PROJECTIVITY OF THE MODULI SPACE OF α-STABLE CURVES AND THE LOG MINIMAL PROGRAM FOR $\overline{M}_{g,n}$

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Abstract. For the moduli stacks of α-stable curves defined in [AFSv14], we prove nefness of natural log canonical divisors generalizing a well-known result of Cornalba and Harris for $\overline{\mathcal{M}}_{g,n}$. We deduce the projectivity of the good moduli spaces of α-stable curves and identify these moduli spaces with the log canonical models of $\overline{\mathcal{M}}_{g,n}$.

Contents

1. Introduction

This is the final part of the trilogy (see also [AFSv14, AFS15]) in which we construct the second flip in the log minimal model program for $\overline{\mathcal{M}}_{g,n}$. In [AFSv14], we construct the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ of α-stable curves (see Definition 2.2). In [AFS15], we prove that the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ admit proper good moduli spaces. The main result of this paper is that good moduli spaces of $\overline{\mathcal{M}}_{g,n}(\alpha)$ are projective and constitute steps in the log minimal model for $\overline{\mathcal{M}}_{g,n}$.

Namely, for $\alpha > 2/3-\epsilon$, let $\overline{\mathbb{M}}_{g,n}(\alpha)$ be the good moduli space of $\overline{\mathcal{M}}_{g,n}(\alpha)$. By [AFS15, Theorem 1.1], $\overline{\mathbb{M}}_{g,n}(\alpha)$ is a proper algebraic space. Consider the following log canonical models of $\overline{\mathcal{M}}_{g,n}$:

$$\overline{M}_{g,n}(\alpha) := \operatorname{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_{g,n}, \lfloor m(K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta + (1-\alpha)\psi)\rfloor). \tag{1.1}$$

We prove that the two independently defined objects, $\overline{\mathbb{M}}_{g,n}(\alpha)$ and $\overline{M}_{g,n}(\alpha)$, are in fact the same:

Theorem 1.1. *For $\alpha > 2/3-\epsilon$, the following statements hold:*

(1) *The line bundle $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$.*

(2) *There is an isomorphism* $\overline{\mathbb{M}}_{g,n}(\alpha) \simeq \overline{M}_{g,n}(\alpha)$.

Corollary 1.2. *The algebraic stack* $\overline{\mathcal{M}}_{g,n}(\alpha)$ *has a projective good moduli space for every* $\alpha > 2/3-\epsilon$.

The key ingredient in the proof of Theorem 1.1 is a positivity result for certain line bundles on $\overline{\mathcal{M}}_{g,n}(\alpha)$ generalizing the following well-known result of Cornalba and Harris:

Theorem ([CH88])**.** *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi \sim 11\lambda - \delta + \psi$$

is nef on $\overline{\mathcal{M}}_{g,n}$ *for all* (g,n), *and has degree* 0 *precisely on the families whose only non-isotrivial components are* A_1*-attached elliptic tails.*

Using [CH88], Cornalba proved that $12\lambda - \delta + \psi$ is in fact ample on $\overline{\mathcal{M}}_{g,n}$ and thus obtained a direct intersection-theoretic proof of the projectivity of $\overline{M}_{g,n}$ [Cor93]. In the introduction to [Cor93], the author says that "... it is hard to see how [these techniques] could be extended to other situations." In what follows, we do precisely that in giving intersection-theoretic proofs of the projectivity for $\overline{\mathbb{M}}_{g,n}(7/10-\epsilon)$ and $\overline{\mathbb{M}}_{g,n}(2/3-\epsilon)$ by proving the following positivity result:

Theorem 1.3 (Positivity of log canonical divisors)**.**

(a) *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}(9/11-\epsilon)} + \frac{7}{10}\delta + \frac{3}{10}\psi \sim 10\lambda - \delta + \psi$$

is nef on $\overline{\mathcal{M}}_{g,n}(9/11-\epsilon)$, *and, if* $(g,n) \neq (2,0)$, *has degree* 0 *precisely on the families whose only non-isotrivial components are* A_1/A_1*-attached elliptic bridges. It is trivial if* $(g,n) = (2,0)$.

(b) *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}(7/10-\epsilon)} + \frac{2}{3}\delta + \frac{1}{3}\psi \sim \frac{39}{4}\lambda - \delta + \psi$$

is nef on $\overline{\mathcal{M}}_{g,n}(7/10-\epsilon)$, *and has degree* 0 *precisely on the families whose only non-isotrivial components are* A_1*-attached Weierstrass chains.*

Our proof of the above theorem is inspired by [Cor93]. We also refer the reader to [ACG11, Chapter 14] for an excellent exposition of the Cornalba's original argument and a comprehensive treatment of intersection-theoretic approaches to the projectivity of $\overline{M}_{g,n}$, many of which make appearance in this paper.

Roadmap. Our proof of Theorems 1.1 and 1.3 is organized as follows. We recall the necessary notions and definitions in Section 2. In Section 3, we develop a theory of simultaneous normalization for families of at-worst tacnodal curves. By tracking how the relevant divisor classes change under normalization, we can reduce Theorem 1.3 to proving a (more complicated) positivity result for families of generically smooth curves. In Section 4, we collect several preliminary positivity results, stemming from three sources:

the Cornalba-Harris inequality, the Hodge Index Theorem, and some ad hoc divisor calculations on $\overline{\mathcal{M}}_{0,n}$. In Sections 5 and 6, we combine these ingredients to prove parts (a) and (b) of Theorem 1.3, respectively. Finally, in Section 7, we apply Theorem 1.3 to obtain Theorem 1.1.

Acknowledgments. We thank Ian Morrison for carefully reading the earlier versions of this paper and for numerous suggestions that improved the exposition. The results of this paper were first presented at the AIM Workshop "Log minimal model program for moduli space" in December 2012. We thank the participants of the workshop, and in particular Gavril Farkas, for questions and discussions on the subject of this paper.

During the preparation of this paper, the first author was partially supported by the Australian Research Council grant DE140101519 and by a Humboldt Fellowship. The second author was partially supported by NSF grant DMS-1259226 and the Australian National University MSRVP fund. The third author was partially supported by the Australian Research Council grant DE140100259. This work was revised when the second author visited the Max-Planck Institute for Mathematics in Bonn.

2. Preliminaries on line bundles on $\overline{\mathcal{M}}_{g,n}(\alpha)$

The following terminology will be in force throughout the paper. We let $\widetilde{\mathcal{U}}_g(A)$ denote the stack of connected curves of arithmetic genus g with only A-singularities, and let $\widetilde{\mathcal{U}}_g(A_\ell) \subset \widetilde{\mathcal{U}}_g(A)$ be the open substack parameterizing curves with at worst A_k, $k \leq \ell$, singularities. Since $\widetilde{\mathcal{U}}_g(A)$ is smooth, we may freely alternate between line bundles and divisor classes on $\widetilde{\mathcal{U}}_g(A)$. In addition, any relation between divisor classes on $\widetilde{\mathcal{U}}_g(A)$ that holds on the open substack of at-worst nodal curves extends to $\widetilde{\mathcal{U}}_g$, because the locus of worse-than-nodal curves has codimension 2.

Let $\pi\colon\ \mathcal{X} \to \widetilde{\mathcal{U}}_g(A)$ be the universal family. We define the *Hodge class* as $\lambda := c_1(\pi_*\omega_\pi)$ and the *kappa class* as $\kappa := \pi_*(c_1(\omega_\pi)^2)$. The divisor parameterizing singular curves in $\widetilde{\mathcal{U}}_g(A)$ is denoted δ. It can be further decomposed as $\delta = \delta_{\mathrm{irr}} + \delta_{\mathrm{red}}$, where δ_{red} is the closed locus of curves with disconnecting nodes. (The fact that δ_{red} is closed follows, for example, from [AFSv14, Corollary 2.11]).

By the preceding remarks, Mumford's relation $\kappa = 12\lambda - \delta$ holds on $\widetilde{\mathcal{U}}_g(A)$. Note that the higher Hodge bundles $\pi_*(\omega_\pi^m)$ for $m \geq 2$ are well-defined on the open locus in $\widetilde{\mathcal{U}}_g(A)$ of curves with nef dualizing sheaf. This open locus is the complement of the closed locus of curves with rational tails. If we restrict to this locus, the Grothendieck-Riemann-Roch formula gives

$$c_1(\pi_*(\omega_\pi^m)) = \lambda + \frac{m^2 - m}{2}\kappa. \tag{2.1}$$

Now let $\mathcal{C} \to B$ be a family of curves in $\widetilde{\mathcal{U}}_g(A)$. If $\sigma\colon\ B \to \mathcal{C}$ is any section of the family, we define $\psi_\sigma := \sigma^*\omega_{\mathcal{C}/B}$. We say that σ is *smooth* if it avoids the relative singular locus of $\mathcal{C}/B$.

From now on, we work only with families $\mathcal{C} \to B$ over a smooth and proper curve B. If $\sigma\colon\ B \to \mathcal{C}$ is generically smooth and the only singularities of fibers that $\sigma(B)$ passes through are nodes, then $\sigma(B)$ is a $\mathbb{Q}$-Cartier divisor on $\mathcal{C}$, and we define the *index of σ*

to be

$$\iota(\sigma) := (\omega_{\mathcal{C}/B} + \sigma) \cdot \sigma. \tag{2.2}$$

Notice that the index $\iota(\sigma)$ is non-negative, and if σ is smooth, then $\iota(\sigma) = 0$. We also have the following standard result:

Lemma 2.1. *Suppose $\mathcal{C} \to B$ is a generically smooth non-isotrivial family of curves in $\widetilde{\mathcal{U}}_g(A)$.*

(1) *If $g \geq 1$ and $\sigma \colon B \to \mathcal{C}$ is a smooth section, then $\sigma^2 < 0$.*
(2) *If $g = 0$ and $\sigma, \sigma', \sigma'' \colon B \to \mathcal{C}$ are 3 smooth sections such that σ is disjoint from σ' and σ'', then $\sigma^2 < 0$.*

Let $\mathcal{C} \to B$ be a one-parameter family of curves in $\widetilde{\mathcal{U}}_g(A)$. If $p \in \mathcal{C}$ is a node of its fiber, then the local equation of $\mathcal{C}$ at p is $xy = t^e$, for some $e \in \mathbb{Z}$ called *the index of p* and denoted $\operatorname{index}(p)$. A *rational tail* (resp., a *rational bridge*) of a fiber is a $\mathbb{P}^1$ meeting the rest of the fiber in exactly one (resp., two) nodes. If $E \subset C_b$ is a rational tail and $p = E \cap \overline{(C_b \setminus E)}$, then *the index of E* is defined to be $\operatorname{index}(p)$. Similarly, if $E \subset C_b$ is a rational bridge and $\{p, q\} = E \cap \overline{(C_b \setminus E)}$, then the index of E is defined to be $\min\{\operatorname{index}(p), \operatorname{index}(q)\}$. We also denote the index of E by $\operatorname{index}(E)$. We say that a rational bridge $E \subset C_b$ is *balanced* if $\operatorname{index}(p) = \operatorname{index}(q)$.

We now recall the notion of α-stability introduced in [AFSv14, Section 2]; see [AFSv14, Definitions 2.1 and 2.2] for the definitions of elliptic tails, bridges, chains, and Weierstrass tails and chains.

Definition 2.2 (α-stability)**.** For $\alpha \in (2/3 - \epsilon, 1]$, we say that an n-pointed curve $(C, \{p_i\}_{i=1}^n)$ is *α-stable* if $\omega_C(\Sigma_{i=1}^n p_i)$ is ample and:

For $\alpha \in (9/11, 1)$: C has only A_1-singularities.
For $\alpha = 9/11$: C has only A_1, A_2-singularities.
For $\alpha \in (7/10, 9/11)$: C has only A_1, A_2-singularities, and does not contain:
- A_1-attached elliptic tails.

For $\alpha = 7/10$: C has only A_1, A_2, A_3-singularities, and does not contain:
- A_1, A_3-attached elliptic tails.

For $\alpha \in (2/3, 7/10)$: C has only A_1, A_2, A_3-singularities, and does not contain:
- A_1, A_3-attached elliptic tails,
- A_1/A_1-attached elliptic chains.

For $\alpha = 2/3$: C has only A_1, A_2, A_3, A_4-singularities, and does not contain:
- A_1, A_3, A_4-attached elliptic tails,
- $A_1/A_1, A_1/A_4, A_4/A_4$-attached elliptic chains.

For $\alpha \in (2/3 - \epsilon, 2/3)$: C has only A_1, A_2, A_3, A_4-singularities, and does not contain:
- A_1, A_3, A_4-attached elliptic tails,
- $A_1/A_1, A_1/A_4, A_4/A_4$-attached elliptic chains,
- A_1-attached Weierstrass chains.

A family of α-stable curves is a flat and proper family whose geometric fibers are α-stable. We let $\overline{\mathcal{M}}_{g,n}(\alpha)$ denote the stack of n-pointed α-stable curves of arithmetic genus g.

Since $\overline{\mathcal{M}}_{g,n}(\alpha)$ parameterizes unobstructed curves, it is a smooth algebraic stack and thus has a canonical divisor $K_{\overline{\mathcal{M}}_{g,n}(\alpha)}$. Because non-nodal curves in $\overline{\mathcal{M}}_{g,n}(\alpha)$ form a closed substack of codimension 2, the standard formula (cf. [Log03, Theorem 2.6]) gives

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha)} = 13\lambda - 2\delta + \psi.$$

Since λ, δ, and ψ are defined everywhere on $\mathcal{U}_{g,n}$, we have the following formula

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_c \pm \epsilon)} = K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}|_{\overline{\mathcal{M}}_{g,n}(\alpha_c \pm \epsilon)}$$

for all $\alpha_c \in \{2/3, 7/10, 9/11\}$.

3. Degenerations and simultaneous normalization

Our first goal is to develop a theory of simultaneous (partial) normalization along generic singularities in families of at-worst tacnodal curves. In contrast to the situation for nodal curves, where normalization along a nodal section can always be performed because a node is not allowed to degenerate to a worse singularity, we must now deal with families where a node degenerates to a cusp or a tacnode, where two nodes degenerate to a tacnode, or where a cusp degenerates to a tacnode.

