ENDOMORPHISM ALGEBRAS OF MODULAR MOTIVES

A Thesis

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by

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DECLARATION.

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Eknath Ghate, at the Tata Institute of Fundamental Research, Mumbai.

(Debargha Banerjee)

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

(Professor Eknath Ghate)

Date:

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THESIS DEDICATED TO MY BABA ARR MAA

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My initial interest in algebraic number theory came from my M. Math days, when Prof. B. Sury taught two courses in number theory. My interest in the subject grew after coming to TIFR.

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Chapter 1

Synopsis

1.1 Weight 2 and more

Let $f = \sum_{n>0} a_n q^n$ be a primitive non-CM cusp form of weight at least 2, level N and character ϵ. Let M_f be the motive attached to f. In weight 2 this is Shimura's abelian variety, and in higher weight it is Scholl's Grothendieck motive. Let $\mathrm{End}(M_f)$ denote the ring of endomorphisms of M_f over $\bar{\mathbb{Q}}$ and $X_f = \mathrm{End}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $E = \mathbb{Q}(a_n)$ be the number field generated by the Fourier coefficients of f and let F be the subfield generated by $a_p^2 \epsilon(p)^{-1}$ for all primes p such that $\gcd(p, N) = 1$. It is known that X_f is a central simple algebra over the the number field F. One also knows that the class of $X = X_f$ in $\mathrm{Br}(F)$ is 2-torsion. K. Ribet has asked [Ri80] for an explicit description of this class.

In view of the exact sequence

$$0 \to {}_2\mathrm{Br}(F) \to \oplus_v \, {}_2\mathrm{Br}(F_v) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

it is enough to consider the local behaviour of X. For each place v of F, let $X_v = X \otimes_F F_v$. A theorem of Momose [Mo81] says that X is totally indefinite if k is even, and totally definite if k is odd, giving complete information about the Brauer class of X_v at the infinite places v. When v is a finite place, we prove that the class of X_v in $\mathrm{Br}(F_v)$ is completely determined in terms of the *parity* of the slope at v of the adjoint lift of f (when this slope is finite).

According to Langlands principle of functoriality, given two reductive algebraic groups H and G over $\mathbb{Q}$ and a homomorphism between their L-groups $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$, there should be a way to lift cuspidal automorphic representations π of $\mathrm{H}(\mathbb{A}_\mathbb{Q})$ to cuspidal automorphic representation Π of $\mathrm{G}(\mathbb{A}_\mathbb{Q})$, so that the Langland's L-functions of π and Π are related by the formula $L(s, \Pi, r) = L(s, \pi, r \circ u)$. In the case that $\mathrm{H} = \mathrm{GL}_2$ and $\mathrm{G} = \mathrm{GL}_3$, and u is the adjoint map, it is (by now) a classical theorem of Gelbart and Jacquet that every cuspidal automorphic form π on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ has a lift $\mathrm{ad}(\pi)$, called the Gelbart-Jacquet adjoint lift, to an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$. If the Satake parameters at an unramified prime p of π are α_p and β_p, then the Satake parameters of the adjoint lift $\mathrm{ad}(\pi)$ are $\frac{\alpha_p}{\beta_p}, 1, \frac{\beta_p}{\alpha_p}$.

Let now $\pi = \pi_f$ be the automorphic representation attached to the non-CM form f as above, and let $\mathrm{Ad}(\pi) = \mathrm{ad}(\pi) \oplus 1$ be the automorphic form on $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ obtained from the Gelbart-Jacquet adjoint lift by adding the trivial representation. Finally let

$$\Pi = \mathrm{Ad}(\pi)(k-1)$$

be the automorphic representation on $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ obtained by taking the $(k-1)$-st twist of $\mathrm{Ad}(\pi)$.

We define the *slope* m_v of Π at $v \mid p$ to be

$$m_v := [F_v \cdot \mathbb{Q}_p] \cdot v(t_p) \in \mathbb{Z} \cup \{\infty\},$$

where v is normalized so that $v(p) = 1$ and $t_p \in F$ is defined to be the sum of the four parameters of Π_p, namely

$$t_p = \left(\frac{\alpha_p}{\beta_p} + 1 + \frac{\beta_p}{\alpha_p} + 1 \right) \cdot p^{k-1} = \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1}.$$

The following theorem may be considered as a summary of all the results of the first part of the thesis (cf. [BG10a]).

Theorem 1.1.1. *Suppose v is a finite place of F such that $m_v \in \mathbb{Z}$ is finite. Then the class of X_v in* $\mathrm{Br}(F_v)$ *is determined by the parity of the slope m_v of* Π *at v.*

For each place v of F, lying over p with $\gcd(p, N) = 1$, one can compute

$$m_v = [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon(p)^{-1}) \in \mathbb{Z} \cup \{\infty\}.$$

We prove the following theorem.

Theorem 1.1.2 (Good reduction)**.** *Let p be a prime with* $\gcd(p, N) = 1$*. Assume* $a_p \neq 0$*. Let v be a place of F lying over p. Then X_v is a matrix algebra over F_v if and only if the finite slope m_v of Π at v is even.*

For p odd this is a theorem of Brown-Ghate [BG04] and Ghate–González-Jiménez–Quer [GGQ05], under a minor technical condition. We prove the above theorem for $p = 2$, and removed this technical condition for all primes p.

Let N_p be the exponent of the exact power of p dividing N. Let C denote the conductor of ϵ and let C_p be the exponent of the exact power of p dividing C. So $N_p \geq C_p$.

We prove a similar result for primes $p|N$. We start with the so called Steinberg case.

Theorem 1.1.3 (Steinberg)**.** *Let $N_p = 1$, $C_p = 0$, and let v be a prime of F lying above p. Then X_v is a matrix algebra over F_v if and only if the slope $m_v = [F_v : \mathbb{Q}_p](k-2)$ of Π at v is even.*

Note that this result is an extension of the previous theorem, as in this case $\epsilon(p)$ makes sense, and it is well known that $a_p^2 = \epsilon(p)p^{k-2}$ (cf. Theorem 4.6.17 [Mi89]).

We now turn to the very interesting case when $N_p = C_p \geq 1$. We decompose $\epsilon = \epsilon' \cdot \epsilon_p$ into its prime-to-p and p parts. We then have:

$$m_v = [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon'(p)^{-1} + 2p^{(k-1)} + \bar{a}_p^2 \epsilon'(p)) \in \mathbb{Z} \cup \{\infty\}.$$

Note that in this case $a_p \bar{a}_p = p^{k-1}$ (cf. Theorem 4.6.17 of [Mi89]). It can be checked that the three term expression in the last line above is indeed an element of F. Note again that $m_v \in \mathbb{Z}$ (unless it is infinite). In view of the two previous theorems, one might conjecture:

Conjecture 1.1.4. If $m_v < \infty$, then X_v is a matrix algebra over F_v if and only if m_v is even.

We prove that the conjecture is essentially true. In particular, when $m_v < [F_v : \mathbb{Q}_p](k-1)$, the conjecture is true.

Theorem 1.1.5 (Ramified principal series unequal slope case)**.** *Let $N_p = C_p$ and let v be a place of F lying above p. Assume that $m_v < [F_v : \mathbb{Q}_p](k-1)$. Then X_v is a matrix algebra over F_v if and only if the finite slope m_v of Π at v is even.*

We remark that while a partial result in the 'if' direction was proved in [GGQ05, Thm. 5.1], Theorem 1.1.5 gives complete information about the ramification of X_v in the unequal slope case.

In the equal slope case, when $m_v \geq [F_v : \mathbb{Q}_p](k-1)$, Conjecture 1.1.4 is, somewhat surprisingly, false. We introduce a set of new quantities m_v^ζ, which may be thought of as replacements of m_v. Let $\mu = \frac{a_p^2}{\epsilon'(p)}$ and $\nu = \bar{\mu} = \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)}$. Let e_v and f_v be the ramification index and residue degree of $v \mid p$, and let G_v be the decomposition subgroup of F at v. For any root of unity ζ in the image of ϵ_p on G_v, let

$$m_v^\zeta := e_v \cdot v\left(\mu^{f_v} \cdot 1/\zeta + 2p^{(k-1)f_v} + \nu^{f_v} \cdot \zeta\right) \in \mathbb{Z} \cup \{\infty\}.$$

The three term expression lies in F, so m_v^ζ is well defined. In particular, we define $m_v^+ := m_v^{+1}$, and if -1 belongs to the image of ϵ_p on G_v, then we define $m_v^- := m_v^{-1}$. We prove:

Theorem 1.1.6 (Ramified principal series equal slope case)**.** *Assume that $v \mid p$ and $N_p = C_p \geq 1$. Suppose $m_v \geq [F_v : \mathbb{Q}_p](k-1)$.*

(i) Let p be an odd prime. Assume that the tame part of ϵ_p on G_v is not quadratic. Let ζ be in the image of the tame part of ϵ_p on G_v. Then the parity of m_v^ζ is independent of ζ when it is finite, and then X_v is a matrix algebra over F_v if and only if $m_v^\zeta \in \mathbb{Z}$ is even.

(ii) If $p = 2$ and ϵ_2 is not quadratic, then there exists an integer n_v mod 2 defined purely in terms of ϵ_2, such that X_v is a matrix algebra over F_v if and only if one of

$$m_v^\pm + n_v \in \mathbb{Z}$$

is even.

(iii) If p is odd and the tame part of ϵ_p is quadratic on G_v, or if $p = 2$ and ϵ_2 is quadratic on G_v, then there is an integer n_v mod 2 defined in terms of the Hilbert symbol $(t, d)_v$, with t depending only on ϵ_p and d on an explicit Fourier coefficient of f, such that X_v is a matrix algebra over F_v if and only if one of

$$m_v^\pm + n_v \in \mathbb{Z}$$

is even.

The above theorem reduces to the previous theorem when the slopes are unequal. Indeed, in the unequal slope case, the quantities $m_v^\zeta = m_v$, and $n_v = 0$ in case (2) and (3). Thus we may think of n_v as an error term to the validity of Conjecture 1.1.4 in the equal slope case.

The above results give a complete answer to Ribet's question on the Brauer class of $X_f = X$ in the cases of finite slope. We note that these cases cover all cases where M_f has either semistable or crystabelian (crystalline over an abelian extension of $\mathbb{Q}$) reduction. The remaining finite places of bad reduction occur when $N_p > C_p$. In such cases $a_p = 0$ and even the slope of f is not finite.

1.2 Weight One

We have also extended some of the above results to modular forms of weight one [BG10b]. One can attach an Artin motive to such a form. We have proved a structure theorem for the endomorphism algebra of this motive. We remark that it is well known that the category of Artin motives can be identified with a category of (finite image) Galois representations.

Let $f \in S_1(\Gamma_0(N), \epsilon)$ be a primitive form without RM or CM. Let $E = \mathbb{Q}(a_n)$ be the number field generated by the Fourier coefficients of f. Let ρ_f be the Galois representation attached to f, as constructed by Deligne-Serre [DS74], i.e.,

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(E).$$

For all $\sigma \in \Sigma = \mathrm{Hom}(E, \bar{\mathbb{Q}})$, let ρ_σ be the Galois representation

$$\rho_\sigma : G_{\bar{\mathbb{Q}}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}})$$

obtained by composing ρ_f with σ. We define the representation ρ by

$$\rho = \bigoplus_{\sigma \in \Sigma} \quad \rho_\sigma.$$

The underlying vector space is

$$V = \bigoplus_{\sigma \in \Sigma} V_\sigma,$$

where V_σ is the underlying vector space of ρ_σ. We view V as a vector space over $\bar{\mathbb{Q}}$.

