\varepsilon$, $X = X_{h_\varepsilon}$ *be the maximizer in Lemma 5.1. If* X *is* C^1, *then* h *is a generalized solution of*

$$\det(h_{ij} + h\delta_{ij}) = -\frac{1}{\lambda} f\varphi'(h) \ ,$$

where by $V(X) = 1$,

$$\lambda = -\frac{1}{n+1}\int_{S^n} fh\varphi' > 0 \ .$$

Proof For any given $\eta \in C^\infty(S^n)$, let K_t, X_t, h_ε as in Lemma 5.2. Let $\alpha(t) > 0$ be such that

$$V\left(\alpha(t)h_t\right) = 1 \ .$$

Then

$$\alpha'(0) = \frac{-1}{n+1}\int_{S^n} \eta d\mu \ .$$

Since X in C^1, it follows from Lemma 5.2 that

$$\lim_{t\to 0^+} \frac{h_t - h}{t} = \eta \ .$$

As h is a maximizer,

$$\lim_{t_j\to 0^+} \frac{\tilde{\mathcal{J}}(t_j) - \tilde{\mathcal{J}}(0)}{t_j} \leqslant 0$$

for any convergent subsequence $\{h_{t_j}\}$, where

$$\tilde{\mathcal{J}}(t) = \inf_\xi \int_{S^n} f\varphi\left(\alpha(t)h_t - \xi\cdot x\right) \ .$$

Suppose the infimum is attained at $\xi(t)$. From the assumption on φ we know that ξ is Lipschitz continuous. Without loss of generality, let's assume

$$\lim_{t_j\to 0}\frac{\xi(t_j)-\xi(0)}{t_j}=\alpha^*.$$

Therefore,

$$\int f\varphi'\left(\alpha'(0)h+\eta+\alpha^*\cdot x\right)\leqslant 0\ .$$

Recall that the infimum of $\int_{S^n} f\varphi(h-\xi\cdot x)$ is attained at $\xi=0$. We have

$$\int_{S^n} f\varphi' x_i=0\ ,\ \ i=1,\cdots,n+1.$$

Therefore,

$$\int_{S^n} f\varphi'\left(\alpha'(0)h+\eta\right)\leqslant 0\ .$$

It follows that

$$\lambda\int_{S^n}\eta d\mu+\int f\varphi'\eta\leqslant 0\ .$$

Replacing η by $-\eta$ we see that

$$\lambda\int\eta d\mu+\int f\varphi'\eta=0$$

for all $\eta\in C^\infty(S^n)$. □

It remains to show that the maximizer X is a C^1-hypersurface.

Lemma 5.5 *The Gauss curvature of X is bounded below in the generalized sense by a positive constant C.*

Proof By a proper rotation of axes, we may assume a fixed point p on X is located on the negative x_{n+1}-axis. Near p, X is the graph of a convex function u. Let D be the projection of X onto $\{x_{n+1}=0\}$. For any closed convex set $\Omega\subset\subset D$ containing the origin, let $\omega\subset X$ be the graph of u over Ω. Let $\omega^*=G(\omega)$, where G is the Gauss mapping of X. Then ω^* is a closed subset in S^n.

Let K be the convex body bounded by X and K_t the convex hull of $K\cup N_t(\omega)$ where $N_t(\omega)=\{p: dist(p,\omega)<t\}$. Then, $X_t=\partial K_t$ and its support function h_t, satisfy

$$\lim_{t\to 0^+}\frac{V(h_t)-V(t)}{t}\geqslant|\omega|\ ,$$

and

$$\lim_{t\to 0^+} \frac{h_t(x)-h(x)}{t} = 1 \ , \ \forall x \in \omega^* \ .$$

Observe that for $y \notin \omega^*$, $h_t(y) = h(y)$ for sufficiently small t. Hence

$$\lim_{t\to 0^+} \frac{h_t(y)-h(y)}{t} = 0 \ , \ \forall y \in S^n \backslash \omega^* \ .$$

Denote $\eta(x) = 1$ for $x \in \omega^*$ and $\eta(x) = 0$ for $x \in S^n\backslash\omega^*$. We have

$$\lim_{t\to 0^+} \frac{h_t(x)-h(x)}{t} = \eta(x) \ .$$

Let $\alpha(t)$ be defined as before. Then

$$\begin{aligned}
\alpha'(0) &= \lim_{t\to 0^+} \frac{\alpha(t)-\alpha(0)}{t} \\
&= \frac{-1}{n+1} \lim_{t\to 0^+} \frac{V(h_t)-V(h)}{t} \\
&\leqslant -\frac{|\omega|}{n+1} \ ,
\end{aligned}$$

where the limit can be taken for any convergent subsequence. Since h is maximizing, we have

$$\int_{S^n} f\varphi'(\alpha'(0)h+\eta) \leqslant 0 \ ,$$

as before. In other words,

$$-\int_{\omega^*} f\varphi' \geqslant -\frac{1}{n+1}\int_{S^n} f\varphi' h|\omega| \ ,$$

and so

$$\frac{|\omega^*|}{|\omega|} \geqslant C \ ,$$

where C depends on the bounds on h, $|\varphi'|$ and f, and hence on ε. □

Lemma 5.6 *The Gauss curvature of X is bounded above in the generalized sense by some constant C.*

Proof Let $X' = \{p \in X : G(p)$ lies in the open south hemisphere$\}$ and $X'' = X\backslash X'$. Then X' is the graph of a convex function u defined inside some D as described in the previous lemma. Let $u^*(x) = h(x,-1)$, where h is the support function of X, be the Legendre transform of u. Denote its graph by X^*. First we prove

$$\det D^2 u^* \geqslant C \ , \tag{5.7}$$

in the generalized sense.

For any closed convex set $C^* \subseteq \mathbb{R}^n$, let $C = N_{u^*}(\Omega^*)$, $\omega = \{(x, u(x)) : x \in C\}$, and $u_t^* = \sup \ell$ where the supremum is taken among all linear functions ℓ satisfying $\ell \leqslant u^*$ in $\mathbb{R}^n$ and $\ell \leqslant u^* - t$ in C^*. Let u_t be the Legendre transform of u_t^* and X_t' its graph, and let K_t be the convex body bounded by X_t' and X'', and denote $X_t = \partial K_t$, $h_t = h_{X_t}$. Since C^* is closed, we have

$$u_t(x) = u(x) + t \ , \ \forall x \in C$$

and

$$u_t(x) = u(x) \ , \ \forall x \notin C$$

for sufficiently small t. Hence

$$\lim_{t \to 0^+} \frac{V(h_t) - V(h)}{t} = -|C| \ .$$

Let $\omega^* = G(\omega)$,i.e.,

$$\omega^* = \{p \in S^n : -\frac{1}{p_{n+1}}(p_1, \cdots, p_n) \in C^*\} \ ,$$

such that C^* is the radial projection of ω^* onto $\{x_{n+1} = -1\}$. We have

$$\begin{aligned} \lim_{t \to 0^+} \frac{h_t(p) - h(p)}{t} &= -p_{n+1} \lim_{t \to 0^+} \frac{u_t^*(p') - u^*(p')}{t} \\ &= p_{n+1} \ , \end{aligned}$$

for any $p \in C^*$ and $p' = (p_1, \cdots, p_n)$. By our construction, h_t is non-increasing in t. Hence

$$\lim_{t \to 0^+} \frac{h_t(p) - h(p)}{t} \leqslant \eta(p) \ ,$$

where $\eta(p) = p_{n+1}$ if $p \in C^*$ and vanishes if $p \in S^n \backslash C^*$.