We begin by describing all possible degenerations of singularities in one-parameter families of tacnodal curves:

Proposition 3.1. *Suppose $\mathcal{C} \to \Delta$ is a family of at-worst tacnodal curves over Δ, the spectrum of a DVR. Denote by $C_{\bar{\eta}}$ the geometric generic fiber and by C_0 the central fiber. Then the only possible limits in C_0 of the singularities of $C_{\bar{\eta}}$ are the following:*

1. *A limit of a tacnode of $C_{\bar{\eta}}$ is necessarily a tacnode of C_0. Moreover, a limit of an outer tacnode is necessarily an outer tacnode.*
2. *A limit of a cusp of $C_{\bar{\eta}}$ is either a cusp or a tacnode of C_0.*
3. *A limit of an inner node of $C_{\bar{\eta}}$ is either a node, a cusp, or a tacnode of C_0.*
4. *A limit of an outer node of $C_{\bar{\eta}}$ is either an outer node of C_0 or an outer tacnode of C_0. Moreover, if an outer tacnode of C_0 is a limit of an outer node, it must be a limit of two outer nodes, necessarily joining the same components.*

Proof. By deformation theory of A-singularities, a cusp deforms only to a node, a tacnode deforms only either to a cusp, or to a node, or to two nodes. Given this, the result follows directly from [AFSv14, Proposition 2.10]. □

We describe the operation of normalization along the generic singularities for each of the following degenerations:

(A) Inner nodes degenerate to cusps and tacnodes (see Proposition 3.3).
(B) Outer nodes degenerate to tacnodes (see Proposition 3.4).
(C) Cusps degenerate to tacnodes (see Proposition 3.5).

We begin with a preliminary result concerning normalization along a collection of generic nodes. Suppose $\pi\colon\ \mathcal{X}\to B$ is a family in $\widetilde{\mathcal{U}}_g(A)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are distinct nodes of $\mathcal{X}_b$ for a generic $b\in B$ and such that $\{\sigma_i(B)\}_{i=1}^k$ do not meet any other generic singularities. (The last condition will be automatically satisfied when $\{\sigma_i\}_{i=1}^k$ is the collection of all inner or all outer nodes.) Let $\nu\colon\ \mathcal{Y}\to\mathcal{X}$ be the normalization of $\mathcal{X}$ along $\cup_{i=1}^k\sigma_i(B)$. Denote by $\{\eta_i^+,\eta_i^-\}$ the two preimages of σ_i (which exist after a base change). Let $R_i^+\colon\ \nu_*\mathcal{O}_{\mathcal{Y}}\to\mathcal{O}_{\sigma_i(B)}$ (resp., $R_i^-\colon\ \nu_*\mathcal{O}_{\mathcal{Y}}\to\mathcal{O}_{\sigma_i(B)}$) be the morphisms of sheaves on $\mathcal{X}$ induced by pushing forward the restriction maps $\mathcal{O}_{\mathcal{Y}}\to\mathcal{O}_{\eta_i^{\pm}(B)}$ and composing with the natural isomorphisms $\nu_*(\mathcal{O}_{\eta_i^{\pm}(B)})\simeq\mathcal{O}_{\sigma_i(B)}$. We let $R_i:=R_i^+-R_i^-$ be the difference map, and set

$$R:=\oplus_{i=1}^k R_i\colon\ \nu_*\mathcal{O}_{\mathcal{Y}}\longrightarrow\oplus_{i=1}^k\mathcal{O}_{\sigma_i(B)}.$$

In this notation, we have the following result.

Lemma 3.2. *There is an exact sequence*

$$0\to\mathcal{O}_{\mathcal{X}}\xrightarrow{\nu^{\#}}\nu_*\mathcal{O}_{\mathcal{Y}}\xrightarrow{R}\oplus_{i=1}^k\mathcal{O}_{\sigma_i(B)}\to\mathcal{K}\to 0,\tag{3.1}$$

where $\mathcal{K}$ is supported on the finitely many points of $\mathcal{X}$ at which the generic nodes $\{\sigma_i(B)\}_{i=1}^k$ degenerate to worse singularities. Consequently,

$$\lambda_{\mathcal{X}/B}=\lambda_{\mathcal{Y}/B}+\operatorname{length}(\pi_*\mathcal{K}).$$

Proof. Away from finitely many points on $\mathcal{X}$ where the generic nodes degenerate, we have $\operatorname{im}(\nu^{\#})=\ker(R)$ and R is surjective. Consider now a point $p\in\mathcal{X}$ where a generic nodes coalesce to an A_{2a-1}-singularity. A local chart of $\mathcal{X}$ around p can be taken to be

$$\operatorname{Spec}\mathbb{C}[[x,y,t]]/\big(y^2-(x-s_1(t))^2\cdots(x-s_a(t))^2f(x,t)\big),$$

where $x=s_i(t)$ are the equations of generic nodes. By assumption on the generic nodes, $f(x,t)$ is a square-free polynomial. Hence

$$\mathcal{Y}=\operatorname{Spec}\mathbb{C}[[x,u,t]]/\big(u^2-f(x,t)\big)$$

and the normalization map is given by

$$y\mapsto u\prod_{i=1}^a(x-s_i(t)).$$

Without loss of generality, the equation of $\eta_i^{\pm}$ is $u=\pm v_i(t)$, where $v_i(t)^2=f(s_i(t),t)$. It follows that $R_i\colon\ \mathbb{C}[[x,u,t]]/(u^2-f(x,t))\to\mathbb{C}[[t]]$ is given by

$$R_i\big(g(x,u,t)\big)=g(s_i(t),v_i(t),t)-g(s_i(t),-v_i(t),t).$$

Write $\mathbb{C}[[x,u,t]]/(u^2-f(x,t))=\mathbb{C}[[x,t]]+u\mathbb{C}[[x,t]]$. Clearly, $\mathbb{C}[[x,t]]\subset\ker(R)\cap\operatorname{im}(\nu^{\#})$. Note that $ug(x,t)\in\ker(R)$ if and only if $R_i\big(ug(x,t)\big)=2v_i(t)g(s_i(t),t)=0$ for every i if and only if $g(x,t)\in(x-s_i(t))$ for every i. Since the generic nodes are distinct, we conclude that $ug(x,t)\in\ker(R)$ if and only if $\prod_{i=1}^a(x-s_i(t))\mid g(x,t)$ if and only if $ug(x,t)\in y\mathbb{C}[[x,t]]\subset\operatorname{im}(\nu^{\#})$. The exactness of (3.1) follows.

Pushing forward (3.1) to B and noting that $c_1((\pi\circ\nu)_*\mathcal{O}_{\mathcal{Y}}) = c_1(\pi_*\mathcal{O}_X) = c_1(\pi_*\mathcal{O}_{s_i(B)}) = 0$, we obtain

$$c_1(R^1(\pi\circ\nu)_*\mathcal{O}_{\mathcal{Y}}) = c_1(R^1\pi_*\mathcal{O}_{\mathcal{X}}) + c_1(\pi_*\mathcal{K}).$$

The formula relating Hodge classes now follows by relative Serre duality. $\square$

Proposition 3.3 (Type A degeneration)**.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are distinct inner nodes of $\mathcal{X}_b$ for a generic $b\in B$, degenerating to cusps and tacnodes over a finite set of points of B. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\cup_{i=1}^k\sigma_i(B)$ and by $\{\eta_i^+,\eta_i^-\}$ the two preimages of σ_i. Then $\{\eta_i^\pm\}$ are sections of $\mathcal{Y}/B$ satisfying:*

1. *If $\sigma_i(b)$ is a cusp of $\mathcal{X}_b$, then $\eta_i^+(b)=\eta_i^-(b)$ is a smooth point of $\mathcal{Y}_b$.*
2. *If $\sigma_i(b)$ is a tacnode of $\mathcal{X}_b$ and $\sigma_j(b)\neq\sigma_i(b)$ for all $j\neq i$, then $\eta_i^+(b)=\eta_i^-(b)$ is a node of $\mathcal{Y}_b$ and $\eta_i^++\eta_i^-$ is Cartier at b.*
3. *If $\sigma_i(b)=\sigma_j(b)$ is a tacnode of $\mathcal{X}_b$ for some $i\neq j$, then (up to $\pm$) $\eta_i^+(b)=\eta_j^+(b)$ and $\eta_i^-(b)=\eta_j^-(b)$ are smooth and distinct points of $\mathcal{Y}_b$.*

Set $\eta_i:=\eta_i^++\eta_i^-$ and $\psi_{\eta_i}:=\omega_{\mathcal{Y}/B}\cdot\eta_i=\psi_{\eta_i^+}+\psi_{\eta_i^-}$. Define

$$\psi_{inner}:=\sum_{i=1}^k\psi_{\eta_i},\quad \delta_{tacn}:=\sum_{i\neq j}\big(\eta_i\cdot\eta_j\big),\quad \textit{and}\quad \delta_{inner}=\sum_{i=1}^k(\eta_i^+\cdot\eta_i^-).$$

Then we have the following formulae:

$$\lambda_{\mathcal{X}/B}=\lambda_{\mathcal{Y}/B}+\frac{1}{2}\delta_{tacn}+\delta_{inner}+\sum_{i=1}^k\iota(\eta_i^+),$$

$$\delta_{\mathcal{X}/B}=\delta_{\mathcal{Y}/B}-\psi_{inner}+4\delta_{tacn}+10\delta_{inner}+10\sum_{i=1}^k\iota(\eta_i^+).$$

A pair of sections $\{\eta_i^+,\eta_i^-\}$ arising from the normalization of a generic inner node will be called inner nodal pair *and $\eta_i^\pm$ will be called* inner nodal transforms.

Proof. The formula for the Hodge class follows from Lemma 3.2, whose notation we keep, once we analyze the torsion sheaf $\mathcal{K}$ on $\mathcal{X}$. Consider the following loci in $\mathcal{X}$:

(a) Cu is the locus of cusps in $\mathcal{X}/B$ which are limits of generic inner nodes.
(b) Tn_1 is the locus of tacnodes in $\mathcal{X}/B$ which are limits of a single generic inner node.
(c) Tn_2 is the locus of tacnodes in $\mathcal{X}/B$ which are limits of two generic inner nodes.

(a) A local chart of $\mathcal{X}$ around a point $p\in\mathrm{Cu}$ can be taken to be

$$\operatorname{Spec}\mathbb{C}[[x,y,t]]/\big(y^2-(x-t^{2m})^2(x+2t^{2m})\big),$$

where $x=t^{2m}$ is the equation of the generic node σ degenerating to the cusp p. Then $\mathcal{Y}=\operatorname{Spec}\mathbb{C}[[x,u,t]]/\big(u^2-x-2t^{2m}\big)$ and the normalization map is $y\mapsto u(x-t^{2m})$. The preimages η^+ and η^- of the generic node σ have equations $u=\sqrt{3}t^m$ and $u=-\sqrt{3}t^m$.

Note that $\mathcal{Y}$ is smooth and the intersection multiplicity of η^+ and η^- at the preimage of p is m. It follows that the contribution of p to δ_{inner} is m.

The elements of $\mathbb{C}[[x,u,t]]/\big(u^2 - x - 2t^{2m}\big)$ that do not lie in $\ker(R)$ are of the form $ug(x,t)$ and we have $R\big(ug(x,t)\big) = 2\sqrt{3}t^m g(t^{2m}, t)$. It follows that $\operatorname{im}(R) = (t^m) \subset \mathbb{C}[[t]]$. Hence $\mathcal{K}_p = \mathbb{C}[[t]]/\operatorname{im}(R)$ has length m.

(b) A local chart of $\mathcal{X}$ around a point $p \in \mathrm{Tn}_1$ can be taken to be

$$\operatorname{Spec} \mathbb{C}[[x,y,t]]/\big(y^2 - (x-t^m)^2(x^2+t^{2c})\big),$$

where $x = t^m$ is the equation of the generic node σ degenerating to the tacnode p. Then

$$\mathcal{Y} = \operatorname{Spec} \mathbb{C}[[x,u,t]]/(u^2 - x^2 - t^{2c})$$

is a normal surface with A_{2c-1}-singularity at the preimage of p, and the normalization map is given by

$$y \mapsto u(x - t^m).$$

The preimage of σ is the bi-section given by the equation $u^2 = t^{2m} + t^{2c}$, which splits into two sections given by the equations $u = \pm v(t)$, where the valuation of $v(t)$ is equal to $\min\{m, c\}$. The map $R\colon \mathbb{C}[[x,u,t]]/(u^2-x^2-t^{2c}) \to \mathbb{C}[[t]]$ sends an element of the form $ug(x,t)$ to $2v(t)g(t^m,t)$ and everything else to 0. We conclude that $\mathcal{K}_p = \mathbb{C}[[t]]/\operatorname{im}(R)$ has length $\min\{m,c\}$.

It remains to show that the contribution of p to $\big(\eta^+\cdot\eta^- + \iota(\eta^+)\big)$ is $\min\{m,c\}$. There are two cases to consider. First, suppose $c \leq m$. Then the equations of η^+ and η^- are $u = \alpha t^c$ and $u = -\alpha t^c$ where $\alpha \neq 0$ is a unit in $\mathbb{C}[[t]]$. The minimal resolution $h\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}$ has the exceptional divisor

$$E_1 \cup \cdots \cup E_{2c-1},$$

which is a chain of (-2)-curves. The strict transforms $\widetilde{\eta}^+$ and $\widetilde{\eta}^-$ meet the central (-2)-curve E_c at two distinct points. Clearly, $h^*\omega_{\mathcal{Y}/B} = \omega_{\widetilde{\mathcal{Y}}/B}$ and a straightforward computation shows that

$$h^*(\eta^+ + \eta^-) = \widetilde{\eta}^+ + \widetilde{\eta}^- + \sum_{i=1}^{c-1} i(E_i + E_{2c-i}) + cE_c.$$

It follows that the contribution of p to $\big(\eta^+\cdot\eta^- + \iota(\eta^+)\big) = (\omega_{\mathcal{X}/B} + \eta^+ + \eta^-)\cdot\eta^+$ is c.

Suppose now that $c > m$. Then the equations of η^+ and η^- are $u = \alpha t^m$ and $u = -\alpha t^m$, respectively, where $\alpha \neq 0$ is a unit in $\mathbb{C}[[t]]$. The exceptional divisor of the minimal resolution $h\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is still a chain of (-2)-curves of length $2c-1$. However, $\widetilde{\eta}^+$ and $\widetilde{\eta}^-$ now meet E_m and E_{2c-m}, respectively. It follows that the contribution of p to $\big(\eta^+\cdot\eta^- + \iota(\eta^+)\big)$ is m.