We say an element $T \in \mathrm{End}(V)$ is defined over a number field K, if T is H-equivariant, where

$$H = \mathrm{Gal}(\bar{\mathbb{Q}}/K).$$

We define:

$$X_f = \mathrm{End}_{\mathrm{Gal}(\bar{\mathbb{Q}}/K_0)}(V),$$

where K_0 is a sufficiently large finite abelian extension of $\mathbb{Q}$. This is the endomorphism algebra (over K_0) of the Artin motive associated to f and its conjugates.

Lemma 1.2.1. *Let*

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

be an odd representation with non-dihedral projective image. If H is an open, normal subgroup of $G_{\mathbb{Q}}$ with $G_{\mathbb{Q}}/H$ abelian, then H acts irreducibly.

Recall that a pair (γ, χ_γ), where $\gamma \in \mathrm{Aut}(E)$ and χ_γ is an E-valued Dirichlet character, is called an extra twist for f, if $f^\gamma = f \otimes \chi_\gamma$, i.e., $a_p^\gamma = a_p \cdot \chi_\gamma(p)$, for all primes $p \nmid N$. Let $F \subset E$ be the field fixed by Γ. We call χ_γ the Dirichlet character attached to $\gamma \in \Gamma$. Let L be a sufficiently large finite Galois extension of $\mathbb{Q}$ (containing E and all the Gauss sums $G(\chi_\gamma^{-1})$, for $\gamma \in \Gamma$). Write $V_{/L}$ for the Artin motive above, but with field of rationality L instead of $\bar{\mathbb{Q}}$, and let $X_{f/L}$ be the corresponding endomorphism algebra (over K_0).

Combining the above lemma with arguments of Ribet [Ri80] we prove the following theorem.

Theorem 1.2.2. *Let f be a non-dihedral primitive cusp form of weight one. The endomorphism algebra $X_{f/L}$ is isomorphic to $X \otimes_{\mathbb{Q}} L$, where X is the algebra given by*

$$X = \bigoplus_{\gamma \in \Gamma} E \cdot x_\gamma,$$

thought of as an algebra over $\mathbb{Q}$, with relations

$$\begin{aligned} x_\gamma \cdot x_\delta &= \frac{G(\chi_\gamma^{-1}) G(\chi_\delta^{-\gamma})}{G(\chi_{\gamma\delta}^{-1})} \cdot x_{\gamma\delta}, \ \textit{and} \\ x_\gamma \cdot e &= \gamma(e) \cdot x_\gamma, \end{aligned}$$

for $\gamma, \delta \in \Gamma$, $e \in E$.

It is a deep fact that the algebra X is also the endomorphism algebra of the motive attached to a primitive non-CM cusp form of weight $k \geq 2$ (as was proved by Ribet [Ri80] and Momose for $k = 2$ and Brown-Ghate [BG04] for $k > 2$). In view of the above theorem, we conclude that the algebra X is important for weight one forms as well.

Note that X is a central simple algebra over F, and again X is 2-torsion in the Brauer group of F. We show that the analogue of Theorem 1.1.2 holds in the weight one setting, namely:

Theorem 1.2.3. *Let $p \nmid N$ be a prime with $a_p \neq 0$, and let v be a place of F lying over p. Then X_v is a matrix algebra over F_v if and only if m_v is even.*

We calculate the possible values of m_v in the theorem above using the adjoint representation and conclude that X_v is a matrix algebra in almost all cases (cf. [BG10b]).

Since, the above result provides no information at the primes $p \mid N$, we also calculated the Brauer class of the algebra X at all primes for forms of *prime* level. In this case Serre [Se77a] has classified such forms into three types (b), (c1) and (c2).

Theorem 1.2.4. *Let $f \in S_1(p, \epsilon)$ be a primitive non-dihedral cusp form. The following are the three possibilities for X in* $\mathrm{Br}(F)$.

(i) If f is of Serre type (b)*, then $X = (-1, -2)$,*

(ii) If f is of Serre type (c1)*, then $X = (-2, -p)$,*

(iii) If f is of Serre type (c2)*, then $X = (-1, -p)$.*

Here, (a, b) is the symbol defined for $a, b \in F^*$.

Chapter 2

Introduction

In this thesis, we study the endomorphism algebras of motives attached to non-dihedral modular forms of weight greater or equal to one.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a primitive non-CM (non dihedral for weight one) cusp form of weight $k \geq 1$, level $N \geq 1$ and character ϵ, and let M_f be the motive attached to f.

- If f has weight 1, M_f is an Artin motive.
- If f has weight 2, M_f is the abelian variety attached to f by Shimura [Sh71].
- If f has weight larger than 2, M_f is the Grothendieck motive attached to f by Scholl [Sc90].

Let $E = \mathbb{Q}(a_n)$ be the Hecke field of f. If f has weight greater than one, then M_f is known to be a pure motive of rank 2, weight $k-1$, with coefficients in E. Let $\mathrm{End}(M_f)$ denote the ring of endomorphisms of M_f defined over $\bar{\mathbb{Q}}$ and let

$$X_f = \mathrm{End}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the $\mathbb{Q}$-algebra of endomorphisms of M_f.

Let $\Gamma \subset \mathrm{Aut}(E)$ be the group of extra twists of f, in all weights. Recall that a pair (γ, χ_γ), where $\gamma \in \Gamma \subset \mathrm{Aut}(E)$ and χ_γ is an E-valued Dirichlet character, is called an extra twist for f, if $f^\gamma = f \otimes \chi_\gamma$, i.e., $a_p^\gamma = a_p \cdot \chi_\gamma(p)$, for all primes $p \nmid N$.

Define the E-valued Jacobi sum 2-cocycle c on Γ by

$$c(\gamma, \delta) = \frac{G(\chi_\delta^{-\gamma}) G(\chi_\gamma^{-1})}{G(\chi_{\gamma \cdot \delta}^{-1})} \in E,$$

for γ, $\delta \in \Gamma$, where $G(\chi)$ is the usual Gauss sum attached to the character χ. Let X be the corresponding crossed product algebra defined by:

$$X = \bigoplus_{\gamma \in \Gamma} E \cdot x_\gamma, \tag{2.0.1}$$

where the x_γ are formal symbols satisfying the relations

$$\begin{aligned} x_\gamma \cdot x_\delta &= c(\gamma, \delta) \cdot x_{\gamma\delta}, \\ x_\gamma \cdot e &= \gamma(e) \cdot x_\gamma, \end{aligned}$$

for γ, $\delta \in \Gamma$ and $e \in E$. Clearly X is a central simple algebra over F, the fixed field of Γ in E. A fundamental result due to Momose [Mo81] and Ribet [Ri80] in weight 2, and Brown and Ghate [BG04] and Ghate, González-Jiménez and Quer [GGQ05] in higher weights, says that $X_f \cong X$ and the class of X_f in the Brauer group $\mathrm{Br}(F)$ of F is 2-torsion. Moreover $F \subset E$ is known to be the subfield generated by $a_p^2 \epsilon^{-1}(p)$, for primes $p \nmid N$.

Ribet has remarked that it seems difficult to describe the class of X_f by pure thought. In this thesis, we give a complete description of the Brauer class of X_f in terms of the slopes of the adjoint lift of f, under a finiteness hypothesis on these slopes [BG10a].

One might also wonder to what extent the above mentioned results extend to the presumably simpler case of non-dihedral modular forms of weight one, where the Grothendieck motive is replaced by an Artin motive, which for practical purposes may be taken to be the direct sum ρ of the finite image Galois representations attached to the conjugates of the non-dihedral modular form f.

We consider a suitable algebra of endomorphisms X_f of ρ whose elements are defined over *abelian* number fields, and prove a similar structure theorem for X_f. More precisely, if the field of definition L of ρ is sufficiently large, then we prove that

$$X_f \simeq X \otimes_{\mathbb{Q}} L.$$

In this thesis, we also completely analyze the Brauer class of X for weight one forms [BG10b].

We start the thesis with some preliminaries. In Chapter 3, we introduce modular forms and Galois representations attached to *newforms*. In Chapter 4, we define motives attached to such *newforms* and the corresponding endomorphism algebras, which are the objects of our study.

In Chapter 5, we recall how, for forms of weight greater than one, the endomorphism algebra X_f satisfies $X_f \cong X$. In this chapter, we also prove that $X_f \simeq X \otimes_{\mathbb{Q}} L$ for modular forms of weight one (*cf.* Theorem 5.1.11).

The above structure theorems show the importance of the central simple algebra X in the study of the endomorphism algebras of modular motives for all weight $k \geq 1$. The main objective of the present thesis is to study the Brauer class of X in ${}_2\mathrm{Br}(F)$. The standard exact sequence from class field theory

$$0 \to {}_2\mathrm{Br}(F) \to \oplus_v \, {}_2\mathrm{Br}(F_v) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

where v runs over all places of F, shows that it is enough to study the class of $X_v = X \otimes_F F_v$ in $\mathrm{Br}(F_v)$, for each place v. It is well known that ${}_2\mathrm{Br}(F_v) \cong \mathbb{Z}/2$ (including if v is infinite since F is totally real), and X_v is a matrix algebra over F_v if the class of X_v is trivial, and a matrix algebra over a quaternion division algebra over F_v, if the class of X_v is non-trivial. A theorem of Momose [Mo81] says that X is totally indefinite if k is even, and totally definite if k is odd, giving complete information about the Brauer class at the infinite places v. When v is a finite place, we shall prove the class of X_v in $\mathrm{Br}(F_v)$ is completely determined in terms of the *parity* of the slope at v of the adjoint lift of f (when this slope is finite).

In Chapter 6, we recall some basic facts about automorphic forms and Langlands principle of functoriality. According to Langlands principle of functoriality, given two reductive algebraic groups H and G over $\mathbb{Q}$ and a homomorphism between their L-groups $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$, there should be a way to lift cuspidal automorphic representations π of $\mathrm{H}(\mathbb{A}_{\mathbb{Q}})$ to cuspidal automorphic representation Π of $\mathrm{G}(\mathbb{A}_{\mathbb{Q}})$, so that the Langlands L-functions of π and Π are related by the formula $L(s, \Pi, r) = L(s, \pi, u \circ r)$. In the case that $\mathrm{H} = \mathrm{GL}_2$ and $\mathrm{G} = \mathrm{GL}_3$, and u is the adjoint map, it is (by now) a classical theorem of Gelbart and Jacquet that every cuspidal automorphic form π on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ has a lift $\mathrm{ad}(\pi)$, called the Gelbart-Jacquet adjoint lift, to an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$. If the Satake parameters at an unramified prime p of π are α_p and β_p, then the Satake parameters of the adjoint lift $\mathrm{ad}(\pi)$ are $\frac{\alpha_p}{\beta_p}, 1, \frac{\beta_p}{\alpha_p}$.

Let now $\pi = \pi_f$ be the automorphic representation attached to the non-CM form

f as above, and let $\mathrm{Ad}(\pi) = \mathrm{ad}(\pi) \oplus 1$ be the automorphic form on $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ obtained from the Gelbart-Jacquet adjoint lift by adding the trivial representation. Finally let

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be the automorphic representation on $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ obtained by taking the $(k-1)$-st twist of $\mathrm{Ad}(\pi)$.

We define the *slope* m_v of Π at $v \mid p$ to be

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where v is normalized so that $v(p) = 1$ and $t_p \in F$ is defined to be the sum of the four parameters of Π_p, namely

$$t_p = \left(\frac{\alpha_p}{\beta_p} + 1 + \frac{\beta_p}{\alpha_p} + 1 \right) \cdot p^{k-1} = \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1}.$$

The following theorem may be considered as a summary of all the results proved in Chapter 7 about the endomorphism algebras of motives attached to modular forms of weight greater than 1.