Let $\alpha(t)$ be defined as above. Hence,

$$\begin{aligned} 0 &\geqslant \lim_{t \to 0^+} \frac{\tilde{\mathcal{J}}(t) - \tilde{\mathcal{J}}(0)}{t} \\ &\geqslant \alpha'(0) \int_{S^n} f\varphi' h + \lim_{t \to 0^+} \int_{S^n} f\varphi' \frac{h_t(p) - h(p)}{t} \\ &\geqslant \frac{|C|}{n+1} \int_{S^n} f\varphi' h + \int_{S^n} f\varphi' \eta \ . \end{aligned}$$

It follows that

$$|C| \geqslant C_1 |\omega^*| \geqslant C_2 |C^*| \ , \tag{5.8}$$

for some constants C_1 and C_2, so (5.7) holds.

Now, the lemma follows from (5.7). Indeed, for any p on X, by a rotation we bring it to the negative x_{n+1}-axis, and represent X as the graph of some convex function u over D. As before, let u^* be the Legendre transform of u and X^* its graph.

Let $\Omega \subset\subset D$ be a closed convex set containing the origin and $\omega \subset X$ be the graph of u over Ω. Let $\Omega^* = N_u(\Omega)$ and ω^* the graph of u^* over Ω^*. Then Ω^* and ω^* are both closed sets. Let $\tilde{\Omega} = N_{u^*}(\Omega^*)$. Then $\tilde{\Omega}$ is closed and $\Omega \subseteq \tilde{\Omega}$. Let $\Omega' = \{x \in \tilde{\Omega}\backslash\Omega : u \text{ is not } C^1 \text{ at } x\}$ and $\Omega'' = \{x \in \tilde{\Omega}\backslash\Omega : u \text{ is } C^1 \text{ at } x\}$. By convexity and a lemma of Aleksandrov, Ω' is of measure zero and $N_u(\Omega'')$ is also of measure zero since u is not strictly convex at any $x \in \Omega''$. It follows that $|\Omega''| = 0$ by (5.8). Hence $|\tilde{\Omega}| = |\Omega|$. So, by (5.7),

$$\frac{|\Omega|}{|\Omega^*|} = \frac{|\tilde{\Omega}|}{|\Omega^*|} \geq C \ .$$

Hence det $D^2u \leq C$ in the generalized sense. □

Now, we can prove Theorem D. Let $\varphi = \varphi_\varepsilon$ be chosen as above. We may further assume that φ satisfies

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(z) = \varphi_0(z) = \begin{cases} -\dfrac{1}{q-1} z^{1-q} \ , & q \neq 1 \\ -\log z \ , & q = 1 \ , \end{cases}$$

and

$$\lim_{\varepsilon\to 0} \varphi'_\varepsilon(z) = z^{-q} \ ,$$

uniformly on every compact subset of $(0, \infty)$. Let $h = h_\varepsilon$ be the maximizer in Lemma 5.1. By Lemmas 5.5 and 5.6, the Gauss curvature of $X = X_\varepsilon$ is pinched between two positive constants. By [C1], X_ε is C^1 and strictly convex. By Corollary 5.4, h_ε satisfies the equation

$$\det(h_{ij} + h\delta_{ij}) = -\frac{1}{\lambda_\varepsilon} f \varphi'_\varepsilon(h) \ , \tag{5.9}$$

when

$$\lambda_\varepsilon = -\frac{1}{n+1} \int_{S^n} f h_\varepsilon \varphi'_\varepsilon > 0 \ .$$

The definition of φ_ε implies that h_ε is positive, and hence the full regularity of h_ε follows from the general theory.

By passing to subsequences, we may suppose that $\lambda_\varepsilon \to \lambda_0$ and $h_\varepsilon \to h_0$ as $\varepsilon \to 0$. Obviously, the volume enclosed by X_0, the hypersurface determined by h_0, is equal to 1 and X_0 has bounded diameter. Also we know that $\inf \int_{S^n} f\varphi_0(h_0 - \xi \cdot x)$ is attained at $\xi = 0$.

We claim $\lambda_0 < \infty$. For, if $\lambda_0 = \infty$, then

$$\det(h_{0ij} + h_0\delta_{ij}) = 0 \text{ on } \{h_0 > 0\} .$$

In other words, the area measure vanishes on $\{h_0 > 0\}$. This is impossible. On the other hand, $\lambda_0 > 0$. For, otherwise we'll have

$$\begin{aligned} -\int_{S^n} f\varphi'_\varepsilon(h_\varepsilon) &= \lambda_\varepsilon \int_{S^n} det(h_{\varepsilon ij} + h_\varepsilon \delta_{ij}) \\ &= \lambda_\varepsilon |X_\varepsilon| \\ &\longrightarrow 0 . \end{aligned}$$

Again this is impossible.

Finally, to show that h_0 solves (1) we look at the Gauss curvature of X_ε, which is given by

$$K_\varepsilon = -\frac{\lambda_\varepsilon}{f}\frac{1}{\varphi'_\varepsilon(h_\varepsilon)} .$$

By the weak convergence of the curvature measure, the Gauss curvature of X_0 satisfies

$$K = \frac{\lambda_0}{f} h_0^q \tag{5.10}$$

Hence, after a suitable scaling αh_0 solves (1) where $\alpha = \lambda_0^{-1/q+n}$.

When $q \in [n, n+2), h_0 > 0$; this follows by taking integration of (5.9) and observing that the integral of $\det(h_{\varepsilon ij} + h_\varepsilon \delta_{ij})$ over S^n is uniformly bounded. The proof of Theorem D is completed.

We remark that the solution in this case is in general not unique. Let's take $n = 1$, $p < 3$ and close to 3, and

$$f(x) = 2 + \cos 4x \ , \ x \in [0, 2\pi)$$

Then f has strict maxima at $x = 0,\ \pi/2,\ \pi,\ 3\pi/2$. If h solves (1), so does $h(x + \pi/2)$. Consider the maximizer of

$$\beta_q = \sup_{h \in \mathcal{S}^+} \left\{ \inf_\xi \int \frac{1}{(h - \xi \cdot x)^{q-1}} \ , \ V(h) = \frac{1}{n+1}\omega_n \right\}$$

By the Blaschke-Santalo inequality

$$\lim_{q\to 3}\beta_q = \sup f = 3\ .$$

By Lemma 5.1, the supremum is attained by h_q. Let X_q be the corresponding convex curve. We have $X_q\subset\{|x|<\delta\}$ or $X_q\subset\{|y|<\delta\}$ with $\delta\to 0$ as $q\to n+2$. Hence $h_q(x)\neq h_q(x+\pi/2)$. So there are at least two solutions. When $q>n+2$, interesting non-uniqueness examples for (1), in the special case $f\equiv 1$, can be found in Andrews[A4].