(c) A local chart of $\mathcal{X}$ around a point $p \in \mathrm{Tn}_2$ can be taken to be

$$\operatorname{Spec} \mathbb{C}[[x,y,t]]/\big(y^2 - (x-t^m)^2(x+t^m)^2\big),$$

where $x = t^m$ and $x = -t^m$ are the equations of the generic nodes $\{\sigma_1, \sigma_2\}$ coalescing to the tacnode p. Then

$$\mathcal{Y} = \operatorname{Spec} \mathbb{C}[[x,u,t]]/(u^2 - 1)$$

is a union of two smooth sheets, and the normalization map is given by

$$y \mapsto u(x-t^m)(x+t^m).$$

The preimages η_1^+ and η_1^- of the generic node σ_1 have equations $\{u=1, x=t^m\}$ and $\{u=-1, x=t^m\}$. The preimages η_2^+ and η_2^- of the generic node σ_2 have equations $\{u=1, x=-t^m\}$ and $\{u=-1, x=-t^m\}$. In particular, $\eta_j^{\pm}$ are smooth sections, with η_1^+ meeting η_2^+, and η_1^- meeting η_2^-, each with intersection multiplicity m. It follows that the contribution of p to δ_{tacn} is $2m$.

The elements of $\mathbb{C}[[x,u,t]]/(u^2-1)$ that do not lie in $\ker(R)$ are of the form $ug(x,t)$ and we have $R(ug(x,t)) = (2g(t^m,t), 2g(-t^m,t)) \in \mathbb{C}[[t]] \oplus \mathbb{C}[[t]]$. It follows that

$$\mathrm{im}\,(R) = \langle (1,1), (t,t), \dots, (t^{m-1}, t^{m-1})\rangle + (t^m) \times (t^m) \subset \mathbb{C}[[t]] \times \mathbb{C}[[t]].$$

Hence $\mathcal{K}_p = (\mathbb{C}[[t]] \oplus \mathbb{C}[[t]])/\mathrm{im}\,(R)$ has length m.

It remains to prove the formula for the boundary classes. To do this, note that $\nu^*\omega_{\mathcal{X}/B} = \omega_{\mathcal{Y}/B}\big(\sum_{i=1}^k (\eta_i^+ + \eta_i^-)\big)$. Therefore,

$$\begin{aligned}\kappa_{\mathcal{X}/B} &= \kappa_{\mathcal{Y}/B} \\ &+ 2\sum_{1\le i<j\le k} \big((\eta_i^+ + \eta_i^-)\cdot(\eta_j^+ + \eta_j^-)\big) + 2\omega_{\mathcal{Y}/B}\cdot\sum_{i=1}^k (\eta_i^+ + \eta_i^-) + \sum_{i=1}^k (\eta_i^+ + \eta_i^-)^2 \\ &= \kappa_{\mathcal{Y}/B} + 2\delta_{tacn} \\ &+ \omega_{\mathcal{Y}/B}\cdot\sum_{i=1}^k (\eta_i^+ + \eta_i^-) + \sum_{i=1}^k \big(\omega_{\mathcal{Y}/B}\cdot\eta_i^+ + (\eta_i^+)^2 + \omega_{\mathcal{Y}/B}\cdot\eta_i^- + (\eta_i^-)^2\big) + 2\sum_{i=1}^k \big(\eta_i^+\cdot\eta_i^-\big) \\ &= \kappa_{\mathcal{Y}/B} + 2\delta_{tacn} + \psi_{inner} + 2\sum_{i=1}^k \iota(\eta_i^+) + 2\delta_{inner}.\end{aligned}$$

Using Mumford's relation $\kappa = 12\lambda - \delta$ and the already established relation between $\lambda_{\mathcal{X}/B}$ and $\lambda_{\mathcal{Y}/B}$, we obtain the desired relation between $\delta_{\mathcal{X}/B}$ and $\delta_{\mathcal{Y}/B}$. □

Proposition 3.4 (Type B degeneration)**.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are outer nodes of $\mathcal{X}_b$ for a generic $b \in B$, degenerating to outer tacnodes over a finite set of points of B. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\cup_{i=1}^k \sigma_i(B)$ and by $\{\zeta_i^+, \zeta_i^-\}$ the two preimages of σ_i. Then $\{\zeta_i^{\pm}\}_{i=1}^k$ are smooth sections of $\mathcal{Y}$ such that ζ_i^+ and ζ_i^- lie on different irreducible components of $\mathcal{Y}$. Setting*

$$\delta_{tacn} := \sum_{i\neq j} (\zeta_i^+ + \zeta_i^-)\cdot(\zeta_j^+ + \zeta_j^-),$$

we have the following formulae:

$$\begin{aligned}\lambda_{\mathcal{X}/B} &= \lambda_{\mathcal{Y}/B} + \frac{1}{2}\delta_{tacn}, \\ \delta_{\mathcal{X}/B} &= \delta_{\mathcal{Y}/B} - \sum_{i=1}^k (\psi_{\zeta_i^+} + \psi_{\zeta_i^-}) + 4\delta_{tacn}.\end{aligned}$$

The sections $\{\zeta_i^+, \zeta_i^-\}_{i=1}^k$ *will be called* outer nodal transforms.

Proof. By Proposition 3.1, outer nodes can degenerate only to outer tacnodes. Moreover, an outer tacnode which is a limit of one outer node is a limit of two outer nodes. The statement now follows by repeating verbatim the proof of Proposition 3.3 (Part (c)), and using Lemma 3.2. □

Proposition 3.5 (Type C degeneration)**.** *Suppose* $\mathcal{X}/B$ *is a family in* $\widetilde{\mathcal{U}}_g$ *with sections* $\{\sigma_i\}_{i=1}^k$ *such that* $\sigma_i(b)$ *is a cusp of* $\mathcal{X}_b$ *for a generic* $b \in B$*, degenerating to a tacnode over a finite set of points in* B*. Denote by* $\mathcal{Y}$ *the normalization of* $\mathcal{X}$ *along* $\cup_{i=1}^k \sigma_i(B)$ *and by* ξ_i *the preimage of* σ_i*. Then* ξ_i *is a section of* $\mathcal{Y}/B$ *such that* $\xi_i(b)$ *is a node of* $\mathcal{Y}_b$ *whenever* $\sigma_i(b)$ *is a tacnode of* $\mathcal{X}_b$ *and* $\xi_i(b)$ *is a smooth point of* $\mathcal{Y}_b$ *otherwise. Moreover,* $2\xi_i$ *is Cartier and we have the following formulae:*

$$\lambda_{\mathcal{X}/B} = \lambda_{\mathcal{Y}/B} - \sum_{i=1}^k \psi_{\xi_i} + 2\sum_{i=1}^k \iota(\xi_i),$$

$$\delta_{\mathcal{X}/B} = \delta_{\mathcal{Y}/B} - 12\sum_{i=1}^k \psi_{\xi_i} + 20\sum_{i=1}^k \iota(\xi_i).$$

The sections ξ_i *will be called* cuspidal transforms*.*

Proof. The proof of this proposition is easier than the previous two results because a generic cusp cannot collide with another generic singularity. In particular, we can consider the case of a single generic cusp σ. Let $\nu\colon \mathcal{Y} \to \mathcal{X}$ be the normalization along σ. Suppose $\sigma(b)$ is a tacnode. Then the local equation of $\mathcal{X}$ around $\sigma(b)$ is

$$y^2 = (x - a(t))^3(x + 3a(t)),$$

where $x = a(t)$ is the equation of the generic cusp. It follows that $\mathcal{Y}$ has local equation $u^2 = (x - a(t))(x + 3a(t))$ and ν is given by $y \mapsto u(x - a(t))$. The preimage of σ is a section $\xi\colon B \to \mathcal{Y}$ given by $x - a(t) = u = 0$. Note that $\xi(b) = \{x = u = t = 0\}$ is a node of $\mathcal{Y}_b$, and consequently ξ is not Cartier at $\xi(b)$.

Clearly, $\nu^*\omega_{\mathcal{X}/B} = \omega_{\mathcal{Y}/B}(2\xi)$ and by duality theory for singular curves

$$\pi_*\omega_{\mathcal{X}/B} = (\pi \circ \nu)_*(\omega_{\mathcal{Y}/B}(2\xi)).$$

Therefore,

$$\kappa_{\mathcal{X}/B} = (\omega_{\mathcal{Y}/B} + 2\xi)^2 = (\omega_{\mathcal{Y}/B})^2 + 4(\xi^2 + \xi \cdot \omega_{\mathcal{Y}/B}) = \kappa_{\mathcal{Y}/B} + 4\iota(\xi),$$

and by Grothendieck-Riemann-Roch formula

$$\lambda_{\mathcal{X}/B} = c_1\big((\pi \circ \nu)_*(\omega_{\mathcal{Y}/B}(2\xi))\big) = \lambda_{\mathcal{Y}/B} - \psi_\xi + 2\iota(\xi).$$

The claim follows. □

4. Assorted positivity results

4.1. **Cornalba-Harris inequality.** We generalize a well-known Cornalba-Harris result on the positivity of divisor classes for generically smooth families of Deligne-Mumford curves to the case of tacnodal curves.

Proposition 4.1 (Cornalba-Harris inequality)**.** *Let $g \geq 2$. Suppose $f\colon \mathcal{C} \to B$ is a generically smooth family in $\widetilde{\mathcal{U}}_g(A_3)$, over a smooth and proper curve B, with $\omega_{\mathcal{C}/B}$ relatively nef. Then*

$$\left(8 + \frac{4}{g}\right) \lambda_{\mathcal{C}/B} - \delta_{\mathcal{C}/B} \geq 0.$$

Moreover, if the general fiber of $\mathcal{C}/B$ is non-hyperelliptic and $\mathcal{C}/B$ is non-isotrivial, then the inequality is strict.

Remark. When the total space $\mathcal{C}$ is *smooth*, this result was proved in [Xia87] and [Sto08, Theorem 2.1], with no restrictions on fiber singularities.

Proof. As in [Sto08, Theorem 2.1], if the general fiber of $\mathcal{C}/B$ is non-hyperelliptic, the result is obtained by the original argument of Cornalba and Harris [CH88], which we now recall.

Suppose C_b for some $b \in B$ is a non-hyperelliptic curve of genus $g \geq 3$. After a finite base change, we can assume that $\lambda \in \operatorname{Pic}(B)$ is g-divisible. Then the line bundle $\mathcal{L} := \omega_{\mathcal{C}/B} \otimes f^*(-\lambda/g)$ on $\mathcal{C}$ satisfies the following conditions:

(1) $\det(f_*(\mathcal{L})) \simeq \mathcal{O}_B$.
(2) $f_*(\mathcal{L}^m)$ is a vector bundle of rank $(2m-1)(g-1)$ for all $m \geq 2$.
(3) $\operatorname{Sym}^m f_*(\mathcal{L}) \to f_*(\mathcal{L}^m)$ is generically surjective for all $m \geq 1$.

For $m \geq 2$ and general $b \in B$, the map $\operatorname{Sym}^m \mathrm{H}^0(C_b, \omega_{C_b}) \to \mathrm{H}^0(C_b, \omega_{C_b}^m)$ defines the m^{th} Hilbert point of C_b. Since the canonical embedding of C_b has a stable m^{th} Hilbert point for some $m \gg 0$ by [Mor09, Lemma 14], the proof of [CH88, Theorem 1.1] gives $c_1(f_*(\mathcal{L}^m)) \geq 0$. Using (2.1), we obtain

$$\left(8 + \frac{4}{g} - \frac{2(g-1)}{gm} + \frac{2}{gm(m-1)}\right) \lambda - \delta = c_1(f_*(\mathcal{L}^m)) \geq 0. \tag{4.1}$$

To conclude we note that $\delta \geq 0$, and if $\delta = 0$, then $\lambda > 0$ for any non-isotrivial family by the existence of the Torelli morphism $\overline{M}_g \to \overline{\mathcal{A}}_g$. We conclude that $(8 + 4/g)\,\lambda - \delta > 0$.

Suppose now that $\mathcal{C} \to B$ is a family of at-worst tacnodal curves with a relatively nef $\omega_{\mathcal{C}/B}$ and a smooth hyperelliptic generic fiber. To prove the requisite inequality, we construct $\mathcal{C}/B$ explicitly as a double cover of a family of $(2g+2)$-pointed curves, and prove a corresponding inequality on families of rational pointed curves.

Suppose that $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^{2g+2})$ is a family of $(2g+2)$-pointed at-worst nodal rational curves where σ_i are smooth sections and no more than 4 sections meet at a point. We say that an irreducible component E in the fiber Y_b of $\mathcal{Y}/B$ is an odd bridge if the following conditions hold:

(1) E meets the rest of the fiber $\overline{Y_b \setminus E}$ in two nodes of equal index,

(2) $E \cdot \sum_{i=1}^{2g+2} \sigma_i = 2$,
(3) the degree of $\sum_{i=1}^{2g+2} \sigma_i$ on each of the connected components of $\overline{Y_b \setminus E}$ is odd.

Suppose $h\colon \mathcal{Y} \to \mathcal{Z}$ is a blow-down of some collection of odd bridges. The image of $\sum_{i=1}^{2g+2} \sigma_i$ in $\mathcal{Z}$ will be denoted by Σ. Note that while the individual images of σ_i's are not Cartier on $\mathcal{Z}$ along the image of blown-down odd bridges, the total class of Σ is Cartier on $\mathcal{Z}$. We say that a node $p \in \mathcal{Z}_b$ (resp., $p \in \mathcal{Y}_b$) is an odd node if the degree of Σ (resp., $\sum_{i=1}^{2g+2} \sigma_i$) on each of the connected component of the normalization of $\mathcal{Z}_b$ (resp., $\mathcal{Y}_b$) at p is odd. We denote by δ_{odd} the Cartier divisor on B associated to all odd nodes of $\mathcal{Z}/B$ (resp., $\mathcal{Y}/B$).