Theorem 2.0.5. *Suppose v is a finite place of F such that $m_v \in \mathbb{Z}$ is finite. Then the class of X_v in $\mathrm{Br}(F_v)$ is determined by the parity of the slope m_v of Π at v.*

Before we proceed further, we wish to remark that the above theorem seems to be another instance of a recurring theme in the theory of the arithmetic of automorphic forms, wherein arithmetic information about an object attached to a form (in this case the endomorphism algebra) is contained in the Fourier coefficients of a suitable lift of the original form(in this case the twisted adjoint lift). The most striking example of this theme occurs in the correspondence between forms of integral weight k and forms of half-integral weight $(k+1)/2$. Here, twisted central critical L-values of the original form on PGL_2 occur as Fourier coefficients of the Shimura-Shintani-Waldspurger lift of this form to the metaplectic group $\widetilde{\mathrm{SL}}_2$ (*cf.* [Sh73] and [Wa81]). The theorem above establishes another instance where this theme is played out.

The slope m_v of Π at a place $v \mid p$ of F in the theorem above is defined to be a suitably normalized v-adic valuation of the sum of certain parameters coming from the local automorphic representation Π at p. The shape of these parameters vary in different cases, but can be made completely precise. As a result we obtain various explicit versions of the above theorem which we state now.

For instance, suppose that $v \mid p$ with $p \nmid N$, so that π_p is an unramified representation. Then the slope m_v of Π at v is the (normalized) v-adic valuation of the sum of the Satake parameters of Π_p. Since $\mathrm{Ad}(\pi)$ has Satake parameters $\frac{\alpha_p}{\beta_p}, 1, \frac{\beta_p}{\alpha_p}, 1$, we have

$$\begin{aligned} m_v &:= [F_v : \mathbb{Q}_p] \cdot v\left((\frac{\alpha_p}{\beta_p} + 1 + \frac{\beta_p}{\alpha_p} + 1) \cdot p^{k-1} \right) = [F_v : \mathbb{Q}_p] \cdot v\left(\frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1} \right) \\ &= [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon^{-1}(p)) \in \mathbb{Z} \cup \{\infty\}, \end{aligned}$$

where v is normalized so that $v(p) = 1$. We remark that F may be considered as the Hecke field of the adjoint lift Π, since it is generated by the quantities $a_p^2 \epsilon^{-1}(p)$, for $p \nmid N$. Moreover, the slope m_v of Π at v is an integer because of the local degree term $[F_v : \mathbb{Q}_p]$ (unless of course $a_p = 0$, in which case m_v is infinite). We prove (*cf.* Theorem 7.1.1):

Theorem 2.0.6 (Spherical case)**.** *Assume* $\gcd(p, N) = 1$. *Let* v *be a place of* F *lying over* p. *Assume* $a_p \neq 0$. *Then* X_v *is a matrix algebra over* F_v *if and only if* $m_v = [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon(p)^{-1}) \in \mathbb{Z}$ *is even.*

The case $k = 2$ and $m_v = 0$ (good, ordinary reduction) is due to Ribet [Ri81]. The general case for odd primes, and for $p = 2$ when $F = \mathbb{Q}$, was proved in [BG04] and [GGQ05, Thm. 2.2], under a minor technical hypothesis. Here we include the case $p = 2$ for all F, and remove this technical hypothesis. We also prove the analogous theorem for weight one forms (*cf.* Theorem 8.1.1).

However, the main point of our thesis is to treat, for $k \geq 2$, the primes of bad reduction, i.e., the primes $v \mid p$ of F with $p \mid N$. Let $N_p \geq 1$ be the exponent of the exact power of p dividing N. Let C denote the conductor of ϵ and let $C_p \geq 0$ be the exponent of the exact power of p dividing C. Note $N_p \geq C_p$. Since $p \mid N$, we no longer have the Satake parameters of π_p at our disposal. However, we can replace these numbers by the corresponding eigenvalues of ℓ-adic Frobenius in the ℓ-adic Weil-Deligne representation corresponding to π_p, for $\ell \neq p$, or equivalently by [Sa97], with the eigenvalues of crystalline Frobenius on the filtered (φ, N)-module attached to π_p as in [GM09], and can still compute the slope of Π at v.

For example, in the case that $N_p = 1$ and $C_p = 0$, it is well known that π_p is an unramified twist of the Steinberg representation. In this case, the eigenvalues of

ℓ-adic Frobenius are nothing but $\alpha_p = a_p$ and $\beta_p = pa_p$, up to multiplication by the same constant. We thus have:

$$\begin{aligned} m_v &:= [F_v : \mathbb{Q}_p] \cdot v\left(\frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1}\right) \\ &= [F_v : \mathbb{Q}_p] \cdot (k-2) \in \mathbb{Z}. \end{aligned}$$

In Theorem 7.2.4 we prove:

Theorem 2.0.7 (Steinberg case)**.** *Let $N_p = 1$, $C_p = 0$, $k \geq 2$ and let v be a prime of F lying above p. Then X_v is a matrix algebra over F_v if and only if the slope $m_v = [F_v : \mathbb{Q}_p](k-2)$ of Π at v is even.*

The proof of Theorem 2.0.7 uses the structure of the ℓ-adic Galois representation attached to f at p, for $\ell \neq p$, due to Langlands. The case $k = 2$ is due to Ribet [Ri81], who in fact showed that the algebra X is trivial in the Brauer group of F, using the fact that the corresponding residual abelian variety has toric reduction. Ribet's result was extended to forms of even weight k in [BG04, Thm. 1.0.6]. In this paper examples were also given of forms of odd weight for which the endomorphism algebra is ramified at Steinberg primes. The above theorem gives a complete criterion for the ramification of X at Steinberg primes in all weights $k \geq 2$.

We now turn to the very interesting case when $N_p = C_p \geq 1$ and π_p is in the ramified principal series. The behaviour of the local Brauer class in this case is mysterious, but has now become possible to treat using the adjoint lift. The eigenvalues of ℓ-adic or crystalline Frobenius are not well-defined in this case since the Weil-Deligne parameter corresponding to π_p is ramified. However, one more or less canonical choice is $\alpha_p = a_p$ and $\beta_p = \bar{a}_p \epsilon'(p)$, where we decompose $\epsilon = \epsilon' \cdot \epsilon_p$ into its' prime-to-p and p parts. We set, $\mu = \frac{a_p^2}{\epsilon'(p)}$ and $\nu = \bar{\mu}$. We then have:

$$\begin{aligned} m_v &:= [F_v : \mathbb{Q}_p] \cdot v\left(\frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1}\right) \\ &= [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon'(p)^{-1} + 2p^{(k-1)} + \bar{a}_p^2 \epsilon'(p)) \in \mathbb{Z} \cup \{\infty\}. \end{aligned}$$

It can be checked that the three term expression in the last line above is indeed an element of F. It is clearly fixed by complex conjugation; it is in fact fixed by all elements of Γ (*cf.* Lemma 7.3.1). Note again that $m_v \in \mathbb{Z}$ (unless it is infinite). In view of the two previous theorems, one might conjecture:

Conjecture 2.0.8. Assume $v \mid p$ and $N_p = C_p \geq 1$. If $m_v < \infty$, then X_v is a matrix algebra over F_v if and only if m_v is even.

We prove that the conjecture is essentially true. In particular, we prove the conjecture is true in Theorem 7.3.10 and Theorem 7.3.15, when $m_v < [F_v : \mathbb{Q}_p](k-1)$ and $k \geq 2$.

Theorem 2.0.9 (Ramified principal series unequal slope case)**.** *Let $N_p = C_p$, $k \geq 2$, and let v be a place of F lying above p. Assume that $m_v < [F_v : \mathbb{Q}_p](k-1)$. Then X_v is a matrix algebra over F_v if and only if the finite slope m_v of Π at v is even.*

We remark that while a partial result in the 'if' direction was proved in [GGQ05, Thm. 5.1], Theorem 2.0.9 gives complete information about the ramification of X_v in the unequal slope case.

In the equal slope case, when $m_v < [F_v : \mathbb{Q}_p](k-1)$, the conjecture 2.0.8 is, somewhat surprisingly, false. Counterexamples for $p = 2$ and $p = 3$ are given in Chapter 9. This is related to the fact that the eigenvalues of the ℓ-adic Frobenius are not well-defined.

We introduce a set of new quantities m_v^ζ, which may be thought of as replacements of m_v. Let $\mu = \frac{a_p^2}{\epsilon'(p)}$ and $\nu = \bar{\mu} = \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)}$. Let e_v and f_v be the ramification index and residue degree of $v \mid p$, and let G_v be the decomposition subgroup of F at v. For any root of unity ζ in the image of ϵ_p on G_v, let

$$m_v^\zeta := e_v \cdot v\left(\mu^{f_v} \cdot 1/\zeta + 2p^{(k-1)f_v} + \nu^{f_v} \cdot \zeta\right) \in \mathbb{Z} \cup \{\infty\}.$$

The three term expression lies in F, so m_v^ζ is well defined. In particular, we define $m_v^+ := m_v^{+1}$, and if -1 belongs to the image of ϵ_p on G_v, then we define $m_v^- := m_v^{-1}$. We prove:

Theorem 2.0.10 (Ramified principal series equal slope case)**.** *Assume $v \mid p$ and $N_p = C_p \geq 1$ and $k \geq 2$. Suppose $m_v \geq [F_v : \mathbb{Q}_p](k-1)$.*

(i) Let p be an odd prime. Assume that the tame part of ϵ_p on G_v is not quadratic. Let ζ be in the image of the tame part of ϵ_p on G_v. Then the parity of m_v^ζ is independent of ζ when it is finite, and then X_v is a matrix algebra over F_v if and only if $m_v^\zeta \in \mathbb{Z}$ is even.

(ii) If $p = 2$ and ϵ_2 is not quadratic on G_v, then there exists an integer n_v mod 2 defined purely in terms of ϵ_2, such that X_v is a matrix algebra over F_v if and only if one of

$$m_v^\pm + n_v \in \mathbb{Z}$$

is even.

(iii) If p is odd and the tame part of ϵ_p is quadratic on G_v, or if $p = 2$ and ϵ_2 is quadratic on G_v, then there is an integer n_v mod 2 defined in terms of the Hilbert symbol $(t, d)_v$, with t depending only on ϵ_p and d on an explicit Fourier coefficient of f, such that X_v is a matrix algebra over F_v if and only if one of

$$m_v^{\pm} + n_v \in \mathbb{Z}$$

is even.

The above theorem reduces to the previous theorem when the slopes are unequal. Indeed, in the unequal slope case, the quantities $m_v^{\zeta} = m_v$, and $n_v = 0$ in case (ii) and (iii). Thus we may think of n_v as an error term to the validity of conjecture 2.0.8 in the equal slope case.

The above results give a complete answer to Ribet's question on the Brauer class of $X_f = X$ in the cases of finite slope and weight $k \geq 2$. We note that these cases cover all cases where M_f has either semistable or crystabelian (crystalline over an abelian extension of $\mathbb{Q}$) reduction. The remaining finite places of bad reduction occur when $N_p > C_p$. In such cases $a_p = 0$ and even the slope of f is not finite. We hope to return to the infinite slope cases in subsequent work.