When $q>n+2$, β_q is unbounded. Instead one may consider the minimization problem

$$\inf_{h\in\mathcal{S}^+}\{\int_{S^n}\frac{f}{h^{q-1}},\quad V(h)=1\}. \tag{5.11}$$

We have

$$V(h)\int_{S^n}\frac{1}{h^{q-1}}\geq C. \tag{5.12}$$

Therefore it is easy to prove there is a minimizer. However a minimizer may not be a solution of (1). Indeed it is known that when $n=1$, the best constant in this inequality is attained by any triangles containing the origin (see Schneider [S]).

§6. Proof of Theorem E

When $-n+1<p<n+1$ and $p\neq 1$, the solution of (1) may become zero somewhere. In this case the Monge-Ampère equation (1) is either degenerate or singular, and the solution may not be smooth even for smooth and positive f, see [GL]. Indeed, let u be the restriction of h on the tangent hyperplane of the n-sphere at the south pole. Then u satisfies the equation

$$\det D^2u = g(x)u^{p-1}, \tag{6.1}$$

in the Euclidean space $I\!R^n$ in the generalized sense, with

$$g(x)=(1+|x|^2)^{-\frac{n}{2}-p}f\big(\frac{x,-1}{\sqrt{1+|x|^2}}\big).$$

Let $u(x)=|x|^{2\alpha}$, $\alpha=n/(n-p+1)$. Then u satisfies equation (6.1) for some positive, smooth g. Namely h satisfies (1) for some positive, smooth f near the south pole. By a suitable extension of h and hence f one obtains a global solution of (1) with some positive, smooth f on the entire sphere.

In the following we prove Theorem E. Technically our proof of part (a) is inspired by [C1]. First we consider the case $1 < p < n+1$ of part (a). Let h be a nonnegative solution of (1). Assume $h = 0$ and h is not C^1 at the south pole. Let u be the restriction of h on the tangent space of S^n at the south pole. Then u is not C^1 at the origin. It follows that $\beta =: \overline{\lim}_{x\to 0} \frac{1}{|x|}(u(x) - u(0)) > 0$. By a rotation of axes we may assume that $x_{n+1} = \beta x_1$ is a supporting hyperplane of u at the origin. Let

$$u_\varepsilon = u - ((\beta - \varepsilon')x_1 + \varepsilon), \qquad \Omega_\varepsilon = \{x \in \mathbb{R}^n : \ u_\varepsilon(x) < 0\}. \tag{6.2}$$

where $\varepsilon, \varepsilon'$ are small positive constants, $\varepsilon' < \varepsilon$. Hence Ω_ε is bounded.

Let E_ε be the minimum ellipsoid of Ω_ε. By John's lemma [C1], $\frac{1}{n}E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$, where tE denotes the t-dilation of E with respect to its center. Denote the right hand side of (6.1) by μ and regard it as a Borel measure. Then we have,

$$\mu(\frac{1}{2}E_\varepsilon) \geq C\mu(E_\varepsilon) \tag{6.3}$$

for some positive C independent of ε.

Let T_ε be the linear transformation such that $T_\varepsilon(E_\varepsilon)$ is the unit ball, and let $w_\varepsilon = \varepsilon^{-1}u_\varepsilon$. Then $w_\varepsilon = 0$ on ∂D_ε, $\inf w_\varepsilon = -1$, and

$$\det D^2 w_\varepsilon = \mu_\varepsilon,$$

for some measure μ_ε satisfying

$$\mu_\varepsilon(\frac{1}{2}D_\varepsilon) \geqslant C\mu_\varepsilon(D_\varepsilon)$$

where $D_\varepsilon = T_\varepsilon(\Omega_\varepsilon)$.

By $\inf w_\varepsilon = -1$ and $w_\varepsilon = 0$ on ∂D_ε, we have $\mu_\varepsilon(\frac{1}{2}D_\varepsilon) \leq C$. On the other hand, we have $w_\varepsilon(0) = -1$ and that $\text{dist}(0, \partial D_\varepsilon) \to 0$ as $\varepsilon \to 0$ by choosing $\varepsilon' > 0$ sufficiently small. Hence the normal image of w_ε over D_ε has unbounded area when $\varepsilon \to 0$. Namely $\mu_\varepsilon(D_\varepsilon)$ is not uniformly bounded in ε. We have arrived at a contradiction. So h must be C^1 near the set $Z = \{h = 0\}$.

Next we show that h is locally strictly convex in $\{h > 0\}$, namely u is locally strictly convex in $\{u > 0\}$. Indeed, if this is not true, then the graph of u, M_u, is not strictly convex in $\{u > 0\}$. Therefore there exists a point $p_0 \in M_u \cap \{u > 0\}$

such that the contact set $\mathcal{C} = P \cap M_u$, where P is the supporting plane of M_u at p_0, contains a line segment. Since u is C^1 near $\{u = 0\}$, there is no extreme points of $\mathcal{C}$ in $\{u = 0\}$. By the convexity of $\mathcal{C}$, this means that $\mathcal{C} \cap \{u = 0\}$ is empty, and so all extreme points of $\mathcal{C}$ lies in $\{u > 0\}$. On the other hand, from the argument in [C1], there is no extreme point of $\mathcal{C}$ at which the right hand side of (6.1) is positive. We reach a contradiction. Hence h is locally strictly convex in $\{h > 0\}$. By Proposition 1.2 it follows that $h \in C^{1,\gamma}(\{h > 0\})$ when f is a bounded positive function and $h \in C^{2,\alpha}(\{h > 0\})$ when f is Hölder continuous.

In the case $-n + 1 < p < 1$, we want to prove that the convex hypersurface X determined by the solution h is C^1. This is equivalent to showing that the set $\{u = 0\}$ contains at most one single point, where u is the restriction of h of any tangent plane of the n-sphere. The proof in this case is in the same spirit as above except the definition of u_ε and Ω_ε in (6.2) should be replaced by

$$u_\varepsilon = u - \varepsilon, \qquad \Omega_\varepsilon = \{u_\varepsilon < 0\},$$

so that (6.3) holds. As above we also have $\mu_\varepsilon(\frac{1}{2}D_\varepsilon) \leq C$. If the set $\{u = 0\}$ contains more than one points, by convexity it contains a line segment (note that by the equation, the set $\{u = 0\}$ must have measure zero). Hence there is a point $x_\varepsilon \in D_\varepsilon$ such that $\text{dist}(x_\varepsilon, \partial D_\varepsilon) \to 0$ as $\varepsilon \to 0$, which implies that $\mu_\varepsilon(D_\varepsilon)$ is not uniformly bounded in ε. We also reach contradiction.