The hyperelliptic involution on the generic fiber of $f\colon \mathcal{C} \to B$ extends to all of $\mathcal{C}$ and realizes $\mathcal{C}/B$ as a double cover of a family $(\mathcal{Z}/B, \Sigma)$ described above in such a way that $\mathcal{C} \to \mathcal{Z}$ ramifies over Σ. Let δ_{odd} be the divisor of odd nodes of $\mathcal{Z}/B$. We have the following standard formulae:

$$\lambda_{\mathcal{C}/B} = \frac{1}{8}\left(\Sigma^2 + 2\omega_{\mathcal{Z}/B} \cdot \Sigma - \delta_{odd}\right)_{\mathcal{Z}/B},$$
$$\delta_{\mathcal{C}/B} = \left(\Sigma^2 + \omega_{\mathcal{Z}/B} \cdot \Sigma + 2\omega_{\mathcal{Z}/B}^2 - \frac{3}{2}\delta_{odd}\right)_{\mathcal{Z}/B}.$$

Consider $h\colon \mathcal{Y} \to \mathcal{Z}$. Then $h^*(\Sigma) = \sum_{i=1}^{2g+2} \sigma_i + E$, where E is a collection of odd bridges, and $h^*\omega_{\mathcal{Z}/B} = \omega_{\mathcal{Y}/B}$. Set $\psi_{\mathcal{Y}/B} := \omega_{\mathcal{Y}/B} \cdot \sum_{i=1}^{2g+2} \sigma_i$, $\delta_{inner} := \sum_{i \neq j} (\sigma_i \cdot \sigma_j)$, and $e := -\frac{1}{2}E^2$. Then

$$\lambda_{\mathcal{C}/B} = \left(\frac{1}{8}(\psi_{\mathcal{Y}/B} + 2\delta_{inner} - \delta_{odd}) + \frac{1}{2}e\right)_{\mathcal{Y}/B},$$
$$\delta_{\mathcal{C}/B} = \left(2\delta_{inner} + 2\delta_{even} + \frac{1}{2}\delta_{odd} + 5e\right)_{\mathcal{Y}/B}.$$

We obtain

$$\left(8 + \frac{4}{g}\right)\lambda_{\mathcal{C}/B} - \delta_{\mathcal{C}/B} = \left(\frac{2g+1}{2g}\psi + \frac{1}{g}\delta_{inner} + \left(\frac{2}{g} - 1\right)e - 2\delta_{even} - \left(\frac{3}{2} + \frac{1}{2g}\right)\delta_{odd}\right)_{\mathcal{Y}/B}.$$

Multiplying by $2g$, we need to show that on $\mathcal{Y}/B$ we have

$$(2g+1)\psi + 2\delta_{inner} - 4g\delta_{even} - (3g+1)\delta_{odd} - (2g-4)e \geq 0.$$

Noting that

$$(2g+1)\psi + 2\delta_{inner} = \sum_{i=2}^{g+1} i(2g+2-i)\delta_i,$$

and using the inequality $2e \leq \delta_{odd}$, we obtain the desired claim. □

Hodge Index Theorem Inequalities. We apply a method of Harris [Har84] to obtain inequalities between the ψ-classes, indices of cuspidal and inner nodal transforms, and the kappa class. In the following lemmas, we use the following variant of Hodge Index Theorem for singular surfaces.

Lemma 4.2. *Let S be a proper integral algebraic space of dimension 2. Suppose H is a $\mathbb{Q}$-Cartier divisor on S such that $H^2 > 0$. Then the intersection pairing on any subspace of $\mathrm{NS}(S)$ containing H has signature $(1, \ell)$.*

Proof. Let $\pi\colon \widetilde{S} \to S$ be the minimal desingularization of the normalization of S. Then $\widetilde{S}$ is a smooth projective surface. Note that $\pi^*\colon\ \mathrm{NS}(S) \to \mathrm{NS}(\widetilde{S})$ is an injection preserving the intersection pairing. The statement now follows from the Hodge Index Theorem for smooth projective surfaces. $\square$

Lemma 4.3. *Suppose $\mathcal{X}/B$ is a family of Gorenstein curves of arithmetic genus $g \geq 2$ with a section ξ. Assume $\mathcal{X}$ is irreducible. Let $\iota(\xi) = (\xi + \omega_{\mathcal{X}/B}) \cdot \xi$ be the index of ξ. Then*

$$\psi_\xi \geq \frac{(g-1)}{g}\iota(\xi) + \frac{\kappa}{4g(g-1)}. \tag{4.2}$$

Proof. Apply the Hodge Index Theorem to the three classes $\langle F, \xi, \omega_{\mathcal{X}/B}\rangle$, where F is the fiber class. Since $\xi + kF$ has positive self-intersection for $k \gg 0$, the determinant of the following intersection pairing matrix is non-negative:

$$\begin{pmatrix} 0 & 1 & 2g-2 \\ 1 & -\psi_\xi + \iota(\xi) & \psi_\xi \\ 2g-2 & \psi_\xi & \kappa \end{pmatrix}.$$

The claim follows by expanding the determinant. $\square$

Lemma 4.4. *Suppose $\mathcal{X}/B$ is a family of Gorenstein curves of arithmetic genus $g \geq 2$ with a pair of sections η^+, η^-. Assume $\mathcal{X}$ is irreducible. Then*

$$\psi_{\eta^+} + \psi_{\eta^-} \geq \frac{2(g-1)}{g+1}\big((\eta^+ \cdot \eta^-) + \iota(\eta^+)\big) + \frac{\kappa}{g^2-1}. \tag{4.3}$$

Proof. Consider the three divisor classes $\langle F, \eta = \eta^+ + \eta^-, \omega_{\mathcal{X}/B}\rangle$, where F is the fiber class. Since $\eta + kF$ has positive self-intersection for $k \gg 0$, the Hodge Index Theorem implies that the determinant of the following intersection pairing matrix is non-negative:

$$\begin{pmatrix} 0 & 2 & 2g-2 \\ 2 & -\psi_{\eta^+} - \psi_{\eta^-} + 2(\eta^+ \cdot \eta^-) + \iota(\eta^+) + \iota(\eta^-) & \psi_{\eta^+} + \psi_{\eta^-} \\ 2g-2 & \psi_{\eta^+} + \psi_{\eta^-} & \kappa \end{pmatrix}.$$

The claim follows by expanding the determinant. $\square$

Lemma 4.5. *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_2(A_3)$ with a smooth section τ. Assume $\mathcal{X}$ is irreducible. Then*

$$8\psi_\tau \geq \kappa. \tag{4.4}$$

Moreover, if $\delta_{\mathrm{red}} = 0$, then the equality is satisfied if and only if $(\mathcal{X}/B, \tau)$ is a family of Weierstrass tails in $\overline{\mathcal{M}}_{2,1}(7/10 - \epsilon)$.

Proof. The inequality follows directly from Lemma 4.3 by taking $g = 2$. Moreover, the proof of Lemma 4.3 shows that equality holds if and only if the intersection pairing on $\langle F, \tau, \omega_{\mathcal{X}/B}\rangle$ is degenerate. Assuming $\delta_{\mathrm{red}} = 0$, there is a global hyperelliptic involution $h\colon \mathcal{X} \to \mathcal{X}$. Hence $\omega_{\mathcal{X}/B} \equiv \tau + h(\tau) + xF$, for some $x \in \mathbb{Z}$. Observe that $\omega_{\mathcal{X}/B} \cdot \tau = \omega_{\mathcal{X}/B} \cdot h(\tau)$ and $F \cdot \tau = F \cdot h(\tau)$. Since no combination of ω and F is in the kernel of the intersection pairing, we conclude that

$$\tau^2 = \tau \cdot h(\tau).$$

However, the intersection number on the left is negative by Lemma 2.1 and the intersection number on the right is non-negative whenever $\tau \neq h(\tau)$. We conclude that equality holds if only if $h(\tau) = \tau$, that is τ is a Weierstrass section. $\square$

We will need special variants of Lemmas 4.3 and 4.4 for the case of relative genus 1 and 0.

Lemma 4.6. *Let $\mathcal{X}/B$ be a family of Gorenstein curves of arithmetic genus 1 with a pair of sections η^+, η^-, and suppose that η^+ and η^- are disjoint from N smooth pairwise disjoint section of $\mathcal{X}/B$. Assume $\mathcal{X}$ is irreducible. Then*

$$(\eta^+ \cdot \eta^-) + \iota(\eta^+) \leq \frac{N+2}{2N}(\psi_{\eta^+} + \psi_{\eta^-}) + \frac{1}{2N^2}\delta_{\mathrm{red}}.$$

Proof. Let Σ be the sum of N pairwise disjoint smooth sections of $\mathcal{X}/B$ disjoint from $\{\eta^+, \eta^-\}$. Then $(\omega_{\mathcal{X}/B} + 2\Sigma)^2 = \omega_{\mathcal{X}/B}^2 = \kappa$. Apply the Hodge Index Theorem to $\langle F, \eta^+ + \eta^-, \omega_{\mathcal{X}/B} + 2\Sigma\rangle$, where F is the fiber class. The determinant of the matrix

$$\begin{pmatrix} 0 & 2 & 2N \\ -\psi_{\eta^+} - \psi_{\eta^-} + 2(\eta^+ \cdot \eta^-) + \iota(\eta^+) + \iota(\eta^-) & \psi_{\eta^+} + \psi_{\eta^-} & \\ 2N & \psi_{\eta^+} + \psi_{\eta^-} & \kappa \end{pmatrix}$$

is non-negative. Therefore

$$-4\kappa + 8N(\psi_{\eta^+} + \psi_{\eta^-}) + 4N^2(\psi_{\eta^+} + \psi_{\eta^-}) \geq 8N^2\big((\eta^+ \cdot \eta^-) + \iota(\eta^+)\big),$$

which gives the desired inequality using $\kappa = -\delta_{\mathrm{red}}$. $\square$

Lemma 4.7. *Let $\mathcal{X}/B$ be a family of Gorenstein curves of arithmetic genus 1 with a section ξ, and suppose that ξ is disjoint from N smooth pairwise disjoint sections of $\mathcal{X}$. Assume $\mathcal{X}$ is irreducible. Then*

$$\iota(\xi) \leq \frac{N+1}{N}\psi_\xi + \frac{1}{4N^2}\delta_{\mathrm{red}}.$$

Furthermore, suppose $N = 1$, with τ being a smooth section disjoint from ξ, and $\delta_{\mathrm{red}} = 0$. Then equality holds if and only if $2\xi \sim 2\tau$.

Proof. Let Σ be the collection of smooth sections of $\mathcal{X}/B$ disjoint from ξ. By the Hodge Index Theorem applied to $\langle F, \xi, \omega_{\mathcal{X}/B} + 2\Sigma\rangle$, the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 2N \\ 1 & -\psi_\xi + \iota(\xi) & \psi_\xi \\ 2N & \psi_\xi & \kappa \end{pmatrix}$$

is non-negative. Therefore

$$\iota(\xi) \leq \psi_\xi + \frac{1}{N}\psi_\xi - \frac{1}{4N^2}\kappa.$$

This gives the desired inequality using $\kappa = -\delta_{\mathrm{red}}$.

To prove the last assertion observe that because $\delta_{\mathrm{red}} = 0$ all fibers of $\mathcal{X}/B$ are irreducible curves of genus 1. In particular, $\omega_{\mathcal{X}/B} = \lambda F$ and it follows from the existence of the group law on the set of sections of $\mathcal{X}/B$ that there exists a section τ' such that $2\xi - \tau = \tau'$. Since $\tau \cap \xi = \emptyset$, we have $\tau' \cap \xi = \emptyset$. If equality holds, then the intersection pairing matrix on the classes F, ξ, τ is degenerate. Hence some linear combination $(x\xi + y\tau + zF)$ intersects F, ξ, τ trivially. Clearly, $y \neq 0$. Intersecting with τ, we obtain $y(\tau \cdot \tau) + z = 0$; and intersecting with τ', we obtain $y(\tau \cdot \tau') + z = 0$. Hence $\tau^2 = \tau \cdot \tau'$. This leads to a contradiction if $\tau \neq \tau'$. □

4.2. **An inequality between divisor classes on $\overline{\mathcal{M}}_{0,N}$.** The proof of Theorem 1.3 will require the following ad hoc effectivity result on $\overline{\mathcal{M}}_{0,N}$.

Lemma 4.8. *Suppose $\{\eta_i^+, \eta_i^-\}_{i=1}^a$ are sections of a family of N-pointed stable rational curves. Let $\psi_{inner} := \sum_{i=1}^a \left(\psi_{\eta_i^+} + \psi_{\eta_i^-}\right)$ and $\delta_{inner} := \sum_{i=1}^a \delta_{\{\eta_i^+, \eta_i^-\}}$. If $a \geq 2$, then for any generically smooth one-parameter family in $\overline{\mathcal{M}}_{0,N}$, we have*

$$\psi_{inner} \geq 4\delta_{inner} + 4\sum_{i=1}^a \sum_{\beta \notin \{\eta_i^+, \eta_i^-\}_{i=1}^a} \delta_{\{\eta_i^+, \eta_i^-, \beta\}} + 2\frac{a-2}{a-1}\sum_{i \neq j} \delta_{\{\eta_i^\pm, \eta_j^\pm\}} + \frac{5a-9}{a-1}\sum_{i=1}^a \sum_{j \neq i} \delta_{\{\eta_i^+, \eta_i^-, \eta_j^\pm\}}.$$

Proof. For any two distinct ψ-classes on $\overline{\mathcal{M}}_{0,N}$, we have the following standard relation:

$$\psi_\sigma + \psi_\tau = \sum_{S:\ \sigma \in S,\ \tau \notin S} \delta_S. \tag{4.5}$$

We apply (4.5) to the right-hand side of

$$(a-1)\psi_{inner} = \sum_{1 \leq i < j \leq a} (\psi_{\eta_i^\pm} + \psi_{\eta_j^\pm}) - (a-1)\sum_{i=1}^a (\psi_{\eta_i^+} + \psi_{\eta_i^-}).$$

This gives us a formula of the following form:

$$(a-1)\psi_{inner} = \sum c_S \delta_S.$$

We now estimate the coefficients of the boundary divisors appearing on the right-hand side. Suppose there are x pairs $\{\eta_i^+, \eta_i^-\}$ such that $\eta_i^+ \in S$ and $\eta_i^- \notin S$, or vice versa, and that S contains y pairs $\{\eta_i^+, \eta_i^-\}$. Set $z = a - x - y$. Then

$$c_S = ((x + 2y)(x + 2z) - x) - (a-1)x = x(y + z) + 4yz.$$

We have that

(1) $c_S \geq 0$ for every S.
(2) If $S = \{\eta_i^+, \eta_i^-\}$ or $S = \{\eta_i^+, \eta_i^-, \beta\}$, where $\beta \notin \{\eta_i^+, \eta_i^-\}_{i=1}^a$, then $x = 0$ and $y = 1$, and so $c_S = 4(a-1)$.
(3) If $S = \{\eta_i^\pm, \eta_j^\pm\}$ for $i \neq j$, then $x = 2$ and $y = 0$, and so $c_S = 2(a-2)$.
(4) If $S = \{\eta_i^+, \eta_i^-, \eta_j^\pm\}$ for $j \neq i$, then $x = 1$ and $y = 1$, and so $c_S = 5a - 9$.