In Chapter 8, we also completely analyze the Brauer class of X for weight one forms. An interpretation of the slope in Theorem 2.0.6 in terms of the adjoint representation allows us to compute it completely, showing that the Brauer class is essentially unramified at all primes of good reduction, whenever the corresponding slope is finite (Theorem 8.1.3). We also express the algebra X in terms of symbols, which allows us to show that at a prime of bad reduction the Brauer class of X is determined in a simple way by the nebentypus of the form. Finally, as an example, we determine the Brauer class of X for all non-dihedral weight one forms of *prime* level (Theorem 8.3.2).

In Chapter 9, we give some numerical examples supporting various theorems proved in Chapter 7. These examples were generated by the program `Endohecke` due to Brown and Ghate which was made by suitably modifying the C++ program `Hecke` created by W. Stein.

Chapter 3

Modular forms and Galois representations

3.1 Modular forms

In this section, we recall some basic results about classical elliptic modular forms. We closely follow the expositions of [DS05].

The principal *congruence* subgroup of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Definition 3.1.1. A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is called a *congruence* subgroup if $\Gamma \supset \Gamma(N)$ for some $N \in \mathbb{N}$. We define N to be the level of the *congruence* subgroup.

We recall the definition of the following *congruence* subgroups of level N.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \right\}, \text{ and}$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \text{ and } a \equiv 1 \equiv d \pmod{N} \right\}.$$

Let Γ be a congruence subgroup of level N. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the complex upper half plane. Now $SL_2(\mathbb{Z})$ acts on $\mathbb{Q} \cup \{\infty\}$. We define the cusps of Γ to be the Γ-equivalence classes of $\mathbb{Q} \cup \{\infty\}$.

Let $\mathrm{GL}_2^+(\mathbb{R})$ denote the set of 2×2 real matrices with determinant greater than zero. We define an action of an element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ on the space of holomorphic functions on $\mathbb{H}$ by

$$f[\alpha]_k(\tau) = (\det(\alpha))^{k-1}(c\tau + d)^{-k} f\Big(\frac{a\tau + b}{c\tau + d}\Big). \tag{3.1.1}$$

The congruence subgroup Γ contains a translation matrix of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some minimal $t \in \mathbb{N}$, because $\Gamma \supset \Gamma(N)$. Let the function f satisfies $f[\gamma]_k(\tau) = f(\tau)$ for all $\gamma \in \Gamma$. Let $D' = \{z : z \in \mathbb{C}, |z| < 1\} \backslash \{0\}$. Since Γ contains the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, so $f(z + t) = f(z)$. The form f will define a holomorphic function $g : D' \to \mathbb{C}$ by $g(z) = f(\exp \frac{2\pi i z}{t})$. The modular form f is said to be holomorphic at ∞, if g extends to a holomorphic function on $D = \{z : z \in \mathbb{C}, |z| < 1\}$. Hence, if f is holomorphic at ∞, g extends holomorphically to 0. We write

$$f(z) = \sum_n a_n q^n,$$

where $q = \exp(2\pi i z)$ and we call the above Fourier expansion to be the *q-expansion of the modular form f at ∞.*

Let $c \in \mathbb{Q} \cup \{\infty\}$ be any cusp, it can be written in the form $c = \alpha(\infty)$ for some α in $SL_2(\mathbb{Z})$. The function $h = f[\alpha]_k$ is a holomorphic function on the upper half plane and satisfies the condition $h[\beta]_k(\tau) = h(\tau)$ for all $\beta \in \alpha^{-1}\Gamma\alpha$. Now $\alpha^{-1}\Gamma\alpha$ is a also congruence subgroup of level N. The modular form f is said to be *holomorphic at the cusp c* if h is holomorphic at ∞.

For the congruence subgroup $\Gamma = \Gamma_1(N)$, it is easy to see that $t = 1$, since $\Gamma_1(N)$ contains the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Definition 3.1.2. A modular cusp form, or simply a cusp form, of weight $k \geq 1$, level N and *nebentypus character $\epsilon : \big(\mathbb{Z}/N\mathbb{Z}\big)^\times \longrightarrow \mathbb{C}^\times$ is a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying*

(i) f is holomorphic as a function $f : \mathbb{H} \to \mathbb{C}$,

(ii) for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *and all* $\tau \in \mathbb{H}$*, we have*

$$f[\alpha]_k(\tau) = \epsilon(d) f(\tau),$$

(iii) $f[\alpha]_k$ *is holomorphic at* ∞ *for all* $\alpha \in SL_2(\mathbb{Z})$*,*

(iv) the constant term of the q*-expansion at each cusp is zero.*

The space of all cusp forms of weight k, level N and character ϵ is denoted by $S_k(N, \epsilon)$. Inside the space of cusp forms there is the space of *newforms*. The space of *newforms* is an orthogonal complement of the space of *oldforms* with respect to the *Petersson inner product*. We do not give a formal definition here but remark that *newforms* are cusp forms which are normalized, i.e., whose first Fourier coefficient is 1 and which do not arise from cusp forms of lower level. We sometimes denote them by *cuspidal newforms*. They are also simultaneous eigenfunctions of all the Hecke operators.

3.1.1 Hecke operators

Let $f(z) = \sum_n a_n q^n$, where $q = \exp(2\pi i z)$, be a modular form in $S_k(N, \epsilon)$. For any $n \in \mathbb{N}$, the Hecke operators T_n on $S_k(N, \epsilon)$ are defined as follows:

$$T_n(f) = \sum_{m=0}^{\infty} \Big(\sum_{d|(m,n)} \epsilon(d) d^{k-1} a_{mn/d^2} \Big) q^m.$$

For $l \nmid N$, the Hecke operators $\langle l \rangle$ on $S_k(N, \epsilon)$ are defined as

$$\langle l \rangle f = \epsilon(l) f.$$

The Hecke operators on $S_k(N, \epsilon)$ form an algebra. We state a theorem regarding Fourier coefficients of newforms.

Theorem 3.1.3. *Let* $f \in S_k(N, \epsilon)$ *be a newform, then the field obtained by adjoining all the Fourier coefficients is a number field.*

Proof. [DS05, p. 234]. □

3.1.2 Complex multiplication

Let $f \in S_k(N, \epsilon)$ be an elliptic modular form and let ϕ be a Dirichlet character. Let $f \otimes \phi$ be defined by

$$f \otimes \phi := \sum_{n=1}^{\infty} \phi(n) a_n q^n.$$

By [Sh71, Proposition 3.64], $f \otimes \phi$ is a modular form of weight k and nebentypus $\epsilon\phi^2$.

Definition 3.1.4 ([Ri76])**.** The modular form f is said to have *complex multiplication* (CM) by ϕ if

$$\phi(p) a_p = a_p$$

for all primes p in a set of primes of density 1.

In this thesis, we are interested in modular forms which don't have complex multiplication. We call them non-CM elliptic modular forms.

3.2 Galois representations attached to newforms

In this section, we recall how we can attach Galois representations to newforms.

3.2.1 Galois representations associated to weight one newforms

In fact, we can attach Galois representations to modular forms, which are simultaneous eigenfunctions of all the Hecke operators.

Theorem 3.2.1 ([DS74])**.** *Let f be a modular form of weight one, level N and nebentypus ϵ as defined above. Suppose f is an eigenfunction for T_p for all $p \nmid N$ with eigenvalue a_p, then there exists a Galois representation*

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

such that $\mathrm{trace}(\rho_f(\mathrm{Frob}_p)) = a_p$ *and* $\det(\rho_f(\mathrm{Frob}_p)) = \epsilon(p)$ *for all primes $p \nmid N$. Moreover, the representation ρ_f is irreducible if and only if f is a cusp form.*

We note that the converse of the above theorem is *Artin's conjecture*. Chandrashekhar Khare and J-P. Wintenberger completed the proof of *Artin's conjecture*, as a consequence of *Serre's modularity conjecture*.

In view of the above, newforms f of weight one are completely determined by the corresponding Galois representations $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. We can project the continuous representation to $\mathrm{PGL}_2(\mathbb{C})$. The image of ρ_f is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ and by [Kl93], the only possible finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are

- the cyclic group of order n, denoted by C_n,
- the dihedral group of order $2n$, denoted by D_n for $n \geq 2$,
- the alternating groups A_4 and A_5 and the symmetric group S_4.

We note that the image of ρ_f in $\mathrm{PGL}_2(\mathbb{C})$ is cyclic if and only if ρ_f is reducible. Since $f \in S_1(N, \epsilon)$, so the projective image of the irreducible Galois representation ρ_f can only be D_n or the groups A_4, A_5 or S_4. From [Se77a, Theorem 7.2.1], the projective image of ρ_f is dihedral if and only if ρ_f is induced from a character of a quadratic subfield. In other words, there exists an extension K such that $[K : \mathbb{Q}] = 2$ and a character $\chi : G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to \mathbb{C}^*$ such that $\rho = \mathrm{Ind}_{K/\mathbb{Q}}\chi$. There are two possibilities, K is imaginary or K is real. In the first case, we say f is a CM modular form and in the second case, we say f is a modular form with real multiplication (RM). In this thesis, we will be interested only in the modular forms of weight one, whose projective images are A_4, S_4 and A_5. In these cases, we call the corresponding modular forms of tetrahedral type, octahedral type and icosahedral type.

3.2.2 Galois representations attached to newforms of weight greater than one

We state without proof the analogue of Theorem 3.2.1 for modular forms of weight greater than one.

Theorem 3.2.2 ([DS74]). *Let f be a modular form of weight greater than one, level N and nebentypus ϵ as defined above. Suppose f is an eigenfunction for T_p for all $p \nmid N$ with eigenvalue a_p. Let $E = \mathbb{Q}(a_n)$ be the Hecke field. Let λ be a finite prime of*

E with residue characteristic ℓ and let E_λ be the completion of E at the prime ideal λ. Then there exists a continuous, semisimple Galois representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$$

satisfying the properties $\mathrm{trace}(\rho_{f,\lambda}(\mathrm{Frob}_p)) = a_p$ *and* $\det(\rho_{f,\lambda}(\mathrm{Frob}_p)) = \epsilon(p)p^{k-1}$ *for all primes $p \nmid N\ell$.*

The Galois representations $\{\rho_{f,\lambda}\}_\lambda$ as constructed in the above theorem, form a so called compatible system of Galois representations. Hence we can remove λ from our notations and call the corresponding Galois representations to be ρ_f.

We state the following theorem of Langlands which describes the local behaviour of ρ_f. Let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$ be an ℓ-adic Galois representation attached to f, for a prime $\lambda \mid \ell$ of E with $\ell \neq p$, as described above. Let $\lambda(x)$ be the unramified character which takes arithmetic Frobenius Frob_p to $x \in E_\lambda$.

Theorem 3.2.3 (Langlands)**.** *The local behaviour of $\rho_f|_{G_p}$ at a decomposition group G_p at p is as follows.*

- *If $p \nmid N$, let α_p and β_p be roots of the polynomial $x^2 - a_p x + \epsilon(p)p^{k-1}$. Then*
$$\rho_f|_{G_p} \sim \begin{pmatrix} \lambda(\beta_p) & 0 \\ 0 & \lambda(\alpha_p) \end{pmatrix}.$$
- *If $N_p = 1$ and $C_p = 0$, let $\alpha_p = a_p$ and $\beta_p = pa_p$. Then*
$$\rho_f|_{G_p} \sim \begin{pmatrix} \lambda(\beta_p) & * \\ 0 & \lambda(\alpha_p) \end{pmatrix}.$$
- *If $N_p = C_p \geq 1$, let $\alpha_p = a_p$ and $\beta_p = \bar{a}_p\epsilon'(p)$. Then*
$$\rho_f|_{G_p} \sim \begin{pmatrix} \lambda(\beta_p)\cdot\epsilon_p & 0 \\ 0 & \lambda(\alpha_p) \end{pmatrix}.$$
- *If $N_p \geq 2 > C_p$ and $p > 2$, and π_p is supercuspidal, then $\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p}\chi$, for a quadratic extension K of $\mathbb{Q}_p$, and a character χ of G_K.*

Remark 3.2.4. If f is a modular form with a complex multiplication by ϕ as defined in section 3.1.2, then by considering the determinant of the corresponding Galois representations, we get $\epsilon\phi^2 = \epsilon$. In other words, ϕ is a quadratic character. Let K be the field cut out by the kernel of ϕ. We say that the f has complex multiplication by K.