Next we show that h is locally strictly convex in $\{h > 0\}$. By the C^1 smoothness of X, for any supporting plane P of M_u, the contact set $\mathcal{C} = P \cap M_u$ must be bounded. If $\mathcal{C}$ contains more than one point, by convexity it contains at least two extreme points. From the last paragraph, $\{u = 0\}$ contains the origin only. Hence there must be an extreme point of $\mathcal{C}$ at which $u > 0$. But this is impossible from the argument in [C1]. This completes the proof of part (a).

Before proceeding to the proof of parts (b) and (c), we remark that when $1 < p < \frac{1}{2}(n + 1)$, the radial function

$$u(x) = \begin{cases} (|x| - r)^{\frac{n+1}{n+1-p}}, & |x| > r, \\ 0, & |x| \leq r, \end{cases} \tag{6.4}$$

where $r > 0$ is a positive constant, is a generalized solution to (6.1) for some positive constant g. Note that $u = 0$ in $B_r(0)$. Hence the corresponding convex hypersurface may not be C^1 when $1 < p < \frac{1}{2}(n + 1)$.

We also remark that the function u in (6.4) is not C^2 when $1 < p < \frac{1}{2}(n+1)$. Hence the the second order derivative estimate in Theorem E(c) cannot be extended to $p \in (1, \frac{1}{2}(n+1))$.

Next we prove part (b). First we consider the case $1 < p < n+1$.

Lemma 6.1 *Let u be a nonnegative solution of (6.1) with $u(0) = 0$. Suppose g is Lipschitz and positive. Then for any $\theta \le 1, \theta \in (0, \frac{2p}{n+1})$, we have the estimate*

$$z =: \frac{|Du|^2}{u^\theta} \le C \quad \textit{near } 0. \tag{6.5}$$

Proof Denote $\Omega = \{u < 1\}$. Suppose $\sup_\Omega z$ is attained at some point x_0. If $x_0 \in \partial\Omega$, z is bounded. If x_0 is an interior point, by a rotation of axes we suppose $|Du| = u_1$ at x_0. Then by the approximation at the end of the section, we have,

$$\begin{aligned} 0 = z_i &= \frac{2u_1u_{1i}}{u^\theta} - \theta\frac{u_1^2u_i}{u^{1+\theta}}, \\ 0 \ge z_{ii} &\ge 2\frac{u_1u_{1ii}}{u^\theta} + 2\frac{u_{1i}^2}{u^\theta} - 4\theta\frac{u_1u_iu_{1i}}{u^{1+\theta}} - \theta\frac{u_1^2u_{ii}}{u^{1+\theta}} + \theta(1+\theta)\frac{u_1^2u_i^2}{u^{2+\theta}} \end{aligned}$$

at x_0. From the first formula we have

$$u_{11} = \frac{\theta}{2}\frac{u_1^2}{u}, \qquad u_{1i} = 0 \quad i > 1$$

Hence by a rotation of axes we may suppose furthermore that the Hessian matrix $\{u_{ij}\}$ is diagonal at x_0. Differentiating equation (6.1) gives

$$\sum_i u^{ii}u_{iik} = \frac{g_k}{g} + (p-1)\frac{u_k}{u} \quad \text{at} \quad x_0, \tag{6.6}$$

where $\{u^{ij}\}$ is the inverse of $\{u_{ij}\}$. Hence

$$\begin{aligned} 0 &\ge \sum u^{ii}z_{ii} \\ &= 2\frac{u_1}{u^\theta}(\frac{g_1}{g} + (p-1)\frac{u_1}{u}) + 2\frac{u_{11}}{u^\theta} - (n+4)\theta\frac{u_1^2}{u^{1+\theta}} + \theta(1+\theta)\frac{u_1^4}{u^{2+\theta}u_{11}} \\ &= 2\frac{u_1}{u^\theta}\frac{g_1}{g} + \frac{u_1^2}{u^{1+\theta}}(2p-(n+1)\theta), \end{aligned}$$

where we have used the estimate $u_{11} = \frac{\theta}{2}\frac{u_1^2}{u}$. When $\theta < (0, \frac{2p}{n+1})$, we obtain $z(x_0) \le C$. □

Let $h(r) = \sup\{u(x) : |x| = r\}$. From Lemma 6.1 we have $h' \le Ch^{\theta/2}$. Hence by $h(0) = 0$ we have $h(r) \le Cr^{2/(2-\theta)}$, namely

$$u(x) \le C|x|^{2/(2-\theta)}. \tag{6.7}$$

It follows that when $p > 0$, the solution u is $C^{1,\alpha}$ smooth near Z for $\alpha \in (0, \frac{\theta}{2-\theta})$. If $p > \frac{n+1}{2}$, we can choose $\theta = 1$.

Next we consider the case $-n+1 < p < 1$. Let K be the convex body determined by h and D the projection of K on $\{x_{n+1} = 0\}$. Let v be the Legendre transform of u. Then v is defined in D, and the graph of v is the lower part of ∂K. Furthermore v satisfies the equation

$$\det D^2 v = \widetilde{g}(Dv)(\Sigma_i x_i v_i - v)^{1-p}, \tag{6.8}$$

where $\widetilde{g} = 1/g$. By choosing proper axes we may also suppose that $v \ge 0$, and that $v = 0$ at the origin. When $-n+1 < p < 1$, we have a similar $C^{1,\alpha}$ estimate for v.

Lemma 6.2 *Let v be a nonnegative solution of (6.8) with $v(0) = 0$. Assume $\widetilde{g}$ is Lipschitz and positive. Then for any $\theta \le 1$, $\theta \in (0, \frac{2}{n+p})$, we have the estimate*

$$z =: \frac{|Dv|^2}{v^\theta} \le C \quad \textit{near } 0. \tag{6.9}$$

Proof The proof is similar to that of Lemma 6.1. Suppose z attains its maximum at some point $x_0 \in \Omega$. Then at x_0 we have by choosing a proper coordinate system that $v_{11} = \frac{\theta}{2}\frac{v_1^2}{v}$, $v_{1i} = 0$ for $i > 1$, and $(D^2 v)$ is diagonal at x_0. Instead of (6.6), we have

$$\sum_i v^{ii} v_{ii1} = (\log \widetilde{g})_1 + (1-p)\frac{x_k v_{1k}}{x_i v_i - v} \quad \text{at} \quad x_0,$$

where

$$(\log \widetilde{g})_1 = \sum_j (\log \widetilde{g})_{v_j} v_{1j} = (\log \widetilde{g})_{v_1} \frac{\theta v_1^2}{2v}.$$

Hence at x_0,

$$v^{ii} v_{ii1} = \frac{\theta}{2}\big((\log \widetilde{g})_{v_1}\frac{v_1^2}{v} + (1-p)\frac{x_1 v_1^2}{v(x_i v_i - v)}\big).$$