The claim follows. □

5. Warm-up: Proof of Theorem 1.3(a)

Notice that $10\lambda - \delta + \psi = 0$ on $\overline{\mathcal{M}}_{2,0}(9/11-\epsilon)$ by the standard relation $10\lambda = \delta_{\mathrm{irr}} + 2\delta_{\mathrm{red}}$ that holds for all families in $\mathcal{U}_2$.

We now prove that $10\lambda - \delta + \psi$ is nef on $\overline{\mathcal{M}}_{g,n}(9/11 - \epsilon)$ and has degree 0 precisely on families whose only non-isotrivial components are A_1/A_1-attached elliptic bridges, for all $(g, n) \neq (2, 0)$. Let $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ be a $(9/11 - \epsilon)$-stable family. The proof proceeds by normalizing $\mathcal{C}$ along generic singularities to arrive at a family of generically smooth curves, where the Cornalba-Harris inequality holds, or at a family of low genus curves, where the requisite inequality is established by ad-hoc methods. Keeping in mind that generic outer nodes and generic cusps of $\mathcal{C}/B$ do not degenerate, but generic inner nodes of $\mathcal{C}/B$ can degenerate to cusps, we begin by normalizing generic outer nodes, then normalize generic cusps, and finally normalize generic inner nodes.

5.1. **Reduction 1: Normalization along generic outer nodes.** Let $\mathcal{X}$ be the normalization of $\mathcal{C}$ along generic outer nodes, marked by nodal transforms. By [AFSv14, Lemma 2.17], every connected component of $\mathcal{X}/B$ is a family of generically irreducible $(9/11 - \epsilon)$-stable curves. By Proposition 3.4, we have

$$(10\lambda - \delta + \psi)_{\mathcal{C}/B} = (10\lambda - \delta + \psi)_{\mathcal{X}/B}.$$

We have reduced to proving $10\lambda - \delta + \psi \geq 0$ for a family with generically irreducible fibers.

5.2. **Reduction 2: Normalization along generic cusps.** Suppose $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n)$ is a family of $(9/11 - \epsilon)$-stable curves with generically irreducible fibers. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ along generic cusps. Denote by $\{\xi_i\}_{i=1}^c$ the cuspidal transforms on $\mathcal{Y}$. Set $\psi_{cusp} := \sum_{i=1}^c \psi_{\xi_i}$ and $\psi_{\mathcal{Y}/B} := \psi_{\mathcal{X}/B} + \psi_{cusp}$. Then by Proposition 3.5, we have

$$(10\lambda - \delta + \psi)_{\mathcal{X}/B} = (10\lambda - \delta + \psi)_{\mathcal{Y}/B} + \psi_{cusp}.$$

We have reduced to proving $10\lambda - \delta + \psi + \psi_{cusp} \geq 0$ for a family $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c)$, where

(1) The fibers are at-worst cuspidal and the generic fiber is irreducible and at-worst nodal.
(2) $\{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c$ are smooth sections and $\omega_{\mathcal{Y}/B}(\sum_{i=1}^n \sigma_i + \sum_{i=1}^c \xi_i)$ is relatively ample.[1]

[1] A priori, only $\omega_{\mathcal{Y}/B}(\sum_{i=1}^n \sigma_i + 2\sum_{i=1}^c \xi_i)$ is relatively ample. However, a rational tail cannot meet just a single cuspidal transform because the original family $\mathcal{X}/B$ cannot have cuspidal elliptic tails.

5.3. **Reduction 3: Normalization along generic inner nodes.** Consider the family $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c)$ as in 5.2. Let a be the number of generic inner nodes of $\mathcal{Y}/B$. We let $\mathcal{Z} \to \mathcal{Y}$ be the normalization and denote by η_i^+ and η_i^- the inner nodal transforms of the i^{th} generic node. We obtain a family

$$\big(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\eta_i^{\pm}\}_{i=1}^a, \{\xi_i\}_{i=1}^c\big),$$

where

(1) The fibers are at-worst cuspidal curves and the generic fiber is smooth.
(2) The sections $\{\sigma_i\}_{i=1}^n, \{\eta_i^{\pm}\}_{i=1}^a, \{\xi_i\}_{i=1}^c$ are all smooth and pairwise disjoint, except that η_i^+ can intersect η_i^- for each i.
(3) $\omega_{\mathcal{Z}/B}\big(\sum_{i=1}^n \sigma_i + \sum_{i=1}^a(\eta_i^+ + \eta_i^-) + \sum_{i=1}^c \xi_i\big)$ is relatively ample.

By Proposition 3.3, we have that

$$(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Y}/B} = (10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B},$$

where $\psi_{cusp} = \sum_{i=1}^c \psi_{\xi_i}$ and $\psi_{\mathcal{Z}/B} = \psi_{\mathcal{Y}/B} + \sum_{i=1}^a(\psi_{\eta_i^+} + \psi_{\eta_i^-})$.

We let $N = n + 2a + c$ be the total number of sections of $\mathcal{Z}/B$, including cuspidal and inner nodal transforms. Our proof that $(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B} \geq 0$ will depend on the relative genus h of $\mathcal{Z}/B$.

5.3.1. *Suppose* $h \geq 2$. Passing to the relative minimal model of $\mathcal{Z}/B$ only decreases the degree of $(10\lambda - \delta + \psi + \psi_{cusp})$. Hence we will assume that $\omega_{\mathcal{Z}/B}$ is relatively nef. We still have N smooth and distinct sections (which can now intersect pairwise). With $\omega_{\mathcal{Z}/B}$ relatively nef, we can apply the Cornalba-Harris inequality. If $h \geq 3$, then $10 > 8 + 4/h$ and so $10\lambda - \delta > 0$ by Proposition 4.1. If $h = 2$, then Proposition 4.1 gives $10\lambda - \delta \geq 0$. Lemma 2.1 gives $\psi + \psi_{cusp} > 0$ since we must have $N \geq 1$ (if $N = 0$, then $\mathcal{C}/B$ was a family in $\overline{\mathcal{M}}_{2,0}(9/11 - \epsilon)$).

5.3.2. *Suppose* $h = 1$. Using relations on the stack on N-pointed Gorenstein genus 1 curves inherited from standard relations in $\mathrm{Pic}(\overline{\mathcal{M}}_{1,N})$ given by [AC98, Theorem 2.2], we have $\lambda = \delta_{\mathrm{irr}}/12$, and $\psi = N\delta_{irr}/12 + \sum_S |S|\delta_{0,S} \geq N\delta_{\mathrm{irr}}/12 + 2\delta_{\mathrm{red}}$. If $N \geq 3$, we obtain

$$10\lambda + \psi - \delta \geq 10\delta_{\mathrm{irr}}/12 + N\delta_{\mathrm{irr}}/12 + 2\delta_{\mathrm{red}} - (\delta_{\mathrm{irr}} + \delta_{\mathrm{red}}) > 0.$$

If $N = 2$, we obtain $10\lambda - \delta + \psi \geq \delta_{\mathrm{red}} \geq 0$ and $\psi_{cusp} \geq 0$. We conclude that $10\lambda - \delta + \psi + \psi_{cusp} \geq 0$ with the equality holding if only if $\psi_{cusp} = \delta_{\mathrm{red}} = 0$. This is possible if and only if all fibers are irreducible and there are no cuspidal transforms (by Lemma 2.1), which implies that $\mathcal{X}/B = \mathcal{Y}/B$ is a family of A_1/A_1-attached elliptic bridges.

5.3.3. *Suppose* $h = 0$. Then all fibers of $\mathcal{Z}/B$ are in fact at-worst nodal. Because $\lambda = 0$, we can write $(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B} = \psi - \delta + \psi_{cusp}$. Blow-up the points of intersection of η_i^+ and η_i^- for each i. We obtain a family $\big(\mathcal{W}/B, \{\sigma_i\}_{i=1}^n, \{\eta_i^{\pm}\}_{i=1}^a, \{\xi_i\}_{i=1}^c\big)$ in $\overline{\mathcal{M}}_{0,N}$. Setting $\delta_{inner} := \sum_{i=1}^a \delta_{\{\eta_i^+, \eta_i^-\}}$, we have

$$(\psi - \delta + \psi_{cusp})_{\mathcal{Z}/B} = (\psi - \delta - \delta_{inner} + \psi_{cusp})_{\mathcal{W}/B}\,.$$

If $a = 0$, then $\delta_{inner} = 0$ and we are done because $\psi - \delta > 0$ for any family of Deligne-Mumford stable rational curves, for example by [KM13, Lemma 3.6]. If $a \geq 2$, then by

Lemma 4.8, $\sum_{i=1}^{a}(\psi_{\eta_i^+}+\psi_{\eta_i^-})\geq 4\delta_{inner}$. In addition, $3\psi\geq 4\delta$ by a similar argument. It follows that $\psi>\delta+\delta_{inner}$ and so we are done.

Finally, if $a=1$, then $\big(\mathcal{Y}/B,\{\sigma_i\}_{i=1}^{n},\{\xi_i\}_{i=1}^{b}\big)$ obtained in 5.2 is a family of arithmetic genus 1 (generically nodal) curves and the proof in the case of $h=1$ above goes through without any modifications to show that $(10\lambda-\delta+\psi+\psi_{cusp})_{\mathcal{Y}/B}\geq 0$ with the equality if and only if $\mathcal{X}/B=\mathcal{Y}/B$ is a (generically nodal) elliptic bridge.

6. Proof of Theorem 1.3(b)

In the remaining part of the paper, we prove Theorem 1.3(b). Let $(\mathcal{C}/B,\{\sigma_i\}_{i=1}^{n})$ be a $(7/10-\epsilon)$-stable generically non-isotrivial family of curves. We begin by dealing with the case when $\mathcal{C}/B$ has a generic rosary, or a generic A_1/A_3 or A_3/A_3-attached elliptic bridge. In both cases, generic tacnodes come into play and we will repeatedly use the following result that explains what happens under normalization of a generic tacnode:

Proposition 6.1. *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g$ with a section τ such that $\tau(b)$ is a tacnode of $\mathcal{X}_b$ for all $b\in B$. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along τ and by τ^+ and τ^- the preimages of τ. Then $\tau^{\pm}$ are smooth sections satisfying $\psi_{\tau^+}=\psi_{\tau^-}$ and we have the following formulae:*

$$\lambda_{\mathcal{X}/B}=\lambda_{\mathcal{Y}/B}-\frac{1}{2}(\psi_{\tau^+}+\psi_{\tau^-}),$$
$$\delta_{\mathcal{X}/B}=\delta_{\mathcal{Y}/B}-6(\psi_{\tau^+}+\psi_{\tau^-}).$$

Proof. This is [Smy11, Proposition 3.4] (although it is stated there only in the case of $g=1$). □

6.1. **Reduction 1: The case of generic rosaries.** Let C be the geometric generic fiber of $\mathcal{C}/B$ and consider a maximal length rosary $R=R_1\cup\cdots\cup R_\ell$ of C (see [AFSv14, Definition 2.27]). Since $\mathcal{C}/B$ is non-isotrivial, the rosary cannot be closed. Let $T:=\overline{C\setminus R}$. The point $T\cap R_1$ (resp., $T\cap R_\ell$) is either an outer node or an outer tacnode, so its limit in every fiber is the same singularity by [AFSv14, Proposition 2.10]. Similarly, the limits of the tacnodes $R_i\cap R_{i+1}$, for $i=1,\dots,\ell-1$, remain tacnodes in every fiber. We then have that $\mathcal{C}=\mathcal{T}\cup\mathcal{R}_1\cup\cdots\cup\mathcal{R}_\ell$, where the geometric generic fiber of $\mathcal{R}_i$ and $\mathcal{T}$ is R_i and T respectively. Let χ_1 (resp., χ_2) be the nodal or tacnodal section along which $\mathcal{T}$ and $\mathcal{R}_1$ (resp., $\mathcal{R}_\ell$) meet. Let τ_i, for $i=1,\dots,\ell-1$, be the tacnodal section along which $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ meet. In the rest of the proof we use the fact that self-intersections of 2 disjoint smooth sections on a $\mathbb{P}^1$-bundle over B are equal of opposite signs. Together with Proposition 6.1, this gives

$$(\psi_{\chi_1})_{\mathcal{R}_1/B}=-(\psi_{\tau_1})_{\mathcal{R}_1/B}=-(\psi_{\tau_1})_{\mathcal{R}_2/B}=(\psi_{\tau_2})_{\mathcal{R}_2/B}=\cdots$$
$$=(-1)^{\ell-1}(\psi_{\tau_{\ell-1}})_{\mathcal{R}_\ell/B}=(-1)^{\ell}(\psi_{\chi_2})_{\mathcal{R}_\ell/B}.$$

In what follows, we set $\psi_{\mathcal{T}/B}=\sum_{i=1}^{n}\psi_{\sigma_i}+\psi_{\chi_1}+\psi_{\chi_2}=\psi_{\mathcal{C}/B}+\psi_{\chi_1}+\psi_{\chi_2}$.

Case 1: R is A_1/A_1-attached rosary. By [AFSv14, Remark 2.28], ℓ must be odd. By Proposition 6.1, we obtain

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}=\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}/B}.$$

Since $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$ is $(7/10-\epsilon)$-stable and $\mathcal{R}/B$ is isotrivial, we reduce to proving Theorem 1.3(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$, which has one less generic rosary than $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$.

Case 2: R is A_1/A_3-attached rosary. Suppose χ_1 is a nodal section and χ_2 is a tacnodal section. By the maximality assumption on R, the irreducible component of T meeting R_ℓ is not a 2-pointed smooth rational curve. It follows by Lemma 2.1 that $(\psi_{\chi_2})_{\mathcal{T}} \geq 0$. By Proposition 6.1, we have

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B} + (\psi_{\chi_1})_{\mathcal{R}_1/B} + \frac{5}{4}(\psi_{\chi_2})_{\mathcal{R}_\ell/B} + \frac{9}{4}\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i}.$$

If ℓ is odd, then $\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i} = 0$ and $\psi_{\chi_1} = -\psi_{\chi_2}$. We thus obtain:

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B} + \frac{1}{4}(\psi_{\chi_2})_{\mathcal{T}/B} \geq \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B}.$$

Noting that $\psi_{\chi_2} = 0$ only if $\mathcal{R}/B$ is isotrivial, we reduce to proving Theorem 1.3(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$.