Chapter 4

Modular motives

4.1 Abelian varieties for cusp forms of weight two

In this section, we will explicitly describe how to associate abelian varieties to newforms of weight two. We refer to [DS05] for more elaborate discussions.

4.1.1 Abelian varieties

Let X be a compact Riemann surface of genus g. The Jacobian of X is defined to be the linear functionals on the holomorphic differentials on X modulo those functionals coming from integration over loops in X. Complex analytically, the Jacobian of X is the torus $\mathbb{C}^g/\Lambda_g$, where $\Lambda_g \cong \mathbb{Z}^g$ is a lattice.

Definition 4.1.1. Let $\Omega_{\mathrm{hol}}(X)$ denote the vector space of holomorphic differentials on X and let $\Omega_{\mathrm{hol}}(X)^*$ denote the dual of the vector space $\Omega_{\mathrm{hol}}(X)$. The Jacobian of X is the quotient group

$$\mathrm{Jac}(X) := \Omega_{\mathrm{hol}}(X)^*/\mathrm{H}_1(X, \mathbb{Z}).$$

Let $Y_1(N) = \Gamma_1(N)\backslash\mathbb{H}$ and let $X_1(N)$ be the compactification of $Y_1(N)$ by adding the cusps of $\Gamma_1(N)$. Let $\mathrm{J}_1(N) = \mathrm{Jac}(X_1(N))$ denotes the Jacobian of the compact Riemann surface $X_1(N)$. A well-known construction, namely

$$f \mapsto f dz$$

identifies $S_2(\Gamma_1(N))$ with the holomorphic differentials on $X_1(N)$. We get another description of complex points of $\mathrm{J}_1(N)$. Namely,

$$\mathrm{J}_1(N) = S_2(\Gamma_1(N))^*/\mathrm{H}_1(X_1(N), \mathbb{Z}).$$

4.1.2 Hecke operators

Definition 4.1.2. For a congruence subgroup Γ of $SL_2(\mathbb{Z})$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we decompose $\Gamma\alpha\Gamma = \sum_j \Gamma\beta_j$. We define the weight-2 double coset operator $\Gamma\alpha\Gamma$ to be an operator, which takes $f \in S_2(\Gamma)$ to

$$f[\Gamma\alpha\Gamma]_2 = \sum_j f[\beta_j]_2.$$

We can define the pullback of the weight-2 $\Gamma\alpha\Gamma$-operator on $S_2(\Gamma)^*$ as in [DS05, p. 228]. This action will induce an action on $\mathrm{J}_1(N)$, as explained in [DS05, p. 228]. The Hecke operators are special cases of double coset operators.

For p prime, the Hecke operator T_p on $S_2(\Gamma_1(N))$ is $[\Gamma_1(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\Gamma_1(N)]_2$. Let l be an integer relatively prime to N. The diamond operator $\langle l \rangle$ on $S_2(\Gamma_1(N))$ is $[\Gamma_1(N)\alpha\Gamma_1(N)]_2$, for $\alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N)$ with $\delta \equiv l \pmod{N}$. For $(l, N) > 1$, we define $\langle l \rangle = 0$.

To define T_n, set $T_1 = 1$ (Identity operator); T_p is already defined for primes p. For prime powers, we define inductively

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1}\langle p \rangle T_{p^{r-2}},$$

for $r \geq 2$.

We extend the definition multiplicatively to T_n for all n, by defining $T_n = \prod_p T_{p^r}$ if $n = \prod_p p^r$.

A small calculation will show that the action of T_n and $\langle l \rangle$ on $S_2(N, \epsilon)$ are the same as that defined in 3.1.1.

Definition 4.1.3. The Hecke algebra $\mathbb{T}_\mathbb{Z}$ is the algebra over $\mathbb{Z}$ of endomorphisms of $S_2(\Gamma_1(N))$, generated over $\mathbb{Z}$ by the Hecke operators T_n and $\langle l \rangle$, for l relatively prime to N. That is,

$$\mathbb{T}_{\mathbb{Z}} = \mathbb{Z}[\{T_n, \langle l \rangle : (l, N) = 1\}].$$

4.1.3 Abelian varieties associated to weight two newforms

Let f be a *cuspidal newform*. Recall, we have a map

$$\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \mathbb{C}$$

defined by $T(f) = \lambda_f(T)f$. We denote the kernel of the map by

$$I_f = \{T \in \mathbb{T}_{\mathbb{Z}} : T(f) = 0\}.$$

Since, $\mathbb{T}_{\mathbb{Z}}$ acts on $\mathrm{J}_1(N)$, the subgroup $\mathrm{I}_f\mathrm{J}_1(N)$ of $\mathrm{J}_1(N)$ makes sense.

Definition 4.1.4. The abelian variety A_f associated to f is defined to be

$$A_f = \mathrm{J}_1(N)/\mathrm{I}_f\mathrm{J}_1(N).$$

Let

$$[f] = \{f^\sigma |\ \sigma : E \to \mathbb{C}\}.$$

The cardinality of this set is the number of embeddings of E in $\mathbb{C}$. Let $V_f = \mathrm{span}[f] \subset S_2(\Gamma_1(N))$ be a subspace of dimension $[E : \mathbb{Q}]$. Restricting the action of the subgroup $\mathrm{H}_1(X_1(N), \mathbb{Z})$ of $S_2(\Gamma_1(N))^*$ to the modular forms of the space V_f, we get a subgroup of the dual space V_f^*,

$$\Lambda_f = \mathrm{H}_1(X_1(N), \mathbb{Z})|_{V_f}.$$

The following theorem identifies the complex torus associated to A_f.

Theorem 4.1.5. *The map*

$$A_f \to V_f^*/\Lambda_f$$

defined by

$$[\phi] + I_f\mathrm{J}_1(N) \to \phi|_{V_f} + \Lambda_f$$

for $\phi \in S_2(\Gamma_1(N))^*$, induces an isomorphism $A_f \cong V_f^*/\Lambda_f$. The right hand side is a complex torus of dimension $[E : \mathbb{Q}]$.

Proof. [DS05, Proposition 6.6.4, p. 242]. □

The abelian variety A_f is defined over $\mathbb{Q}$. The $\mathbb{Q}$-algebra of endomorphisms defined over $\mathbb{Q}$ is known to be E. Let $\mathrm{End}(A_f)$ denote the algebra of endomorphisms of A_f, which are defined over $\bar{\mathbb{Q}}$.

Definition 4.1.6. We define the algebra X_f to be

$$X_f := \mathrm{End}(A_f) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We will study this algebra in the next few chapters.

4.2 Modular motives for weight greater than 2

In the section, we will discuss some basic facts about Grothendieck motives and the endomorphism algebras of these motives. We refer to [BG04] for further details.

4.2.1 Grothendieck motives

Let $K \subset \mathbb{C}$ be a number field with a fixed embedding to $\mathbb{C}$. Let X and Y be nonsingular projective varieties defined over K and suppose each geometrically irreducible component of X and Y has dimension d. Let $Z(X \times Y)$ be the rational vector space generated by irreducible subvarieties of $X \times Y$, defined over K, of pure codimension d. We fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\mathrm{H}_B^{2d}(X \times Y)(d)$ denote the Betti cohomology group with coefficients in $(2\pi i)^d\mathbb{Q}$. Let $c_B : Z(X \times X) \to \mathrm{H}_B^{2d}(X \times X)(d)$ denote the cycle class map.

Let $Z_h(X \times Y)$ be the quotient

$$Z_h(X \times Y) := Z(X \times Y)/\sim$$

where $\sim$ is the cohomological equivalence relation, i.e., for any $Z \in Z_h(X \times Y)$, Z is trivial if and only if the image of Z in $\mathrm{H}_B^{2d}(X \times Y)(d)$ under the cycle class map is 0. We endow $Z_h(X \times Y)$ with the multiplication defined by the composition product of correspondences.

An effective motive is a pair $M = (X, p)$, where X is a nonsingular projective variety and $p \in Z_h(X \times X)$ is a projector, that is it satisfies $p^2 = p$. If $N = (Y, q)$ is also an efffective motive, then $\mathrm{Hom}(M, N)$ is defined to be

Definition 4.2.1.

$$\mathrm{Hom}(M,N) := \frac{\{Z \in Z_h(X \times Y) : Z \circ p = q \circ Z\}}{\{Z \in Z_h(X \times Y) : Z \circ p = q \circ Z = 0\}}.$$

In this category of effective motive defined over K, we define the tensor product

$$M \otimes N := (X \times Y, p \times q).$$

Let L be the effective motive $(\mathbb{P}^1, 1 - Z)$, where $Z \in Z_h(\mathbb{P}^1 \times \mathbb{P}^1)$ is the class of the cycle $\mathbb{P}^1 \times \{\text{point}\}$. The functor sending M to $M \otimes L$ is fully faithful, so it can be formally inverted.

Definition 4.2.2. A motive is a pair (M, a), where M is an effective motive and $a \in \mathbb{Z}$. (M, a), sometime denoted by $M(a)$, is the a-fold Tate twist of M. If $N(b)$ is another effective motive, then $\mathrm{Hom}(M(a), N(b))$ is defined by

$$\mathrm{Hom}(M(a), N(b)) := \mathrm{Hom}(M \otimes L^{r-a}, N \otimes L^{r-b})$$

for any $r \geq \max\{a, b\}$.

This defines the category of Grothendieck motives over K with cohomological equivalence. An effective motive has a realisation in each of the standard cohomology theories.

4.2.2 Modular motives attached to modular forms of weight ≥ 3

In this section, we recall how we can associate Grothendieck motives to modular forms of weight greater than 2 and level $N = n$. We start with some definitions from classical algebraic geometry. We refer to [DR72], [KM85], [Sc98], [Br07] for more elaborate discussions on these topics.

An elliptic curve over a scheme S is a proper, smooth morphism of schemes $p : E \to S$, whose fibers are geometrically connected curves of genus one with a section $O \in E(S)$.

The category $\mathcal{E}$ of elliptic curves is a category whose objects are elliptic curves $(p : E \to S, O)$. We denote the objects in this category by E/S. The morphism in this category from the object $(p : E \to S, O)$ to $(p_1 : E_1 \to S_1, O_1)$, consisting of maps $(f : S \to S_1, g : E \to E_1)$ such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & E_1 \\ \Big\downarrow & & \Big\downarrow \\ S & \longrightarrow & S_1 \end{array}$$

is commutative and the induced morphisms of S_1-schemes $E \to E_1 \times_{S_1} S$ is an isomorphism of elliptic curves over S.