Hence we have

$$
\begin{aligned}
0 &\geq \sum v^{ii} z_{ii} \\
&= \theta \frac{v_1}{v^\theta}\left((\log \widetilde{g})_{v_1} \frac{v_1^2}{v} + (1-p)\frac{x_1 v_1^2}{v(x_i v_i - v)}\right) \\
&\qquad + 2\frac{v_{11}}{v^\theta} - (n+4)\theta \frac{v_1^2}{v^{1+\theta}} + \theta(1+\theta)\frac{v_1^4}{v^{2+\theta} v_{11}} \\
&= \theta(1-p)\frac{x_1 v_1}{x_i v_i - v}\frac{v_1^2}{v^{1+\theta}} + \frac{v_1^2}{v^{1+\theta}}\big(2 - (n+1)\theta + \theta v_1 (\log \widetilde{g})_{v_1}\big).
\end{aligned}
$$

Note that at x_0, $x_1 v_1 = (x_i v_i - v) + v$. We obtain

$$
\begin{aligned}
0 &\geq \frac{\theta(1-p)v}{x_i v_i - v}\frac{v_1^2}{v^{1+\theta}} + \frac{v_1^2}{v^{1+\theta}}\big(2 - (n+p)\theta + \theta v_1 (\log \widetilde{g})_{v_1}\big) \\
&\geq \frac{v_1^2}{v^{1+\theta}}\big(2 - (n+p)\theta + \theta v_1 (\log \widetilde{g})_{v_1}\big)
\end{aligned}
$$

as $p < 1$ and $v \geq 0$. Hence z must attains its maximum on the boundary $\partial\Omega$ if we first choose $\theta > 0$ small. It follows that v is $C^{1,\alpha}$ smooth for some $\alpha \in (0,1)$.

To prove (6.9) for any $\theta \in (0, \frac{2}{n+p})$, $\theta \leq 1$, it suffices to consider the supremum $\sup z(x)$ in the domain $\{v < \delta\}$ for some $\delta > 0$ small, such that v_1 is sufficiently small. We again conclude that z attains its maximum on the boundary $\partial\{v < \delta\}$. Hence Lemma 6.2 holds. □

Finally we prove part (c).

Lemma 6.3 *Let u be as in Lemma 6.1. Suppose $g \in C^{1,1}$ and is positive . If $p \in (\frac{n+1}{2}, n+1)$, then*

$$
|D^2 u| \leq C \quad \text{near } 0. \tag{6.10}
$$

Proof Let $z = \log u_{\xi\xi} + \frac{|Du|^2}{u}$. Suppose $\sup\{z(x) : \xi \in S^{n-1}, x \in \Omega\}$ is attained at x_0 and $\xi = (1, 0, \cdots, 0)$. If $x_0 \in \partial\Omega$, we have $z \leq C$. Otherwise at x_0 we have

$$
\begin{aligned}
0 = z_i &= \frac{u_{11i}}{u_{11}} + \frac{2u_k u_{ki}}{u} - \frac{u_k^2 u_i}{u^2} \\
0 \geq z_{ii} &= \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} + \frac{2u_{ki}^2}{u} + \frac{2u_k u_{kii}}{u} - \frac{4u_k u_i u_{ki}}{u^2} - \frac{u_k^2 u_{ii}}{u^2} + \frac{2u_k^2 u_i^2}{u^3}
\end{aligned}
$$

By a rotation of axes we may suppose $(D^2 u)$ is diagonal at x_0. Differentiating equation (6.1) we have

$$
\sum_i u^{ii} u_{ii11} = u^{ii} u^{jj} u_{ij1}^2 + (\log g)_{11} + (p-1)(\frac{u_{11}}{u} - \frac{u_1^2}{u^2}).
$$

Hence we obtain

$$
\begin{aligned}
0 \geq \sum_i u^{ii} z_{ii} \geq & C + (p-1)\left(\frac{1}{u} - \frac{u_1^2}{u^2 u_{11}}\right) + \frac{2u_{ii}}{u} \\
& + \frac{2u_k}{u}\left(\frac{g_k}{g} + (p-1)\frac{u_k}{u}\right) - (n+4)\frac{u_k^2}{u^2} + \frac{2u_k^2 u_i^2}{u^3 u_{ii}}.
\end{aligned}
$$

By Lemma 6.1, u_k^2/u is bounded. Hence we obtain $u_{11}(x_0) \leq C$. $\square$

For estimate (6.10), one may also work on equation (1) and use the auxiliary function $z = \log(h_{\xi\xi} + h) + \frac{|\nabla h|^2}{h}$, so that the proof of parts (b) and (c) is independent of part (a).

In the above proofs one needs to use approximation by smooth solutions. For this purpose one chooses a small constant $\delta > 0$ and consider the unique smooth, positive solution u_δ of

$$
\begin{aligned}
\det D^2 v &= (1-\delta) g(x) v^{p-1}, \\
v &= 1 \quad \text{on} \quad \partial\{u < 1\}.
\end{aligned}
$$

Note that in the above proofs, the assumption $u(0) = 0$ is not needed, rather one just needs to assume the domain $\{u < 1\}$ is bounded. Therefore the estimates (6.5) and (6.10) holds for u_δ. Letting $\delta \to 0$, one obtains (6.5) and (6.10) for u. For estimate (6.9), let u_δ be as above and let v_δ be the Legendre transformation of u_δ. Then (6.9) holds for v_δ. Sending $\delta \to 0$ we obtain (6.9) for $v = \lim_{\delta \to 0} v_\delta$.

§7. The Centroaffine Minkowski Problem

In his Erlangen programme F. Klein pointed out that geometry is the study of properties invariant under a group of transformations on the space. Thus for any transformation group acting on the space there is a corresponding geometry. We shall use this point of view to develop centroaffine geometry and interpret (1), $p = -n-1$, that is,

$$\det(h_{ij} + \delta_{ij}h) = \frac{f(x)}{h^{n+2}} \tag{7.1}$$

as the equation describing the Minkowski problem in centroaffine geometry.