If ℓ is even, then $\psi_{\chi_1} = \psi_{\chi_2}$ and $\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i} + \psi_{\chi_2} = 0$, so that

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B}.$$

Furthermore, we observe that $\mathcal{R}/B$ is isotrivial and we reduce to proving Theorem 1.3(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$.

Case 3: R is A_3/A_3-attached rosary. By the maximality assumption on R, neither $T \cap R_1$ nor $T \cap R_2$ lies on a 2-pointed rational component of T. It follows by Lemma 2.1 that $(\psi_{\chi_1})_{\mathcal{T}}, (\psi_{\chi_2})_{\mathcal{T}} \geq 0$. However, $\psi_{\chi_1} = (-1)^\ell \psi_{\chi_2}$. Therefore, either $\psi_{\chi_1} = \psi_{\chi_2} = 0$, in which case $\mathcal{R}/B$ is an isotrivial family, or ℓ is even and $\psi_{\chi_1} = \psi_{\chi_2} > 0$. In either case, Proposition 6.1 gives

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B} + \frac{1}{4}(\psi_{\chi_2})_{\mathcal{R}/B} \geq \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B},$$

and the inequality is strict if $\mathcal{R}$ is not isotrivial. Thus we reduce to proving Theorem 1.3(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$.

6.2. **Reduction 2: The case of generic A_1/A_3 or A_3/A_3-attached elliptic bridges.** Suppose the geometric generic fiber of $\mathcal{C}/B$ can be written as $C = T_1 \cup E \cup T_2$, where E is an A_1/A_3-attached elliptic bridge. Let $q_1 = T_1 \cap E$ be a node and $q_2 = T_2 \cap E$ be a tacnode. By [AFSv14, Definition 2.10], the limit of q_1 (resp., q_2) remains a node (resp., a tacnode) in every fiber. Thus we can write $\mathcal{C} = (\mathcal{T}_1, \tau_0) \cup (\mathcal{E}, \tau_1, \tau_2) \cup (\mathcal{T}_2, \tau_3)$, where $\tau_0 \sim \tau_1$ are glued nodally and $\tau_2 \sim \tau_3$ are glued tacnodally. Since A_1/A_1-attached elliptic bridges are disallowed, fibers of $\mathcal{E}$ have no separating nodes and so $(\mathcal{E}, \tau_1, \tau_2)$ is a family of elliptic bridges. By [AFSv14, Lemma 2.17], $(\mathcal{T}_1, \tau_0)$ is $(7/10-\epsilon)$-stable. Also, $(\mathcal{T}_2, \tau_3)$ is $(7/10-\epsilon)$-stable because τ_3 cannot lie on an A_1-attached elliptic tail in $\mathcal{T}_2$.

Set $\mathcal{C}' = (\mathcal{T}_1, \tau_0) \cup (\mathcal{T}_2, \tau_3)$, where we glue by $\tau_0 \sim \tau_3$ nodally. Then $(\mathcal{C}'/B, \{\sigma_i\}_{i=1}^n)$ is a $(7/10-\epsilon)$-stable family by [AFSv14, Lemma 2.18]. By Proposition 6.1, we have

$$\begin{aligned}\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B} &= \left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}_1/B}+\left(\frac{39}{4}\lambda-\delta+\psi_{\tau_1}+\frac{5}{4}\psi_{\tau_2}\right)_{\mathcal{E}/B}+\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}_2/B} \\ &= \left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}_1/B}+\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}_2/B} = \left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}'/B},\end{aligned}$$

where we have used relations $(\psi_{\tau_1})_{\mathcal{E}/B} = (\psi_{\tau_2})_{\mathcal{E}/B} = \lambda_{\mathcal{E}/B}$ and $\delta_{\mathcal{E}/B} = 12\lambda_{\mathcal{E}/B}$, both of which hold because $(\delta_{\mathrm{red}})_{\mathcal{E}/B} = 0$.

Note that $(\mathcal{E}/B, \tau_1, \tau_2)$ is trivial if and only if $\psi_{\tau_2} = \psi_{\tau_3} = 0$. Thus we have reduced to proving the requisite inequalities for the family $\mathcal{C}'/B$ with one less generic A_1/A_3-attached elliptic bridge. Moreover, the equality for $\mathcal{C}'/B$ holds if and only if the equality for $\mathcal{C}/B$ holds and $\mathcal{C}'/B$ is obtained by replacing a generic node of $\mathcal{C}'$ by a family of elliptic bridges A_1/A_3-attached along the nodal transforms.

Similarly, if the generic fiber of $\mathcal{C}/B$ has an A_3/A_3-attached elliptic bridge, then we can remove the bridge and recrimp the two remaining components of $\mathcal{C}$ along a generic tacnode. The calculation similar to the above shows that the degree of $\left(\frac{39}{4}\lambda-\delta+\psi\right)$ does not change under this operation.

Replacing an attaching node of a Weierstrass chain of length ℓ by an A_1/A_3-attached elliptic bridge in a way that preserves $(7/10-\epsilon)$-stability gives a Weierstrass chain of length $\ell+1$. Similarly, replacing a tacnode in a Weierstrass chain of length ℓ by an A_3/A_3-attached elliptic bridge gives a Weierstrass chain of length $\ell+1$. In what follows, we will prove that for a non-isotrivial $(7/10-\epsilon)$-stable family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ *with no generic* A_1/A_3 *or* A_3/A_3*-attached elliptic bridges*, we have $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B} \geq 0$ and equality holds if and only if $\mathcal{C}/B$ is a family of *Weierstrass tails*. This implies that for every non-isotrivial $(7/10-\epsilon)$-stable family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$, we have $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B} \geq 0$ and equality holds if and only if $\mathcal{C}/B$ is a family of *Weierstrass chains*.

6.3. **Reduction 3: Normalization along generic tacnodes.** Consider now a family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ of $(7/10-\epsilon)$-stable curves with no generic rosaries and no generic A_1/A_3 or A_3/A_3-attached elliptic bridges. Let $\mathcal{X}$ be the normalization of $\mathcal{C}$ along generic tacnodes. Denote by $\{\tau_i^{\pm}\}_{i=1}^d$ the preimages of the generic tacnodes, and call them tacnodal transforms. Set $\psi_{tacn} := \sum_{i=1}^d (\psi_{\tau_i^+} + \psi_{\tau_i^-})$ and $\psi_{\mathcal{X}/B} := \psi_{\mathcal{C}/B} + \psi_{tacn}$. Applying Proposition 6.1 we have

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda-\delta+\psi+\frac{1}{8}\psi_{tacn}\right)_{\mathcal{X}/B}.$$

If we now treat each tacnodal transform $\tau_i^{\pm}$ as a marked section, then every connected component of $\mathcal{X}$ is a generically $(7/10-\epsilon)$-stable family (there are no generic A_1/A_3 or A_3/A_3-attached elliptic bridges). Blowing-down all rational tails meeting a single tacnodal transform and no other marked sections does not change $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{X}/B}$ but makes $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n, \{\tau_i^{\pm}\}_{i=1}^d)$ into a $(7/10-\epsilon)$-stable family. We still have $\psi_{tacn} \geq 0$ by Lemma 2.1, with strict inequality if $d \geq 1$. Thus, we have reduced to proving Theorem 1.3(b) for a $(7/10-\epsilon)$-stable family with no generic tacnodes.

6.4. **Reduction 4: Normalization along generic outer nodes.** Consider a $(7/10-\epsilon)$-stable family $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n)$ with no generic tacnodes. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ along the generic outer nodes and let $\{\zeta_i^+, \zeta_i^-\}_{i=1}^b$ be the transforms of the generic outer nodes. Set $\delta_{tacn} := \sum_{i\neq j}(\zeta_i^\pm \cdot \zeta_j^\pm)$ and $\psi_{\mathcal{Y}/B} := \psi_{\mathcal{X}/B} + \sum_{i=1}^b(\psi_{\zeta_i^+} + \psi_{\zeta_i^-})$. Then by Proposition 3.4, we have

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{X}/B} = \left(\frac{39}{4}\lambda - \delta + \psi + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Y}/B}.$$

6.5. **Reduction 5: Normalization along generic cusps.** Let $\mathcal{Y}$ be as in 6.4 and let $\mathcal{Z}$ be the normalization of (a connected component of) $\mathcal{Y}$ along generic cusps and let $\{\xi_i\}_{i=1}^c$ be the cuspidal transforms on $\mathcal{Z}$. Then the family $(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c)$ satisfies the following properties:

(1) The generic fiber is irreducible and at-worst nodal.
(2) The sections $\{\sigma_i\}_{i=1}^n$ are smooth, pairwise non-intersecting and disjoint from $\{\zeta_i\}_{i=1}^b$.
(3) The sections $\{\zeta_i\}_{i=1}^b$ are smooth and at most two of them can meet at any given point of $\mathcal{Z}$.
(4) The sections $\{\xi_i\}_{i=1}^c$ are pairwise non-intersecting and disjoint from $\{\zeta_i\}_{i=1}^b$ and $\{\sigma_i\}_{i=1}^n$.

Set $c(B) := 2\sum_{i=1}^c \iota(\xi_i)$, where $\iota(\xi_i)$ is the index of the cuspidal transform ξ_i, and $\psi_{cusp} := \sum_{i=1}^c \psi_{\xi_i}$. Then we have by Proposition 3.5

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Y}/B} = \left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B}. \tag{6.1}$$

Our goal for the rest of the section is to prove that the expression on the right-hand side of (6.1) is non-negative and equals 0 if and only if the only non-isotrivial components of the family $\mathcal{X}/B$ from 6.4 are A_1-attached Weierstrass tails.

Let h be the geometric genus of the generic fiber of $\mathcal{Z}$ and let a be the number of generic inner nodes of $\mathcal{Z}$. Our further analysis breaks down according to the following possibilities:

(A) $h \geq 3$; see §6.5.1.
(B) $h = 2$, or $(h, a) = (1, 1)$, or $(h, a) = (0, 2)$; see §6.5.2.
(C) $h = 1$ and $a \neq 1$, or $(h, a) = (0, 1)$; see §6.5.3.
(D) $h = 0$ and $a \geq 3$, or $(h, a) = (0, 0)$; see §6.5.4.

6.5.1. *Case A: Relative geometric genus $h \geq 3$.* Suppose $\mathcal{Z}/B$ is a family as in 6.5. Let $\mathcal{W}$ be the normalization of $\mathcal{Z}$ along the generic inner nodes. Let $\{\eta_i^+, \eta_i^-\}_{i=1}^a$ be the inner nodal transforms on $\mathcal{W}$. Then $(\mathcal{W}/B, \{\sigma_i\}_{i=1}^n, \{\eta_i^\pm\}_{i=1}^a, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c)$ satisfies the following properties:

(1) The generic fiber is a smooth curve of genus $h \geq 3$.
(2) Sections $\{\sigma_i\}_{i=1}^n$ are smooth, non-intersecting, and disjoint from $\{\eta_i^\pm\}_{i=1}^a$, $\{\zeta_i\}_{i=1}^b$, and $\{\xi_i\}_{i=1}^c$.
(3) Inner nodal transforms $\{\eta_i^\pm\}_{i=1}^a$ are disjoint from $\{\zeta_i\}_{i=1}^b$ and $\{\xi_i\}_{i=1}^c$. Their properties are described by Proposition 3.3.

(4) Outer nodal transforms $\{\zeta_i\}_{i=1}^b$ are disjoint from $\{\xi_i\}_{i=1}^c$. Their properties are described by Proposition 3.4.
(5) Cuspidal transforms $\{\xi_i\}_{i=1}^c$ have properties described by Proposition 3.5.

We let $\psi_{\mathcal{W}/B} := \psi_{\mathcal{Z}/B} + \sum_{i=1}^a (\psi_{\eta_i^+} + \psi_{\eta_i^-})$ and $(\delta_{tacn})_{\mathcal{W}/B} := (\delta_{tacn})_{\mathcal{Z}/B} + \sum_{i\neq j}(\eta_i^\pm \cdot \eta_j^\pm)$. We set $\delta_{inner} := \sum_{i=1}^a (\eta_i^+ \cdot \eta_i^-)$ and $n(B) := \sum_{i=1}^a \iota(\eta_i^+)$, where $\iota(\eta_i^+)$ is the index of the inner nodal transform η_i^+. Then by Proposition 3.3:

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B} = \left(\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{W}/B}. \tag{6.2}$$

Passing to the relative minimal model of $\mathcal{W}/B$ does not increase the degree of the divisor on the right-hand side of (6.2). Hence we will assume that $\omega_{\mathcal{W}/B}$ is relatively nef. Then by Proposition 4.1, we have $(8 + 4/h)\,\lambda - \delta \geq 0$. Since $h \geq 3$ and $\delta \geq 0$, we obtain $\frac{39}{4}\lambda - \delta > 0$ (when $\delta = 0$, we have $\lambda > 0$ by the existence of the Torelli morphism). We proceed to estimate the remaining terms of (6.2). Clearly, $\delta_{tacn} \geq 0$. Since $h \geq 3$ and $\kappa = 12\lambda - \delta > 0$, the inequalities of Lemmas 4.3 and 4.4 give

$$\psi_{cusp} = \sum_{i=1}^c \psi_{\xi_i} \geq \frac{(h-1)}{h}\sum_{i=1}^c \iota(\xi_i) + c\frac{\kappa}{4h(h-1)} = \frac{h-1}{2h}c(B) + c\frac{\kappa}{4h(h-1)} > \frac{1}{3}c(B),$$

$$\sum_{i=1}^a (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{2(h-1)}{h+1}\sum_{i=1}^a ((\eta_i^+ \cdot \eta_i^-) + \iota(\eta_i^+)) + a\frac{\kappa}{h^2-1} > \delta_{inner} + n(B).$$

Summarizing, we conclude that the right hand side of (6.2) is strictly positive.

6.5.2. *Case B: Relative genus* 2. Suppose $\mathcal{Z}/B$ is a family as in 6.5 with relative geometric genus $h = 2$. Let $\mathcal{W}$ be the normalization of $\mathcal{Z}$ along the generic inner nodes. As in 6.5.1, we reduce to proving that

$$\left(\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{W}/B} \geq 0, \tag{6.3}$$

under the assumption that $\omega_{\mathcal{W}/B}$ is relatively nef.