Let $\mathcal{S}$ denote the category of sets. Let $\mathcal{C}$ be any category, and let $F : \mathcal{C} \to \mathcal{S}$ be any contravariant functor. We recall that a functor F is said to be representable, if there exists an object X of $\mathcal{C}$, such that F is isomorphic to the contravarient functor

$$T \to \mathrm{Hom}_{\mathcal{C}}(T, X).$$

A *moduli* problem for elliptic curves is a contravariant functor $\mathrm{P} : \mathcal{E} \to \mathcal{S}$. Now we define a particular type of moduli problem, called rigid moduli problem [KM85, p. 109]. Let P be a moduli problem. For every elliptic curve $f : E \to S$, the group $\mathrm{Aut}_S(E) = \{g : E \simeq E : fg = f\}$ acts from the right on the set $\mathrm{P}(E/S)$ by the functoriality of P. P is said to be a rigid moduli problem if $\mathrm{Aut}_S(E)$ acts freely on $\mathrm{P}(E/S)$. In other words, for every elliptic curve E/S and every $\alpha \in \mathrm{P}(E/S)$, the only element of $\mathrm{Aut}_S(E)$ fixing α is identity. We note that any representable moduli problem is a rigid moduli problem.

Definition 4.2.3. [Br07] Let P be a rigid moduli problem and now consider a functor from the category of schemes Sch to $\mathcal{S}$, which associates to a scheme S, isomorphism classes of pairs $(E/S, \alpha)$, where $\alpha \in \mathrm{P}(E/S)$. Let us assume this functor is representable, and represented by M. Then there exists an elliptic curve E/M, such that the object E/M represents the moduli problem P. This object E/M is called the *universal elliptic curve* for the moduli problem P.

We consider the following moduli problem on $\mathbb{Z}[\frac{1}{n}]$-schemes, so called ($\Gamma_1(n)$-structure) moduli problem: $\mathcal{E} \to \mathcal{S}$,

$$[\Gamma_1(n)] : E/S \mapsto \{\text{points of order } n \text{ in } E(S)\}.$$

We consider a functor from the category of schemes over $\mathbb{Z}[\frac{1}{n}]$ to $\mathcal{S}$, which associates to any $\mathbb{Z}\left[\frac{1}{n}\right]$- scheme S, isomorphism classes of elliptic curves E over S with a $\Gamma_1(n)$-structure on E.

This is a representable functor for $n \geq 3$ by [KM85, p. 104, Theorem 3.7.1]. Let M_n denote the affine, smooth curve over $\mathbb{Z}[\frac{1}{n}]$ which represents the functor. Let $j : M_n \to \bar{M}_n$ denote the smooth compactification of M_n. Let $\pi : X_n \to M_n$ denote the universal elliptic curve which represents the ($\Gamma_1(n)$-structure) moduli problem.

Let $\bar{\pi} : \bar{X}_n \to \bar{M}_n$ denote the universal generalized elliptic curve, as constructed by Deligne-Rapoport [DR72]. We have a commutative diagram

$$\begin{array}{ccc} X_n & \longrightarrow & \bar{X}_n \\ \downarrow & & \downarrow \\ M_n & \longrightarrow & \bar{M}_n \end{array}.$$

Let $\pi_k : \bar{X}_n^k \to \bar{M}_n$ denote the k-fold product over $\bar{M}_n$. Let $\bar{\bar{X}}_n^k$ denote the canonical desingularisation, constructed by Deligne.

The group $(\mathbb{Z}/n\mathbb{Z})^2$ acts on $\bar{X}_n$ by translation in the fibers and μ_2 acts on $\bar{X}_n$ by the inversion in the fibers. This induces an action of the semi-direct product $(\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mu_2$ on $\bar{X}_n$. Let Σ_k-denote the symmetric group on k letters. The group

$$\Gamma_k = ((\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mu_2)^k \rtimes \Sigma_k$$

acts on $\bar{X}_n^k$. By the canonical nature of the desingularisation the action of Γ_k extends to an action on $\bar{\bar{X}}_n$.

Let $\epsilon : \Gamma_k \to \{\pm 1\}$ be a homomorphism which is trivial on $(\mathbb{Z}/n\mathbb{Z})^{2k}$, is the product map on μ_2^k and is the sign character on Σ_k. Let

$$\Pi_\epsilon = \frac{1}{|\Gamma_k|} \sum_{g \in \Gamma_k} \epsilon(g) \cdot g \in \mathbb{Z}\left[\frac{1}{2nk!}\right][\Gamma_k]$$

be the projector attached to ϵ and let $W_n = (\bar{\bar{X}}_n^k, \Pi_\epsilon)$ be the associated effective motive.

The group $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on X_n by

$$(E, \sigma) \mapsto (E, \gamma \circ \sigma)$$

for $\gamma \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

Let Γ_n^1 be the subgroup of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Let I_f be the annihilator of f in the Hecke algebra. Scholl [Sc90] defines M_f to be the sub-motive of $W_n^{\Gamma_n^1}$, which is the kernel of I_f acting on $W_n^{\Gamma_n^1}$. M_f is a motive over $\mathbb{Q}$ with coefficients in the Hecke field $E = \mathbb{Q}(a_n)$ of f.

Definition 4.2.4. Let $\mathrm{End}(M_f)$ denote the ring of endomorphisms of M_f defined over $\bar{\mathbb{Q}}$ and let

$$X_f = \mathrm{End}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the $\mathbb{Q}$-algebra of endomorphisms of M_f.

In the thesis, we study the algebra X_f. One knows that X_f is a central simple algebra over a subfield F of E, and that the class of X_f in the Brauer group $\mathrm{Br}(F)$ of F is 2-torsion.

4.3 The category of Artin motives

4.3.1 Artin motives with coefficients

The category of Artin motives over $\mathbb{Q}$ with coefficients in E is the Karoubian envelope of the dual of the category $\mathcal{C}$ with objects consisting of varieties over $\mathbb{Q}$ of dimension 0, and morphisms consisting of correspondences defined over $\mathbb{Q}$ with coefficients in E.

This category can be made explicit. Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. A variety over $\mathbb{Q}$ of dimension 0 is the spectrum of a finite product of number fields. A correspondence of a variety X of dimension 0, in another Y, is a formal linear combination of connected components of $X \times Y$, with coefficients in E. Thus if $\{Z_i\}$ are the connected components of $X \times Y$, then

$$\mathrm{Cor}(X, Y) = \{\Sigma_i \, a_i Z_i \mid a_i \in E\}.$$

Let $\chi_{Z_i(\bar{\mathbb{Q}})}$ be the characteristic function of $Z_i(\bar{\mathbb{Q}}) \subset (X \times Y)(\bar{\mathbb{Q}})$. The map induced by

$$\Sigma_i a_i Z_i \mapsto \Sigma_i a_i \chi_{Z_i(\bar{\mathbb{Q}})}$$

clearly identifies correspondences of X in Y with E-valued $G_{\mathbb{Q}}$-invariant functions on $(X \times Y)(\bar{\mathbb{Q}}) = X(\bar{\mathbb{Q}}) \times Y(\bar{\mathbb{Q}})$. Note such a $G_{\mathbb{Q}}$-invariant function determines an E-valued matrix, whose rows and columns are indexed by $X(\bar{\mathbb{Q}})$ and $Y(\bar{\mathbb{Q}})$ respectively, and 'composition of correspondences' corresponds to 'matrix products'.

For example, if X and Y are each the spectrum of a single number field, say

$$X = \mathrm{Spec}(K) = \mathrm{Spec}(\mathbb{Q}[x]/(f(x))), \quad \text{and} \quad Y = \mathrm{Spec}(L) = \mathrm{Spec}(\mathbb{Q}[x]/(g(x))),$$

where f, g are irreducible polynomials with coefficients in $\mathbb{Q}$, then

$$X \times Y = \mathrm{Spec}(K \otimes L) = \mathrm{Spec}(K[x]/(g(x))) = \coprod_i \mathrm{Spec}(M_i)$$

where $g(x) = \prod g_i(x)$ is a decomposition of $g(x)$ in $K[x]$ into irreducible factors, $M_i = M[x]/(g_i(x))$, and $Z_i = \mathrm{Spec}(M_i)$ are the connected components of $X \times Y$. Then

$$\mathrm{Cor}(X, Y) = \{\Sigma_i \, a_i \, \mathrm{Spec}(M_i) | a_i \in E\}.$$

Let $\mathrm{Hom}(K, \bar{\mathbb{Q}}) = \{\sigma_i\}$ and $\mathrm{Hom}(L, \bar{\mathbb{Q}}) = \{\tau_j\}$ be the embeddings of K and L into $\bar{\mathbb{Q}}$ respectively. Let $F \in \mathrm{Cor}(X, Y)$ be a correspondence, and let f be the corresponding $G_{\mathbb{Q}}$-invariant function as above. The E-valued matrix corresponding to $F = (F_{\sigma_i,\tau_j})$ and the function f are related by the formula $f(\sigma_i, \tau_j) = F_{\sigma_i,\tau_j}$.

Now consider the dual (opposite) category $\mathcal{C}^\circ$ and its Karoubian envelope. This is formally obtained by adjoining projectors to $\mathcal{C}^\circ$: its objects consist of pairs (X, p), where X is a variety of dimension 0 and $p : X \to X$ is a correspondence such that $p^2 = p$, and morphisms are defined by

$$\mathrm{Hom}((X, p), (Y, q)) = \{Z \in \mathrm{Cor}(Y, X) | Z \circ q = p \circ Z\}.$$

4.3.2 Rational representations

Note that if X is a variety of dimension 0 over $\mathbb{Q}$, then the E-vector space $E^{X(\bar{\mathbb{Q}})}$ has a natural $G_{\mathbb{Q}}$-action, and defines an E-rational representation of $G_{\mathbb{Q}}$. Moreover, every correspondence $F : X \to Y$ (with matrix F) gives rise to a $G_{\mathbb{Q}}$-linear map $E^{Y(\bar{\mathbb{Q}})} \to E^{X(\bar{\mathbb{Q}})}$, which is defined by the matrix F^t. In particular a projector $p : X \to X$ gives a $G_{\mathbb{Q}}$-linear map $p^t : E^{X(\bar{\mathbb{Q}})} \to E^{X(\bar{\mathbb{Q}})}$.

Consider the contravariant functor from the Karoubian envelope of $\mathcal{C}$ to the category of E-rational $G_{\mathbb{Q}}$-representations induced by

$$(X, p) \mapsto \mathrm{Image}(p^t) \quad \text{and} \quad F \mapsto F^t.$$

This functor is clearly fully faithful. It is also essentially surjective since every representation ρ of $G_{\mathbb{Q}}$ with kernel corresponding to the number field K is cut out by a

$G_{\mathbb{Q}}$-linear projector $E^{X(\bar{\mathbb{Q}})} \to E^{X(\bar{\mathbb{Q}})}$, with $X = \mathrm{Spec}(K)$. More precisely, we can look at ρ as a representation of the finite group $\mathrm{Gal}(K/\mathbb{Q})$. The vector space $E^{X(\bar{\mathbb{Q}})}$ is a space on which $G_{\mathbb{Q}}$ acts, and is the regular representation of $\mathrm{Gal}(K/\mathbb{Q})$. There is an explicit $G_{\mathbb{Q}}$-linear, E-rational projection map to the isotypical component containing ρ (*cf.* [Se77b, Theorem 8, p. 21]). Now further project to the E-vector space on which ρ acts. The projection is given by a block matrix consisting of one identity block and other zeros, which is clearly $G_{\mathbb{Q}}$-linear. The composition of these two projections is $G_{\mathbb{Q}}$-linear and cuts out ρ.

Hence the above functor yields an equivalence of categories between the category of Artin motives and the category of E-rational representations of $G_{\mathbb{Q}}$. We identify these categories from now on.