To begin with let's examine the classical differential geometry of hypersurfaces. It is the Klein geometry associated to the Euclidean group of rigid motions in $\mathbb{R}^{n+1}$. Let $f : U \to \mathbb{R}^{n+1}$ where U is an open set in $\mathbb{R}^n$ be an immersion of a hypersurface and let ν be a chosen continuous unit normal vector field on the hypersurface. The immersion and the normal vector field induce the Levi-Civita connection $\overline{\nabla}$ and the second fundamental form b on the hypersurface by the Gauss formula

$$D_X Y = \overline{\nabla}_X Y - b(X, Y)\nu \ , \ X, Y \in TU \ , \tag{7.2}$$

where D is the flat connection in $\mathbb{R}^{n+1}$. Notice that we have identified X with $f_* X$. We also have the Weingarten equation

$$D_X \nu = -SX \ , \tag{7.3}$$

where S defines the shape operator. It is well-known that the Levi-Civita connection is uniquely determined by the first fundamental form, g_{ij}, as written in local coordinates, and $S = g^{ik} b_{kj}$. We may take the first and second fundamental forms as the basic geometric data in classical differential geometry. By cross differentiating (7.2) and (7.3) we obtain the following two compatibility conditions, namely, Gauss Equation and Codazzi-Mainardi equation,

$$\begin{aligned} &R(X, Y)Z = b(Y, Z)SX - b(X, Z)SY \ , \\ &(\overline{\nabla}_X b)(Y, Z) = (\overline{\nabla}_Y b)(X, Z) \ , \end{aligned}$$

where $R(X, Y) = \overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - [X, Y]$ is the Riemann curvature tensor. A classical theorem of Bonnet states that any symmetric tensors g_{ij} and b_{ij} on U, where g_{ij} is positive definite, satisfying these two compatibility conditions, must be locally the first and second fundamental forms of an immersion. Furthermore, the immersion is uniquely determined up to a rigid motion.

In [NS] Nomizu describes a new approach to affine geometry which is in many ways parallel to the above description of classical differential geometry. It also works well for centroaffine geometry. To describe it again let $f : U \to \mathbb{R}^{n+1}$ be an immersion of a hypersurface X. Such an immersion may be called a parametrization of the hypersurace X. The crux is to choose a suitable vector field on X to play the role of the Euclidean normal. In a general setting, let G be a Lie group of transformations acting linearly on $\mathbb{R}^{n+1}$. We call a transversal vector field ξ_f a G-normal vector field if (1) it is defined intrinsically, i.e., independent of the parametization, and (2) it is invariant in the sense

$$\xi_f(f(x)) = \xi_{g\circ f}(gf(x)) \ , \ \forall g \in G \ .$$

In centroaffine geometry the transformation group is $SL(n+1)$, which consists of all unimodular transformations acting linearly on $\mathbb{R}^{n+1}$, and its natural objects are star-shaped hypersurfaces. For any star-shaped hypersurface we can take the centro-affine normal vector field to be the negative of the position vector, $-X$. As in (7.2) we use it to induce the centroaffine connection ∇ and centroaffine fundamental form h

$$D_X Y = \nabla_X Y + h(X,Y)\xi \ , \ \xi = -X \ . \tag{7.4}$$

One can check that the Weingarten equation becomes

$$D_X \xi = -X,$$

so the shape operator is the identity. Since we do not have a first fundamental form, we may take ∇ and h as our basic geometric data. The compatibility conditions are

$$\begin{aligned} &R(X,Y)Z = h(Y,Z)X - h(X,Z)Y \ , \\ &(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) \ . \end{aligned} \tag{7.5}$$

An additional quantity is the volume form ω given by

$$\omega(X_1, \cdots, X_n) = \det(f_* X_1, \cdots, f_* X_n, -X) \ .$$

One can verify that ∇ is torsion-free, and this is equivalent to

$$\nabla \omega = 0 \ . \tag{7.6}$$

It turns out that the triple (∇, h, ω) completely characterizes the immersion up to a unimodular transformation. More precisely, let ∇ be a torsion-free connection, h be a symmetric tensor, and ω a volume form on U so that (7.5) and (7.6) are satisfied. Then there locally exists an immersion $f : U \to \mathbb{R}^{n+1}$ and a parallel volume form θ in $\mathbb{R}^{n+1}$ so that ∇ and h are respectively the centroaffine connection and fundamental form of this immersion, and

$$\omega(X_1, \cdots, X_n) = \theta(f_* X_1, \cdots, f_* X_n, -X) \ .$$

Moreover, the immersion is unique up to a unimodular transformation.

Having characterized centroaffine geometry locally, we consider its curvature. Recall that in classical differential geometry of hypersurfaces the curvature of a hypersurface is, roughly speaking, a function defined intrinsically on the hypersurface whose values do not change under rigid motions. Furthermore, the expression defining the curvature involves derivatives of the immersion up to second order. In general, let's call a function defined intrinsically on the hypersurface a G-differential invariant if (1) its values remain unchanged under any $g \in G$, and (2) it is defined by an expression which depends on the derivatives of the immersion up to some finite order. So, all elementary symmetric functions of the principal curvatures, as well as those functions obtained by covariant differentiating these functions and taking contractions are differential invariants for the Euclidean group. We understand that a curvature is a differential invariant which has the least order in the derivatives. The order of derivatives in the curvature usually depends on the dimension of G, and it increases with the dimension. For example, the order of derivatives in the Euclidean curvature function is 2, but it is 4 for affine curvature functions. The dimension of $SL(n+1)$ is n^2+n, larger than the Euclidean group which is equal to $\frac{1}{2}(n+1)(n+2)+n+1$. It is remarkable that it has a curvature function of order 2.

Proposition 7.1 *Let $f : U \to \mathbb{R}^{n+1}$ be a star-shaped immersion. Then*

$$C = \frac{\det h(X_i, X_j)}{\omega(X_1, \cdots, X_n)} \ , \ X_i = f_* \left(\frac{\partial}{\partial x_i} \right) \ ,$$

is an $SL(n+1)$-differential invariant.

Proof From (7.2) and (7.4) we have

$$h_{ij} = \frac{b_{ij}}{X \cdot \nu}$$

Letting g_{ij} be the first fundamental form of X, we have

$$\det h_{ij} = \frac{K}{(X \cdot \nu)^n} \det g_{ij} \ ,$$

where K is the Gauss curvature of X. On the other hand,

$$\begin{aligned}\omega(X_1, \cdots, X_n) &= \det\,(X_1, \cdots, X_n, X) \\ &= (X \cdot \nu)\,|X_1 \times \cdots \times X_n|\ .\end{aligned}$$

Recall that $X_1 \times \cdots \times X_n$ is defined by

$$\langle X_1 \times \cdots \times X_n, v\rangle = \det(X_1, \cdots, X_n, v),$$

for all v. We may also assume

$$\nu = \frac{X_1 \times \cdots \times X_n}{|X_1 \times \cdots \times X_n|}.$$

Using the formula

$$\det g_{ij} = |X_1 \times \cdots \times X_n|^2\,,$$

we obtain

$$C = \frac{K}{(X \cdot \nu)^{n+2}}\ . \tag{7.7}$$

This formula shows that C is intrinsic on the hypersurface. Now, under a unimodular transformation A, X goes over to $\tilde{X} = AX$. We have, in obvious notation,

$$\begin{aligned}\tilde{b}_{ij} &= -\langle\tilde{\nu}, \tilde{X}_{ij}\rangle \\ &= \frac{-1}{|\tilde{X}_1 \times \cdots \times \tilde{X}_n|}\det\,(\tilde{X}_1, \cdots, \tilde{X}_n, \tilde{X}_{ij}) \\ &= \left(\frac{|X_1 \times \cdots \times X_n|}{|\tilde{X}_1 \times \cdots \times \tilde{X}_n|}\right) b_{ij}\ , \\ \det \tilde{g}_{ij} &= \left(\frac{|X_1 \times \cdots \times X_n|}{|\tilde{X}_1 \times \cdots \times \tilde{X}_n|}\right)^2 \det g_{ij}\ ,\end{aligned}$$

and

$$\tilde{X} \cdot \tilde{\nu} = \frac{|X_1 \times \cdots \times X_n|}{|\tilde{X}_1 \times \cdots \times \tilde{X}_n|}\ X \cdot \nu\ .$$