For any family $\mathcal{W}/B$ of arithmetic genus 2 curves with a relatively nef $\omega_{\mathcal{W}/B}$, we have

$$10\lambda = \delta_{\mathrm{irr}} + 2\delta_{\mathrm{red}}, \tag{6.4}$$

This relation implies that $\delta \leq 10\lambda$ for any generically irreducible family and, consequently, $\kappa = 12\lambda - \delta \geq 2\lambda$, with the equality achieved only if $\delta_{\mathrm{red}} = 0$, i.e., if there are no fibers where two genus 1 components meet at a node. It follows that $\frac{39}{4}\lambda - \delta \geq -\lambda/4$, with the equality only if $\delta_{\mathrm{red}} = 0$.

By Lemma 4.4, we have

$$\sum_{i=1}^a (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{2}{3}(\delta_{inner} + n(B)) + a\frac{\kappa}{3}.$$

By Lemma 4.3, we have

$$\psi_{cusp} \geq \frac{1}{4}c(B) + c\frac{\kappa}{8}.$$

Putting these inequalities together and using $\kappa \geq 2\lambda$, we obtain

$$\frac{9}{4}\psi_{cusp} + \sum_{i=1}^{a}(\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{1}{4}\delta_{inner} + \frac{1}{4}n(B) + \frac{1}{4}c(B) + \left(\frac{2a}{3} + \frac{9c}{16}\right)\lambda.$$

If $a + c \geq 1$, we obtain a strict inequality in (6.3) at once. Suppose $a = c = 0$. So far, we have that

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{W}/B} \geq \sum_{i=1}^{n}\psi_{\sigma_i} + \sum_{i=1}^{b}\psi_{\zeta_i} - \frac{1}{4}\lambda.$$

We now invoke Lemma 4.5 that gives

$$\sum_{i=1}^{n}\psi_{\sigma_i} + \sum_{i=1}^{b}\psi_{\zeta_i} \geq \frac{(n+b)}{8}\kappa \geq \frac{(n+b)}{4}\lambda.$$

Since $n + b \geq 1$ (otherwise, $\mathcal{W}/B$ is an unpointed family of genus 2 curves, which is impossible), we conclude that $\sum_{i=1}^{n}\psi_{\sigma_i} + \sum_{i=1}^{b}\psi_{\zeta_i} - \lambda/4 \geq 0$ and that equality is achieved if and only if $n+b = 1$, $\delta_{\text{red}} = 0$, and equality is achieved in Lemma 4.5. This is precisely the situation when $\mathcal{Y}/B = \mathcal{W}/B$ is a family of A_1-attached Weierstrass genus 2 tails.

Finally, if $(h, a) = (1, 1)$ or $(h, a) = (0, 2)$, we proceed exactly as above but without normalizing the inner nodes: For a family $(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c)$ as in 6.5, where the relative arithmetic genus of $\mathcal{Z}/B$ is 2, we need to prove

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B} \geq 0.$$

Applying (6.4) to estimate δ, Lemma 4.3 to estimate ψ_{cusp}, and Lemma 4.5 to estimate $\sum_{i=1}^{n}\psi_{\sigma_i} + \sum_{i=1}^{b}\psi_{\zeta_i}$ (all of which apply even if the total space $\mathcal{Z}$ is not normal), we obtain

$$\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn} \geq -\frac{1}{4}\lambda + \frac{4n+4b+9c}{16}\lambda + \frac{5}{16}c(B) + \frac{7}{8}\delta_{tacn} \geq 0.$$

Moreover, equality is achieved if and only if $\delta_{\text{red}} = 0$, $c = 0$, and $n + b = 1$, which is precisely the situation when $\mathcal{Y}/B = \mathcal{Z}/B$ is a family of A_1-attached (generically nodal) Weierstrass genus 2 tails.

6.5.3. *Case C: Relative genus* 1. Suppose $\mathcal{Z}/B$ is a family as in 6.5 of relative genus 1 and with a generic inner nodes, where $a \neq 1$. We consider the case $a \geq 2$ first. Let $\mathcal{W}$ be the family obtained from $\mathcal{Z}$ by the following operations:

(1) Normalize $\mathcal{Z}$ along all generic inner nodes to obtain inner nodal pairs $\{\eta_i^+, \eta_i^-\}_{i=1}^a$.
(2) Blow-up all cuspidal and inner nodal transforms to make them Cartier divisors.
(3) Blow-up points of $\eta_i^{\pm} \cap \eta_j^{\pm}$ for all $i \neq j$.
(4) Blow-up points of $\zeta_i \cap \zeta_j$ for all $i \neq j$.

As a result, the sections of $\mathcal{W}/B$ do not intersect pairwise with the only possible exception that η_i^+ is allowed to meet η_i^-. A node of $\mathcal{Z}$ through which ξ_i passes is replaced in $\mathcal{W}$ by a balanced rational bridge meeting the strict transform of ξ_i, which we continue to denote by ξ_i. We say that such a bridge is a *cuspidal bridge associated to* ξ_i. Moreover, if we let $c(\xi_i)$ be the sum of the indices of all bridges associated to ξ_i, then

$$2\iota(\xi_i)_{\mathcal{Z}/B} = c(\xi_i)_{\mathcal{W}/B}.$$

Suppose $\{\eta_i^+, \eta_i^-\}$ is an inner nodal pair of $\mathcal{Z}/B$. Then a node of $\mathcal{Z}$ through which η_i^+ and η_i^- both pass is replaced in $\mathcal{W}$ by a balanced rational bridge meeting the strict transforms of η_i^+ and η_i^-, which we continue to denote by η_i^+ and η_i^-. We say that such a bridge is an *inner nodal bridge associated to* $\{\eta_i^+, \eta_i^-\}$. Moreover, if we let $n(\eta_i)$ be the sum of the indices of all bridges associated to $\{\eta_i^+, \eta_i^-\}$, then

$$((\eta_i^+ \cdot \eta_i^-) + \iota(\eta_i^+))_{\mathcal{Z}/B} = ((\eta_i^+ \cdot \eta_i^-) + n(\eta_i))_{\mathcal{W}/B}.$$

On $\mathcal{W}/B$, we define

$$\delta_{inner} := \sum_{i=1}^{a} (\eta_i^+ \cdot \eta_i^-), \quad \delta_{tacn} := \sum_{i \neq j} \delta_{0,\{\eta_i^\pm, \eta_j^\pm\}} + \sum_{i \neq j} \delta_{0,\{\zeta_i, \zeta_j\}},$$

and let $n(B)$ (resp., $c(B)$) be the sum of the indices of all inner nodal (resp., cuspidal) bridges. We reduce to proving that

$$\left(\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} - \frac{1}{8}\delta_{tacn}\right)_{\mathcal{W}/B} \geq 0.$$

We will make use of the standard relations for pointed families of genus 1 curves and Lemmas 4.6 and 4.7. Let $N = n + 2a + b + c$ be the total number of marked sections of $\mathcal{W}/B$. Clearly, $N \geq 2$. We consider first the case when $N \geq 3$. Then by Lemma 4.6, we have

$$\delta_{inner} + n(B) \leq \frac{N}{2(N-2)} \sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-}) + \frac{a}{2(N-2)^2}\delta_{\text{red}}.$$

Applying Lemma 4.7, we obtain

$$c(B) \leq \frac{2N}{N-1}\psi_{cusp} + \frac{c}{4(N-1)^2}\delta_{\text{red}}.$$

Using the above two inequalities and rewriting $\delta = 12\lambda + \delta_{\text{red}}$, we see that

$$\begin{aligned} &\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\left(\delta_{inner} + n(B) + c(B)\right) + \frac{5}{4}\psi_{cusp} - \frac{1}{8}\delta_{tacn} \\ &\geq -\frac{9}{4}\lambda + \left(\frac{5}{4} - \frac{N}{2(N-1)}\right)\psi_{cusp} + \psi - \frac{N}{8(N-2)} \sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \\ &\qquad - \left(1 + \frac{a}{8(N-2)^2} + \frac{c}{16(N-1)^2}\right)\delta_{\text{red}} - \frac{1}{8}\delta_{tacn}. \end{aligned} \tag{6.5}$$

We rewrite each ψ-class on the right-hand side of (6.5) using the standard relation on families of arithmetic genus 1 curves:

$$\psi_\sigma = \lambda + \sum_{\sigma \in S} \delta_{0,S}.$$

The coefficient of λ in the resulting expression for the right-hand side of (6.5) is

$$-\frac{9}{4}+c\left(\frac{5}{4}-\frac{N}{2(N-1)}\right)+N-\frac{aN}{4(N-2)}. \tag{6.6}$$

Using $N \geq 2a+c$ and the assumption $N \geq 3$, it is easy to check that (6.6) is always positive.

A similarly straightforward but tedious calculation shows that each boundary divisor $\delta_{0,S}$ appears in the resulting expression for the right-hand side of (6.5) with a positive coefficient. Thus we have shown that the right-hand side of (6.5) is positive for every non-isotrivial family with $N \geq 3$.

We consider now the case of $N = 2$. Since $\mathcal{C}/B$ in 6.3 has no generic elliptic bridges (nodally or tacnodally attached), we must have $c = 1$ and $n + b = 1$. Let ξ be the corresponding cuspidal transform and τ be either a marked smooth section (if $n = 1$) or an outer nodal transform (if $b = 1$). We trivially have $\delta_{inner} = n(B) = \delta_{tacn} = \delta_{\mathrm{red}} = 0$. Using $\delta_{\mathrm{irr}} = 12\lambda$ and the inequality $c(B) \leq 4\psi_{cusp}$ from Lemma 4.7, we obtain:

$$\begin{aligned}\frac{39}{4}\lambda-\delta+\psi-\frac{1}{4}\delta_{inner}-\frac{1}{4}n(B)-\frac{1}{4}c(B)+\frac{5}{4}\psi_{cusp}-\frac{1}{8}\delta_{tacn}\\ =\frac{39}{4}\lambda-\delta+\psi-\frac{1}{4}c(B)+\frac{5}{4}\psi_{cusp}\geq\frac{39}{4}\lambda-12\lambda+\psi+\frac{1}{4}\psi_{cusp}\\ =\frac{39}{4}\lambda-12\lambda+2\lambda+\frac{1}{4}\lambda=0.\end{aligned}$$

Moreover, equality holds only if equality holds in Lemma 4.7. This happens if and only if $2\xi \sim 2\tau$ and implies that $\mathcal{Y}/B$ in 6.4 is a generically cuspidal family of A_1-attached Weierstrass genus 2 tails. We are done with the analysis in the case $g = 1$ and $a \neq 1$.

If $(g, a) = (0, 1)$, we proceed exactly as above, but without normalizing the inner node.

6.5.4. *Case D: Relative geometric genus* 0. Suppose $\mathcal{Z}/B$ is a family as in 6.5 of relative geometric genus 0 and with a generic inner nodes, where either $a \geq 3$ or $a = 0$. We consider the case $a \geq 3$ first. Let $\mathcal{W}$ be the family obtained from $\mathcal{Z}$ by the following operations:

(1) Normalize $\mathcal{Z}$ along all generic inner nodes to obtain inner nodal pairs $\{\eta_i^+, \eta_i^-\}_{i=1}^a$.
(2) Blow-up all cuspidal and inner nodal transforms to make them Cartier divisors. This operation introduces cuspidal or nodal bridges as in 6.5.3.
(3) Blow-up points of $\eta_i^{\pm} \cap \eta_j^{\pm}$ for all $1 \leq i < j \leq a$.
(4) Blow-up points of $\zeta_i \cap \zeta_j$ for all $1 \leq i < j \leq b$.
(5) Blow-up points of $\eta_i^+ \cap \eta_i^-$ for all $1 \leq i \leq a$.
(6) Blow-down all rational tails marked by a single section (such tails are necessarily adjacent either to cuspidal or inner nodal bridges).

As a result, $\mathcal{W}/B$ is a family in $\overline{\mathcal{M}}_{0,N}$, where $N = n + 2a + b + c$ and $a \geq 3$. On $\mathcal{W}/B$, we define

$$\delta_{inner} := \sum_{i=1}^{a} \delta_{\{\eta_i^+, \eta_i^-\}}, \qquad \delta_{tacn} := \sum_{i \neq j} \delta_{\{\eta_i^\pm, \eta_j^\pm\}} + \sum_{i \neq j} \delta_{\{\zeta_i, \zeta_j\}},$$
$$\delta_3^{NB} := \sum_{i=1}^{a} \sum_{\beta \neq \eta_i^+, \eta_i^-} \delta_{\{\eta_i^+, \eta_i^-, \beta\}}, \qquad \delta_2^{CB} := \sum_{i=1}^{c} \sum_{\beta \neq \xi_i} \delta_{\{\xi_i, \beta\}},$$

and let $n(B)$ (resp., $c(B)$) be the sum of the indices of all inner nodal (resp., cuspidal) bridges. Then

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B}$$
$$= \left(\psi + \frac{5}{4}\psi_{cusp} - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + c(B) + \delta_3^{NB} + \delta_2^{CB}) - \frac{1}{8}\delta_{tacn}\right)_{\mathcal{W}/B}.$$

We are going to prove that a (strict!) inequality

$$\psi + \frac{5}{4}\psi_{cusp} - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + c(B) + \delta_3^{NB} + \delta_2^{CB}) - \frac{1}{8}\delta_{tacn} > 0$$

always holds on $\mathcal{W}/B$. In doing so, we will use the following standard relation on $\overline{\mathcal{M}}_{0,N}$:

$$\psi = \sum_{r \geq 2} \frac{r(N-r)}{N-1}\delta_r. \tag{6.7}$$

First we deal with the case of a family with 3 inner nodal pairs and no other marked sections, i.e., $a = 3$ and $N = 6$. The desired inequality in this case simplifies to

$$\psi - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + \delta_3^{NB}) - \frac{1}{8}(\delta_2 - \delta_{inner}) > 0.$$

We have an obvious inequality $2n(B) \leq \delta_2$. Thus we reduce to proving

$$\psi > \frac{5}{4}\delta_2 + \frac{9}{8}\delta_{inner} + \delta_3 + \frac{1}{4}\delta_3^{NB}. \tag{6.8}$$

For $a \geq 3$, Lemma 4.8 gives

$$\psi \geq 4\delta_{inner} + \delta_{tacn} + 3\delta_3^{NB} = 3\delta_{inner} + \delta_2 + 3\delta_3^{NB}.$$

Combining this with the standard relation $5\psi = 8\delta_2 + 9\delta_3$ gives

$$8\psi \geq 9\delta_{inner} + 11\delta_2 + 9\delta_3 + 9\delta_3^{NB}.$$

This clearly implies (6.8) as desired.