4.3.3 Artin motive attached to a cusp form of weight one

Let $f = \sum_n a_n q^n \in S_1(N, \epsilon)$ be a non-dihedral normalized newform of weight 1. Let $E = \mathbb{Q}(a_n)$ be the number field generated by the Fourier coefficients of f. Let ρ_f be the finite image Galois representation attached to f, as constructed in [DS74, Thm. 4.1], so

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E),$$

and $\mathrm{trace}(\rho_f(\mathrm{Frob}_p)) = a_p$ and $\det(\rho_f(\mathrm{Frob}_p)) = \epsilon(p)$, for all primes $p \nmid N$. That ρ_f has a model over E follows from the oddness of ρ_f [DS74, p. 521, footnote]. We also note that the extension $E/\mathbb{Q}$ is abelian, since E is generated by the traces of matrices of finite order. For each $\sigma \in \Sigma = \mathrm{Hom}(E, \bar{\mathbb{Q}})$, let ρ_σ be the Galois representation attached to the modular form f^σ. We view ρ_σ as a representation

$$\rho_\sigma : G_{\mathbb{Q}} \to \mathrm{GL}_2(L),$$

where $L \subset \bar{\mathbb{Q}}$ is a sufficiently large number field containing E.

In view of the subsection 4.3.1, the Artin motive attached to f should be the Galois representation ρ_f. However, at least naïvely, this object is not so interesting as far as a study of its endomorphism algebra is concerned. Indeed, if H is a normal subgroup of $G_{\mathbb{Q}}$ and H acts irreducibly (true if $H = G_{\mathbb{Q}}$ since f is a cusp form), then by Schur's lemma, $\mathrm{End}_H(\rho_f) = E$. If H doesn't act irreducibly, and H is sufficiently small (e.g., $H \subset \mathrm{Ker}(\rho_f)$), then $\mathrm{End}_{E[H]}(\rho_f) = \mathrm{End}_E(\rho_f) = \mathrm{M}_{2\times 2}(E)$. Thus for all subgroups H,

$$1 \leq \dim_E \operatorname{End}_{E[H]}(\rho_f) \leq 4,$$

and so this dimension is essentially independent of $|\Gamma|$, where Γ is the group of extra twists contained in $\operatorname{Aut}(E/\mathbb{Q})$, i.e.,

$$\Gamma = \{\gamma \in \operatorname{Aut}(E) \mid f^\gamma = f \otimes \chi_\gamma, \text{ for some } E\text{-valued Dirichlet character } \chi_\gamma\}.$$

However for weight $k \geq 2$, the endomorphism algebra of the motive attached to a cusp form of weight k is usually dependent on $|\Gamma|$. Hence, we make instead the following definition.

Definition 4.3.1. Define the Artin motive attached to f to be the $G_{\mathbb{Q}}$-representation ρ, where

$$\rho := \bigoplus\nolimits_{\sigma \in \Sigma} \rho_\sigma.$$

Let V_σ be the two dimensional vector space over L which affords the representation ρ_σ. The underlying vector space of ρ is then

$$V = \bigoplus\nolimits_{\sigma \in \Sigma} V_\sigma,$$

which is a $2[E : \mathbb{Q}]$-dimensional vector space over L. Thus ρ is an Artin motive with coefficients in L (not E).

We remark that what we have called the Artin motive attached to f is really the sum of the motives attached to each of the conjugates of f. The motive ρ is a closer analogue in weight one than ρ_f of Shimura's abelian variety A_f in weight two, or Scholl's motive M_f in higher weight, since its L-function satisfies $L(s, \rho) = \prod_{\sigma \in \Sigma} L(s, f^\sigma)$, as does the L-function of A_f and M_f.

We now define an algebra of endomorphisms of the Artin motive ρ. Say an element $T \in \operatorname{End}_L(V)$ is defined over a number field K, if T is H-equivariant, where $H = \operatorname{Gal}(\bar{\mathbb{Q}}/K)$, i.e., $T \in \operatorname{End}_{L[H]}(V)$.

Definition 4.3.2. We define

$$X_f := \operatorname{End}_{L[H]}(V)$$

where $H = \operatorname{Gal}(\bar{\mathbb{Q}}/K_0)$ and K_0 is a sufficiently large finite *abelian* extension of $\mathbb{Q}$.

The requirement that the elements of X_f are defined over abelian extensions of $\mathbb{Q}$ seems necessary in order to adapt several arguments from higher weight to the weight one situation.

Chapter 5

Extra twists and crossed product algebras

Let $\Gamma \subset \mathrm{Aut}(E)$ be the group of extra twists of f. Recall that a pair (γ, χ_γ), where $\gamma \in \Gamma \subset \mathrm{Aut}(E)$ and χ_γ is an E-valued Dirichlet character, is called an extra twist for f, if $f^\gamma = f \otimes \chi_\gamma$, i.e., $a_p^\gamma = a_p \cdot \chi_\gamma(p)$, for all primes $p \nmid N$. Define the E-valued Jacobi sum 2-cocycle c on Γ by

$$c(\gamma, \delta) = \frac{G(\chi_\delta^{-\gamma})G(\chi_\gamma^{-1})}{G(\chi_{\gamma\cdot\delta}^{-1})} \in E,$$

for γ, $\delta \in \Gamma$, where $G(\chi)$ is the usual Gauss sum attached to the character χ. Let X be the corresponding crossed product algebra defined by:

$$X = \bigoplus\nolimits_{\gamma\in\Gamma} E \cdot x_\gamma, \tag{5.0.1}$$

where the x_γ are formal symbols satisfying the relations

$$\begin{aligned} x_\gamma \cdot x_\delta &= c(\gamma, \delta) \cdot x_{\gamma\delta}, \\ x_\gamma \cdot e &= \gamma(e) \cdot x_\gamma, \end{aligned}$$

for γ, $\delta \in \Gamma$ and $e \in E$. Clearly X is a central simple algebra over F, the fixed field of Γ in E.

5.1 Structure theorems

5.1.1 Structure theorem for weight greater than one

In this section we will mention a theorem which explicitly describes the crossed-product structure of endomorphism algebras of motives attached to non-CM modular forms.

Theorem 5.1.1. *Let f be a non-CM modular form of weight $k \geq 2$, then*

$$X_f \simeq X.$$

Proof.

- For $k = 2$, *cf.* [Ri80, Theorem 5.1].
- For $k \geq 3$, *cf.* [BG04, Theorem 2.3.8] and [GGQ05, Theorem 2.0.3].

$\square$

5.1.2 Structure theorem for weight one

We recall the definition of the endomorphism algebra for a modular form of weight one.

Definition 5.1.2. We define

$$X_f := \mathrm{End}_{L[H]}(V)$$

where $H = \mathrm{Gal}(\bar{\mathbb{Q}}/K_0)$ and K_0 is a sufficiently large finite *abelian* extension of $\mathbb{Q}$.

Write elements of V in the form $v = (v_\sigma)_{\sigma\in\Sigma}$. Then $\mathrm{End}_L(V)$ can be thought of as an E-algebra using the following map:

$$\iota : E \to \mathrm{End}_L(V)$$

$$e \mapsto \{(v_\sigma)_{\sigma\in\Sigma} \mapsto (\sigma(e)v_\sigma)_{\sigma\in\Sigma}\}.$$

Lemma 5.1.3. *The action of E defined above commutes with the action of $G_{\mathbb{Q}}$.*

Proof. Let $g \in G_{\mathbb{Q}}$. We compute

$$\rho(g)e \cdot (v_\sigma)_{\sigma\in\Sigma} = \rho(g)(\sigma(e)v_\sigma)_{\sigma\in\Sigma} = (\sigma(e)\rho_\sigma(g)v_\sigma)_{\sigma\in\Sigma},$$

while

$$e \cdot \rho(g)(v_\sigma)_{\sigma\in\Sigma} = e \cdot (\rho_\sigma(g)v_\sigma)_{\sigma\in\Sigma} = (\sigma(e)\rho_\sigma(g)v_\sigma)_{\sigma\in\Sigma}.$$

□

In particular the elements of E are defined over $\mathbb{Q}$.

Assume now that L contains all the Gauss sums $G(\chi_\gamma^{-\sigma})$, for $\gamma \in \Gamma$ and $\sigma \in \Sigma$.

Definition 5.1.4. For each $\gamma \in \Gamma$, we define an endomorphism $\eta_\gamma \in \mathrm{End}_L V$ by

$$\eta_\gamma((v_\sigma)_{\sigma\in\Sigma}) = (G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

Lemma 5.1.5. *If H is a subgroup of $G_{\mathbb{Q}}$ such that $H \subset \ker(\chi_\gamma)$, then η_γ is H-equivariant. In particular, η_γ is defined over the fixed field cut out by χ_γ, an abelian extension of $\mathbb{Q}$.*

Proof. If $h \in H$, we have

$$\eta_\gamma\rho(h)(v_\sigma)_{\sigma\in\Sigma} = \eta_\gamma(\rho_\sigma(h)v_\sigma)_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})\rho_{\sigma\gamma}(h)v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

On the other hand

$$\rho(h)\eta_\gamma(v_\sigma)_{\sigma\in\Sigma} = \rho(h)(G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})\rho_\sigma(h)v_{\sigma\gamma})_{\sigma\in\Sigma}.$$

Since $f^{\gamma\cdot\sigma} = f^\sigma \otimes \chi_\gamma^\sigma$, it follows

$$\rho_{\sigma\gamma}(g) = \rho_\sigma(g)\chi_\gamma^\sigma(g),$$

for all $g \in G_{\mathbb{Q}}$. In particular, if $h \in H \subset \ker(\chi_\gamma^\sigma)$, then $\rho_{\sigma\gamma}(h) = \rho_\sigma(h)$, and the two expressions above coincide. □

Define the E-valued 2-cocycle c on Γ by

$$c(\gamma,\delta) \quad = \quad \frac{G(\chi_\gamma^{-1})G(\chi_\delta^{-\gamma})}{G(\chi_{\gamma\cdot\delta}^{-1})} \in E,$$

for γ, $\delta \in \Gamma$. We have:

Lemma 5.1.6. *η_γ defined as above satisfies the relations*

$$\eta_\gamma \cdot e = \gamma(e) \cdot \eta_\gamma,$$

$$\eta_\gamma \cdot \eta_\delta = c(\gamma, \delta) \cdot \eta_{\gamma\delta}.$$

Proof. Again, we compute:

$$\eta_\gamma(e \cdot (v_\sigma)_{\sigma\in\Sigma}) = \eta_\gamma(\sigma(e)v_\sigma)_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})(\sigma\gamma)(e)v_{\sigma\gamma})_{\sigma\in\Sigma},$$

whereas

$$\gamma(e) \cdot (\eta_\gamma(v_\sigma)_{\sigma\in\Sigma}) = \gamma(e) \cdot (G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma} = ((\sigma\gamma)(e)G(\chi_\gamma^{-\sigma})v_{\sigma\gamma})_{\sigma\in\Sigma},$$

proving the first relation. For the second, we have:

$$\eta_\gamma\eta_\delta(v_\sigma)_{\sigma\in\Sigma} = \eta_\gamma(G(\chi_\delta^{-\sigma})v_{\sigma\delta})_{\sigma\in\Sigma} = (G(\chi_\gamma^{-\sigma})G(\chi_\delta^{-\gamma\cdot\sigma})v_{\sigma\gamma\delta})_{\sigma\in\Sigma}.$$

On the other hand

$$c(\gamma, \delta) \cdot \eta_{\gamma\delta}(v_\sigma)_{\sigma\in\Sigma} = c(\gamma, \delta) \cdot (G(\chi_{\gamma\delta}^{-\sigma})(v_{\sigma\gamma\delta}))_{\sigma\in\Sigma} = (\sigma(c(\gamma, \delta))G(\chi_{\gamma\delta}^{-\sigma})v_{\sigma\gamma\delta})_{\sigma\in\Sigma}.$$

So $\eta_\gamma \cdot \eta_\delta = c(\gamma, \delta) \cdot \eta_{\gamma\delta}$, if for all $\sigma \in \Sigma$,

$$\sigma(c(\gamma, \delta)) = \frac{G(\chi_\gamma^{-\sigma})G(\chi_\delta^{-\gamma\cdot\sigma})}{G(\chi_{\gamma\delta}^{-\sigma})}.$$

But this is proved in [Sh76, Lemma 8, p. 797]. □

Let X be the crossed product associated to the cocycle $c(\gamma, \delta)$ defined in the introduction of this chapter.