Therefore $\tilde{C} = C$. □

Formula (7.7) shows C is equal to K/h^{n+2} where h is the support function of X. The centroaffine invariance of C was first discovered by Tzitzéica [T] in 1908, and rediscovered by Loewner-Nirenberg in [LN]. In texts on affine geometry, for example, [NS], usually it is called the affine distance. However, in the context of centroaffine geometry, we prefer to call it the *centroaffine Gauss curvature.*

Now, we can formulate the centroaffine Minkowski problem as the exact analog of the Euclidean Minkowski problem.

Given a positive function f on S^n, find necessary and sufficient conditions on f so that it is the centroaffine Gauss curvature of a convex, closed hypersurface containing the origin as a function of its centroaffine normal direction.

In fact, from the above discussion the natural objects in this Minkowski problem are star-shaped hypersurfaces containing the origin in their interiors. However, to avoid working on an equation with mixed type (see (7.8) below) and to link the problem to (1) for critical p we restrict ourselves to convex hypersurfaces. To write down the equation for this problem we let $X = \{\rho(x)x : x \in S^n\}$ be the solution hypersurface. Then we have

$$\begin{aligned}
\nu &= \frac{1}{\sqrt{\rho^2 + |\nabla\rho|^2}}(\rho x - e^{ij}\nabla_i\rho\nabla_j x)\ , \\
X \cdot \nu &= \rho(x)x \cdot \nu \\
&= \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}\ , \\
g_{ij} &= \rho^2 e_{ij} + \nabla_i\rho\nabla_j\rho\ ,
\end{aligned}$$

and

$$b_{ij} = \frac{1}{\sqrt{\rho^2 + |\nabla\rho|^2}}(-\rho\nabla_i\nabla_j\rho + 2\nabla_i\rho\nabla_j\rho + \rho^2 e_{ij})\ .$$

where e_{ij} is the standard metric on the sphere. So ρ satisfies the equation

$$\frac{\det\,(-\rho\nabla_i\nabla_j\rho + 2\nabla_i\rho\nabla_j\rho + \rho^2 e_{ij})}{\det e_{ij}} = \rho^{2-4n} f \tag{7.8}$$

To compare (7.8) with (7.1) we let $h(x) = \rho^{-1}(x)$ be the support function of the polar body of X. By a straightfoward computation we have

$$g_{ij} = \frac{\nabla_i h\nabla_j h + h^2 e_{ij}}{h^4}\ ,$$

and

$$b_{ij} = \frac{1}{h\sqrt{h^2 + |\nabla h|^2}}(\nabla_i\nabla_j h + he_{ij})$$

So, in terms of h, (7.8) becomes (7.1). We conclude that *the polar body of the convex body determined by the solution of (7.1) solves the centroaffine Minkowski problem.* Hence in the convex category the centroaffine Minkowski problem is equivalent to the solvability of (7.1)

Now, we study the invariance properties of this equation. First of all, any unimodular transformation gives a projective transformation on S^n. Indeed, let $g \in SL(n+1)$, we define $\varphi_g : S^n \to S^n$ by

$$\varphi_g(x) = \frac{gx}{|gx|} .$$

Since det $g = 1$, it is clearly that the correspondence between g and φ_g is one-to-one. We call the Lie group $\{\varphi_g : g \in SL(n+1)\}$ the projective group on S^n. One can easily verify that it is isomorphic to $SL(n+1)$, i.e., $\varphi_{g_1}\varphi_{g_2} = \varphi_{g_1 g_2}$. The Lie algebra of the projective group, when regarded as vector fields on S^n, are of the form

$$(A_{\alpha\beta}x_\beta - A_{\lambda\mu}x_\lambda x_\mu x_\alpha)\frac{\partial}{\partial x_\alpha} \ , \ \alpha, \beta, \lambda, \mu = 1, \cdots, n+1 ,$$

where the matrix $(A_{\alpha\beta}) \in s\ell(n+1)$, i.e., $trA = 0$. Unimodular transformations also induce projective transformations on $\mathbb{R}^n$ when it is viewed as a tangent space of S^n. Let's focus on $\mathbb{R}^n = \{(x_1, \cdots, x_n, x_{n+1}) : x_{n+1} = -1\}$. For any $g \in SL(n+1)$, we set

$$\psi_g(x) = \frac{a_{ij}x_j - b_i}{d - c_i x_i} \ , \ x \in \mathbb{R}^n ,$$

where $g = (a_{\alpha\beta})$, and $a_{jn+1} = b_j$, $a_{n+1j} = c_j$ and $a_{n+1 \ n+1} = d$. Its Lie algebra consists of vector fields of the form

$$(A_{ij}x_j - B_i + C_j x_j x_i - D x_i)\frac{\partial}{\partial x_i} \ , \ \sum A_{ii} + D = 0 .$$

Let π be the sterographic projection from the south hemisphere to $\mathbb{R}^n$

$$\pi(x_i) = \frac{x_i}{|x_{n+1}|}$$

One can easily verify that $\pi_\circ \varphi_g = \psi_g \circ \pi$.

The projective group also acts on functions defined in S^n and $\mathbb{R}^n$. Let f be a function on $\mathbb{R}^{n+1}$. We let

$$f_g(x) = f(gx) \ , \ g \in SL(n+1) .$$

For any function f in S^n we extend it to be a function of homogeneous degree 1 in $\mathbb{R}^{n+1}$. So

$$\begin{aligned} f_g(x) &= f(gx) \\ &= |gx| f(\varphi_g x) , \end{aligned}$$

is the rule of transformation on S^n. Similarly, when restricted to $\mathbb{R}^n = \{x_{n+1} = -1\}$ we have

$$f_g(x) = (c_j x_j - d) f(\psi_g x) \ .$$

Let ρ be any positive function on S^n. Proposition 7.1 and (7.7) show that, for any projective transformation φ_g,

$$\begin{aligned} &\rho^{4n-2} \frac{\det(-\rho \nabla_i \nabla_j \rho + 2\nabla_i \rho \nabla_j \rho + \rho^2 e_{ij})}{\det e_{ij}} \\ =&\tilde{\rho}^{4n-2} \frac{\det(-\tilde{\rho} \nabla_i \nabla_j \tilde{\rho} + 2\nabla_i \tilde{\rho} \nabla_j \tilde{\rho} + \tilde{\rho}^2 e_{ij}(\tilde{x}))}{\det e_{ij}(\tilde{x})} \end{aligned}$$

where $\tilde{x} = \varphi_g x$ and $\tilde{\rho} = \rho_g(\tilde{x})$. Letting $h = 1/\rho$, we then have

$$h^{n+2} \det(h_{ij} + \delta_{ij} h) = \tilde{h}^{n+2} \det(\tilde{h}_{ij} + \delta_{ij} \tilde{h}) \ .$$

So, we have the following invariance properties of (7.1): Let h be a solution of (7.1). Then $\tilde{h} = h_g(\tilde{x})$ solves

$$\det(\tilde{h}_{ij} + \delta_{ij} \tilde{h}) = f(\phi_g^{-1} \tilde{x}) h^{\tilde{-n-2}} \ .$$

Now, we prove Proposition F.