Next, we consider the case of $N \geq 7$. In this case, every inner nodal or cuspidal bridge is adjacent to a node from $\sum_{r\geq 3}\delta_r$. As a result, we have $n(B) + c(B) \leq 2\sum_{r \geq 3}\delta_r$. Furthermore, $\frac{1}{4}\delta_2^{CB} + \frac{1}{8}\delta_{tacn} + \frac{1}{4}\delta_{inner} \leq \frac{1}{4}\delta_2$ (because a node from δ_2 can contribute only to one of the δ_{inner}, δ_{tacn}, or δ_2^{CB}). Hence we reduce to proving

$$\psi + \frac{5}{4}\psi_{cusp} - \frac{5}{4}\delta_2 - \delta_{inner} - \frac{3}{2}\sum_{r \geq 3}\delta_r - \frac{1}{4}\delta_3^{NB} > 0 \tag{6.9}$$

We combine the inequality of Lemma 4.8 with the standard relation (6.7), and the obvious $\psi \geq \psi_{inner}$ to obtain

$$3\left(\psi - \sum_{r\geq 2}\frac{r(N-r)}{N-1}\delta_r\right) + \left(\psi_{inner} - 4\delta_{inner} - 3\delta_3^{NB}\right) + \left(\psi - \psi_{inner}\right) \geq 0.$$

This gives the estimate

$$4\psi \geq 4\delta_{inner} + \frac{6(N-2)}{N-1}\delta_2 + \frac{9(N-3)}{N-1}\sum_{r\geq 3}\delta_r + 3\delta_3^{NB}.$$

Using $N \geq 7$ and $\psi_{cusp} \geq 0$, we finally get

$$\psi + \frac{5}{4}\psi_{cusp} \geq \delta_{inner} + \frac{5}{4}\delta_2 + \frac{3}{2}\sum_{r\geq 3}\delta_r + \frac{3}{4}\delta_3^{NB}.$$

Moreover, the equality could be achieved only if $N = 7$ and $\psi - \psi_{inner} = 0$ which is impossible because $\psi = \psi_{inner}$ implies that all sections are inner nodal transforms and so N must be even. Hence we have established (6.9) as desired.

At last, we consider the case of $a = 0$. Because the family $\mathcal{W}/B$ is non-isotrivial, we must have $N \geq 4$. In addition, if $N = 4$, then there exists a unique family of 4-pointed Deligne-Mumford stable rational curves. The requisite inequality is easily verified for this family by hand. If $N \geq 5$, then using the inequality $2c(B) \leq \delta$, we reduce to proving

$$\psi + \frac{5}{4}\psi_{cusp} - \frac{9}{8}\delta - \frac{1}{4}\delta_2^{CB} - \frac{1}{8}\delta_{tacn} > 0.$$

The standard relation (6.7) gives

$$\psi \geq \frac{3}{2}\sum_{r\geq 2}\delta_r > \frac{11}{8}\delta_2 + \frac{9}{8}\sum_{r\geq 3}\delta_r > \frac{9}{8}\delta + \frac{1}{4}\delta_2.$$

Finally, the inequality $\delta_2 \geq \delta_2^{CB} + \delta_{tacn}$ gives the desired result.

This completes the proof of Theorem 1.3 (b).

7. Projectivity from positivity

Throughout this section, we make use of the following standard abuse of notation: Whenever $\mathcal{L}$ is a line bundle on $\overline{\mathcal{M}}_{g,n}(\alpha)$ that descends to the good moduli space, we denote the corresponding line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$ also by $\mathcal{L}$. In this situation, pullback defines a natural isomorphism $\mathrm{H}^0\big(\overline{\mathbb{M}}_{g,n}(\alpha), \mathcal{L}\big) \simeq \mathrm{H}^0\big(\overline{\mathcal{M}}_{g,n}(\alpha), \mathcal{L}\big)$.

Proposition 7.1. *Let $\alpha > 2/3 - \epsilon$. Suppose that $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi$ descends to $\overline{\mathbb{M}}_{g,n}(\alpha)$ for some $\beta \leq \alpha$. Then we have*

$$\operatorname{Proj} R\big(\overline{\mathbb{M}}_{g,n}(\alpha), K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi\big) \simeq \overline{M}_{g,n}(\beta).$$

Proof. Consider the rational map $f_\alpha \colon \overline{M}_{g,n} \dashrightarrow \overline{\mathbb{M}}_{g,n}(\alpha)$. If $\alpha > 9/11$, then f_α is an isomorphism. If $7/10 < \alpha \leq 9/11$, then $f_\alpha|_{\overline{M}_{g,n}\setminus\delta_{1,0}}$ is an isomorphism onto the complement of the codimension 2 locus of cuspidal curves in $\overline{\mathbb{M}}_{g,n}(\alpha)$. If $\alpha \leq 7/10$, then $f_\alpha|_{\overline{M}_{g,n}\setminus(\delta_{1,0}\cup\delta_{1,1})}$ is an isomorphism onto the complement of the codimension 2 locus of cuspidal and tacnodal curves in $\overline{\mathbb{M}}_{g,n}(\alpha)$. (If $n=0$, then $\delta_{1,1}=\emptyset$). It follows that we have a discrepancy equation

$$f_\alpha^*\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha)}+\beta\delta+(1-\beta)\psi\big)\simeq K_{\overline{\mathcal{M}}_{g,n}}+\beta\delta+(1-\beta)\psi+c_0\delta_{1,0}+c_1\delta_{1,1}, \tag{7.1}$$

where $c_0=0$ if $\alpha>9/11$ and $c_1=0$ if $\alpha>7/10$.

Let $T_1\subset\overline{\mathcal{M}}_{g,n}$ be a non-trivial family of elliptic tails and $T_2\subset\overline{\mathcal{M}}_{g,n}\setminus\delta_{1,0}$ be a non-trivial family of 1-pointed elliptic tails. Then f_α is regular along T_1, and for $\alpha\leq 9/11$ contracts T_1 to a point. Similarly, f_α is regular along T_2, and for $\alpha\leq 7/10$ contracts T_2 to a point. By intersecting both sides of (7.1) with T_1 and T_2, we obtain $c_0=11\beta-9\leq 0$ if $\alpha\leq 9/11$, and $c_1=10\beta-7\leq 0$ if $\alpha\leq 7/10$. It follows that

$$\operatorname{Proj} R\big(\overline{\mathbb{M}}_{g,n}(\alpha),K_{\overline{\mathcal{M}}_{g,n}(\alpha)}+\beta\delta+(1-\beta)\psi\big)\simeq\operatorname{Proj} R\big(\overline{M}_{g,n},K_{\overline{\mathcal{M}}_{g,n}}+\beta\delta+(1-\beta)\psi\big).$$

□

Proposition 7.2. *Fix $\alpha_c\in\{\alpha_1=9/11,\alpha_2=7/10,\alpha_3=2/3\}$ and take $\alpha_0=1$. Suppose that for all $0<\epsilon\ll 1$,*

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+(\alpha_{c-1}-\epsilon)\delta+(1-\alpha_{c-1}+\epsilon)\psi$$

descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. In addition, suppose that

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$$

is nef on $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)$ and all curves on which it has degree 0 are contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)\to\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Then $K_{\overline{\mathcal{M}}_{g,n}(\alpha)}+\alpha\delta+(1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$ for all $\alpha\in[\alpha_c,\alpha_{c-1})$.

Proof. By [AFSv14, Proposition 3.28], for any α_c-closed curve $(C,\{p_i\}_{i=1}^n)$, the action of $\operatorname{Aut}(C,\{p_i\}_{i=1}^n)^\circ$ on the fiber of $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is trivial. It follows that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ descends to $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Consider the open immersion of stacks $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)\hookrightarrow\overline{\mathcal{M}}_{g,n}(\alpha_c)$ and the induced map on the good moduli spaces $j\colon\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)\to\overline{\mathbb{M}}_{g,n}(\alpha_c)$. We have that

$$j^*\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi\big)=K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi.$$

It follows by assumption that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$ descends to a nef line bundle on the projective variety $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. First, we show that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$ is semiample on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. To bootstrap from nefness to semiampleness, we first consider the case $n=0$ and $g\geq 3$. By Proposition 7.1, the section ring of $K_{\overline{\mathcal{M}}_g(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ on $\overline{\mathbb{M}}_g(\alpha_{c-1}-\epsilon)$ is identified with the section ring of $K_{\overline{\mathcal{M}}_g}+\alpha_c\delta$ on $\overline{M}_g$. The latter line bundle is big, by standard bounds on the effective cone of $\overline{M}_g$, and finitely generated by [BCHM10, Corollary 1.2.1]. We conclude that $K_{\overline{\mathcal{M}}_g(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ is big, nef, and finitely generated, and so is semiample by [Laz04, Theorem 2.3.15]. When $n\geq 1$, simply note that $K_{\overline{\mathcal{M}}_{g+hn}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ pulls back to $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$ under the morphism $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)\to\overline{\mathcal{M}}_{g+nh}(\alpha_{c-1}-\epsilon)$ defined by attaching a fixed general curve of genus $h\geq 3$ to every marked point.

We have established that

$$j^*\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi\big) = K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$$

is semiample on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. By assumption, it has degree 0 only on curves contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$. We conclude that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is semiample and is positive on all curves in $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Therefore, $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is ample on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$.

The statement for $\alpha \in (\alpha_c, \alpha_{c-1})$ follows by interpolation. □

Proposition 7.3. *Fix $\alpha_c \in \{9/11, 7/10, 2/3\}$. Suppose that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Then for all $0 < \epsilon \ll 1$,*

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)} + (\alpha_c-\epsilon)\delta_c + (1-\alpha_c+\epsilon)\psi$$

descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.

Proof. Consider the proper morphism $\pi\colon \overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$ given by [AFS15, Theorem 4.25]. Our assumption implies that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$ descends to a line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$ which is a pullback of an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$ via π. To establish the proposition, it suffices to show that a positive multiple of $\psi-\delta$ on $\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)$ descends to a π-ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.

For every $(\alpha_c-\epsilon)$-stable curve $(C, \{p_i\}_{i=1}^n)$, the induced character of $\mathrm{Aut}(C,\{p_i\}_{i=1}^n)^\circ$ on $\delta-\psi$ is trivial by [AFS15, Proposition 3.27]. It follows by [Alp13, Theorem 10.3] that a positive multiple of $\delta-\psi$ descends to a line bundle $\mathcal{N}$ on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.

To show that $\mathcal{N}^\vee$ is relatively ample over $\overline{\mathbb{M}}_{g,n}(\alpha_c)$, consider the commutative cube

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl, "f"'] \arrow[dd] & & \mathcal{W}_\chi^- \arrow[ll, hook'] \arrow[dl] \arrow[dd] \\
\overline{\mathcal{M}}_{g,n}(\alpha_c) \arrow[dd, "\phi_{\alpha_c}"] & & \overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon) \arrow[ll, hook'] \arrow[dd, "\phi_{\alpha_c-\epsilon}" near start] & \\
 & W/\!\!/G \arrow[dl] & & W_\chi^-/\!\!/G \arrow[ll] \arrow[dl] \\
\overline{\mathbb{M}}_{g,n}(\alpha_c) & & \overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon) \arrow[ll, "\pi"'] &
\end{tikzcd} \tag{7.2}$$

where $\mathcal{W} = [\operatorname{Spec} A/G] \to W/\!\!/G = \operatorname{Spec} A^G$ and $\mathcal{W}^-_{\chi_{\delta-\psi}} \to W^-_{\chi_{\delta-\psi}}/\!\!/G = \operatorname{Proj} \bigoplus_{d\geq 0} A_d$ are the good moduli spaces as in [AFS15, Proposition 3.6]. Since the vertical arrows are good moduli spaces, by Proposition [AFS15, Proposition 4.6] and [AFS15, Lemmas 3.18 and 4.7], after shrinking $\mathcal{W}$ by a saturated open substack such that f sends closed points to closed points and is stabilizer preserving at closed points, we may assume that the left and right faces are Cartesian. The argument in the proof of [AFS15, Theorem 4.2] concerning Diagram (7.2) shows that the bottom face is Cartesian.

The restriction of $\mathcal{N}^\vee$ to $\mathcal{W}^-_{\chi_{\delta-\psi}}$ descends to the relative $\mathcal{O}(1)$ on $W^-_{\chi_{\delta-\psi}}/\!\!/G$. Therefore, the pullback of $\mathcal{N}^\vee$ on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$ to $W^-_\chi/\!\!/G$ is $\mathcal{O}(1)$ and, in particular, is relatively ample over $W/\!/G$. Since the bottom face is Cartesian, it follows by descent that $\mathcal{N}^\vee$ is relatively ample over $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. The proposition follows. ☐

We proceed to prove Theorem 1.1 using Propositions 7.1, 7.2, 7.3 and Theorem 1.3.

Proof of Theorem 1.1. First, we show that Part (2) follows from Part (1). Indeed, suppose $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$. Then

$$\overline{\mathbb{M}}_{g,n}(\alpha) \simeq \operatorname{Proj} R\big(\overline{\mathbb{M}}_{g,n}(\alpha), K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi\big) \simeq \overline{M}_{g,n}(\alpha),$$

where the second isomorphism is given by Proposition 7.1 below.

The proof of Part (1) proceeds by descending induction on α beginning with the known case $\alpha > 9/11$, when $\overline{\mathcal{M}}_{g,n}(\alpha) = \overline{\mathcal{M}}_{g,n}$. Let $\alpha_c \in \{\alpha_1 = 9/11, \alpha_2 = 7/10, \alpha_3 = 2/3\}$ and take $\alpha_0 = 1$. Suppose we know Part (1) for all $\alpha \geq \alpha_{c-1} - \epsilon$. By Theorem 1.3, the line bundle $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$ is nef on $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)$ and all curves on which it has degree 0 are contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$. It follows by Proposition 7.2 that the statement of Part (1) holds for all $\alpha \geq \alpha_c$. Finally, Proposition 7.3 gives the statement of Part (1) for $\alpha \geq \alpha_c - \epsilon$. ☐

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