Lemma 7.3 *For any function u defined in $\mathbb{R}^n$,*

$$\frac{\partial^2 Q}{\partial x_i \partial x_j} c_{ij} u + Q \det D^2 u + div(\xi u \det D^2 u) = 0 \ , \tag{7.9}$$

where $\xi^k = C_j x_j x_k + (A_{kj} - \delta_{kj} D) x_j - B_k$, $A_{ii} + D = 0$, $\sigma = C_j x_j - D$, $Q = \sigma u - \xi^k u_k$ and c_{ij} is the (i,j)-entry of the cofactor matrix of $D^2 u$.

Proof Keep using the divergence free property $c_{ij,j} = 0$ and $c_{ik} u_{kj} = \delta_{ij}$, we have

$$\begin{aligned} &\frac{\partial^2 Q}{\partial x_i \partial x_j} c_{ij} u + Q \det D^2 u + \mathrm{div}(\xi u \det D^2 u) \\ =& (\sigma_{ij} u + \sigma_i u_j + \sigma_j u_i + \sigma u_{ij} - \xi^k_{ij} u_k - \xi^k_i u_{kj} - \xi^k_j u_{ki} - \xi^k u_{kij}) u c_{ij} \\ &+ (\sigma u - \xi^k u_k) \det D^2 u + (\mathrm{div} \xi) u \det D^2 u + \xi^i u_i \det D^2 u \\ &+ \xi^k u c_{ij} u_{ijk} \\ =& n \sigma u \det D^2 u - (\mathrm{div} \xi) u \det D^2 u + \sigma u \det D^2 u \\ =& 0 \ . \end{aligned}$$

$\square$

Integrating (7.9) over $B_R = \{|x| < R\}$ we have

$$\int_{B_R} \big(Q_{ij} c_{ij} u + Q \det D^2 u + \operatorname{div}(\xi u \det D^2 u)\big)\, dx = 0.$$

Using $c_{ij,j} = 0$ we obtain, after performing integration by parts,

$$(n+1) \int_{B_R} Q \det D^2 u = - \int_{\partial B_R} (\xi \cdot \nu) u \det D^2 u + \int_{\partial B_R} Q c_{ij} u_j \nu_i - \int_{\partial B_R} Q_i c_{ij} \nu_j u. \tag{7.10}$$

When h satisfies (7.1), its restriction to $\{x_{n+1} = -1\}, u$, satisfies the equation

$$u^{n+2} \det D^2 u = f(x) \ ,$$

where f is regarded as a function of homogeneous degree 0. We have

$$\int_{B_R} (\sigma u - \xi^k u_k) \frac{f(x)}{u^{n+2}} = \int_{B_R} (\sigma - \frac{\operatorname{div} \xi}{n+1}) \frac{f}{u^{n+1}} - \int_{B_R} \frac{\xi^k f_k}{n+1} \frac{1}{u^{n+1}} + \int_{\partial B_R} \frac{\xi^k \nu_k}{n+1} \frac{f(x)}{u^{n+2}} \ .$$

Hence

$$\int_{B_R} \xi^k f_k \frac{1}{u^{n+1}} = \int_{\partial B_R} (\xi \cdot \nu) u \det D^2 u + \int_{\partial B_R} Q c_{ij} u_j \nu_i - \int_{\partial B_R} Q_i c_{ij} \nu_j u + \int_{\partial B_R} (\xi \cdot \nu) \frac{f}{u^{n+2}} \ . \tag{7.11}$$

Using the sterographic projection the integral on the left hand side of this identity becomes

$$\int_{\hat{B}_R} (\nabla_\xi f) \frac{1}{h^{n+1}}$$

where $\hat{B_R} = \{x \in S^n, |\pi(x)| < R\}$, and the boundary integrals are over circles on S^n. As $R \to \infty$, these circles tend to the equator.

Similar consideration can be applied to the tangent space of S^n at the north pole, and the resulting identity is similar to (7.10). By adding up these two identities and then let $R \to \infty$, the left hand side becomes

$$\int_{S^n} (\nabla_\xi f) \frac{1}{h^{n+1}}$$

and the right hand side would become zero. To see how the boundary terms cancel out each other let's look at

$$\int_{\partial B_R} (\xi \cdot \nu) u \det D^2 u ds \ .$$

we have

$$\xi \cdot \nu = \frac{A_{ij}x_jx_i - B_ix_i + C_jx_j|x|^2 - D|x|^2}{|x|}$$
$$= \frac{A_{ij}p_ip_j(1+|x|^2) + B_ip_ip_{n+1}(1+|x|^2) - A_{n+1j}p_jp_{n+1}(1+|x|^2)|x|^2 - D|x|^2}{|x|}$$

where $\pi(p) = x$,

$$u \det D^2u = (1+|x|^2)^{-\frac{n+1}{2}} h \det(h_{ij} + h\delta_{ij})$$

and

$$ds(x) = (1+|x|^2)^{\frac{n-1}{2}} ds(p) \ .$$

Therefore,

$$\lim_{R\to\infty} \int_{\partial B_R} (\xi \cdot \nu) u \det D^2 u ds = -\int_{\{x_{n+1}=0\}} A_{n+1j}p_j \frac{f(p)}{h^{n+1}(p)} ds(p) \ .$$

On the other hand, the sterographic projection from S^n to $\{x_{n+1} = 1\}$ is given by $\pi(p) = p_i/p_{n+1}$. Hence

$$\lim_{R\to\infty} \int_{\partial B_R} (\xi \cdot \nu) u \det D^2 u ds = \int_{\{x_{n+1}=0\}} A_{n+1j}p_j \frac{f(p)}{h^{n+1}(p)} ds(p) \ .$$

So they cancel each other. Similar cancellations hold for other boundary integrals. The proof of the proposition is completed.

From the discussion of the subcritical case in Section 5 one can show by approximation that (7.1) is solvable when f is invariant under certain discrete groups of rotations on the sphere. For example, when the orbit of some point under the group actions has non-empty intersection with any open hemisphere, any subcritical approximation cannot collapse or concentrate, and so it must subconverge to a solution of the critical case. When $n = 1$, some sufficient conditions for solvability without symmetry conditions on f can be found in [ACW]. In this paper one can also find some further discussion on the obstruction.

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