Let K_0 be the fixed field of $H = \cap \ker \chi_\gamma$. Note K_0 is an abelian number field. By Lemma 5.1.5, each η_γ is defined over K_0, and so there is a natural map:

$$X \otimes_{\mathbb{Q}} L \to X_f \tag{5.1.1}$$

induced by

$$x_\gamma \mapsto \eta_\gamma$$

$$e \mapsto \iota(e).$$

We will prove that the above map is an isomorphism. The proof uses several arguments from higher weight (see especially Ribet's proof in weight two [Ri80]), with a few modifications, which we explain below. We start with two general lemmas.

Lemma 5.1.7. *Let $\rho_V : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ and $\rho_W : G_{\mathbb{Q}} \to \mathrm{GL}(W)$ be two representations where V and W are vector spaces over L. Let H be an open, normal subgroup of $G_{\mathbb{Q}}$ such that H acts irreducibly on V and W. Then $V \simeq W$ as H-modules iff $V \simeq W \otimes \phi$ as $G_{\mathbb{Q}}$-modules, for some finite order character $\phi : G_{\mathbb{Q}} \to L^*$ which is trivial on H.*

Proof. Let $T : V \to W$ be an H-isomorphism with inverse $S : W \to V$. For $g \in G_{\mathbb{Q}}$, define an element of $\mathrm{End}_L(W)$ by

$$\phi(g) = T \cdot \rho_V(g) \cdot S \cdot \rho_W(g)^{-1}.$$

Since H is a normal subgroup of $G_{\mathbb{Q}}$, $\phi(g)$ is H-equivariant. Hence $\phi(g) \in \mathrm{End}_{L[H]}(W)$. Evidently ϕ has an inverse as well, so $\phi(g) \in \mathrm{Aut}_{L[H]}(W)$. Since H acts irreducibly, by Schur's lemma, $\mathrm{End}_{L[H]}(W) = L$, hence $\phi(g) \in L^*$. Now ϕ is a homomorphism because

$$\begin{aligned} \phi(g_1 g_2) &= T \cdot \rho_V(g_1 g_2) \cdot S \cdot \rho_W(g_2^{-1} g_1^{-1}) \\ &= T \cdot \rho_V(g_1) \cdot S \cdot T \cdot \rho_V(g_2) \cdot S \cdot \rho_W(g_2^{-1}) \cdot \rho_W(g_1^{-1}) \\ &= \phi(g_1)\phi(g_2). \end{aligned}$$

Finally, since ϕ is trivial on H, and H is a subgroup of finite index of $G_{\mathbb{Q}}$, ϕ has finite order. The definition of ϕ shows $V \simeq W \otimes \phi$ as $G_{\mathbb{Q}}$-modules, proving one direction of the lemma. The other direction is clear. □

Lemma 5.1.8. *Let*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

be an irreducible representation with non-dihedral projective image. If H is an open, normal subgroup of $G_{\mathbb{Q}}$ with $G_{\mathbb{Q}}/H$ abelian, then H acts irreducibly.

Proof. Let $K = \bar{\mathbb{Q}}^{\ker(\rho)}$ be the field cut out by ρ and let $L = \bar{\mathbb{Q}}^H$ be the fixed field of H. Suppose H acts reducibly. Then $\mathrm{Image}(\rho|_H) \xrightarrow{\sim} \mathrm{Gal}(KL/L) \xrightarrow{\sim} \mathrm{Gal}(K/K \cap L)$ is an abelian group. Since $G_{\mathbb{Q}}/H \xrightarrow{\sim} \mathrm{Gal}(L/\mathbb{Q})$ is abelian by hypothesis, the quotient group $\mathrm{Gal}(K \cap L/\mathbb{Q})$ is also abelian. Projecting the exact sequence

$$1 \to N := \mathrm{Gal}(K/K \cap L) \to G := \mathrm{Gal}(K/\mathbb{Q}) \to G/N \xrightarrow{\sim} \mathrm{Gal}(K \cap L/\mathbb{Q}) \to 1$$

to $\mathrm{PGL}_2(\mathbb{C})$ we obtain another exact sequence

$$1 \to \tilde{N} \to \tilde{G} \to \tilde{G}/\tilde{N} \to 1 \tag{5.1.2}$$

where $\tilde{G}$ and $\tilde{N}$ are the projective images of ρ and $\rho|_H$ respectively. Since N is abelian, so is $\tilde{N}$. In fact, since H acts reducibly, $\tilde{N}$ is a cyclic group. Similarly, $\tilde{G}/\tilde{N}$ is an abelian group, since it is a quotient of G/N.

But an exact sequence such as (5.1.2) does not exist. Indeed, (5.1.2) shows that $\tilde{G}$ is solvable, so the case $\tilde{G} \xrightarrow{\sim} A_5$ does not occur. The normal subgroups of S_4 are the trivial group, the Klein 4-group, A_4 and S_4, with respective quotients S_4, S_3, $\mathbb{Z}/2$, and the trivial group. In particular one of $\tilde{N}$ or $\tilde{G}/\tilde{N}$ must always be non-abelian. Thus $\tilde{G} \simeq S_4$ is also not possible. Finally the normal subgroups of A_4 are the trivial group, the Klein 4-group and A_4, with respective quotients A_4, $\mathbb{Z}/3$, and the trivial group. Even though the Klein 4-group and $\mathbb{Z}/3$ are both abelian, the former group is not cyclic, so the case $\tilde{G} \simeq A_4$ is also ruled out.

Thus the assumption that H acts reducibly does not hold. $\square$

Remark 5.1.9. We remark that while the analogous lemma is true in higher weight without the abelianess assumption on $G_{\mathbb{Q}}/H$ (a result of Ribet says that that every non-dihedral cuspidal modular ℓ-adic Galois representation of weight 2 or more is irreducible on each open subgroup of H of $G_{\mathbb{Q}}$), it is clearly false for weight one forms without this assumption (take, e.g, $H = \ker(\rho)$). The utility of the above lemma in adapting higher weight arguments to the weight one setting is one reason why we only consider endomorphisms defined over abelian number fields.

We return to the proof of the main result of this section. We recall some notation that is used in higher weight. If H is a subgroup of $G_{\mathbb{Q}}$, then let

$$\Gamma_H := \{\gamma \in \Gamma \mid \chi_\gamma \text{ is trivial on } H\}$$

and let F_H be the fixed field of Γ_H in E. The following theorem (compare with [Ri81, Theorem 4.4]) is important for dimension counting reasons.

Theorem 5.1.10. *If H is an open, normal subgroup of $G_{\mathbb{Q}}$ such that $G_{\mathbb{Q}}/H$ is abelian, then*

$$\mathrm{End}_{L[H]}V \simeq \mathrm{End}_{F_H}E \otimes_{\mathbb{Q}} L.$$

Proof. By Lemma 5.1.8, V is a sum of modules V_σ, each of which is simple as an H-module, and hence satisfies

$$\mathrm{End}_{L[H]}V_\sigma = L.$$

Thus to compute $\mathrm{End}_{L[H]}V$ we only have to determine when V_σ and V_τ are isomorphic as H-modules, for any two embeddings σ, τ of E into L. By Lemma 5.1.7, this happens iff there is a character of finite order

$$\phi : G_{\mathbb{Q}} \to L^*$$

trivial on H, such that V_σ and $V_\tau \otimes \phi$ are isomorphic as $G_{\mathbb{Q}}$-modules. Taking traces, this happens iff

$$\sigma(a_p) = \tau(a_p) \cdot \phi(p),$$

for almost all primes p, i.e., $\sigma = \tau\gamma$ for some $\gamma \in \Gamma_H$. Recalling Σ is the set of all embeddings $\sigma : E \to L$, we obtain

$$\mathrm{End}_{L[H]}V = \prod\nolimits_{\sigma\in\Sigma/\Gamma_H} \mathrm{End}_{L[H]} \left(\prod\nolimits_{\gamma\in\Gamma_H} V_{\sigma\cdot\gamma}\right) \simeq \mathrm{M}_{a\times a}(L)^b,$$

where

$$a = |\Gamma_H|, \quad \text{and} \quad b = |\Sigma/\Gamma_H|.$$

This proves the theorem, since

$$\mathrm{End}_{F_H} E \otimes_{\mathbb{Q}} L \simeq \mathrm{M}_{a\times a}(F_H) \otimes_{\mathbb{Q}} L \simeq \mathrm{M}_{a\times a}(L)^b.$$

□

We can now prove the following structure theorem for the endomorphism algebra X_f of the Artin motive attached to f.

Theorem 5.1.11. *Suppose f is a cuspidal newform of weight one of non-dihedral type. Let L be a sufficiently large number field which contains E and all $G(\chi_\gamma^{-\sigma})$, for all $\gamma \in \Gamma$, $\sigma \in \Sigma$. Then the endomorphism algebra X_f of f satisfies:*

$$X_f \simeq X \otimes_{\mathbb{Q}} L.$$

Proof. Let K_0 be a sufficiently large abelian extension of $\mathbb{Q}$. Here 'sufficiently large' means that K_0 contains the fixed field of $\cap_{\gamma\in\Gamma} \ker(\chi_\gamma)$, or equivalently, if $H = \mathrm{Gal}(\bar{\mathbb{Q}}/K_0)$, then

$$H \subset \cap_{\gamma\in\Gamma} \ker(\chi_\gamma). \tag{5.1.3}$$

Then, by definition, $X_f = \mathrm{End}_{L[H]}(V)$.

Consider the natural map defined in (5.1.1):

$$X \otimes_{\mathbb{Q}} L \to X_f.$$

This map is clearly injective since $X \otimes_{\mathbb{Q}} L = \prod_{\sigma: F \hookrightarrow L} (X \otimes_{F,\sigma} L)$ and the restriction of the above map to each factor is clearly injective (each factor is a central simple algebra over L, so has no two-sided ideals).

Now X has dimension $[E : F]$ over E, and hence has dimension $[E : F][E : \mathbb{Q}]$ over $\mathbb{Q}$. Hence $X \otimes_{\mathbb{Q}} L$ has dimension $[E : F][E : \mathbb{Q}]$ over L. If we can prove $X_f = \mathrm{End}_{L[H]}(V)$ also has dimension $[E : F][E : \mathbb{Q}]$ over L then the above inclusion will be an isomorphism. Since $G_{\mathbb{Q}}/H = \mathrm{Gal}(K_0/\mathbb{Q})$ is abelian, by Theorem 5.1.10, we have

$$\dim_L \mathrm{End}_{L[H]}(V) = [E : F_H]^2 [F_H : \mathbb{Q}].$$

Since $F = F_H$ by (5.1.3), this dimension is $[E : F][E : \mathbb{Q}]$, as desired. □

Remark 5.1.12. We expect to eliminate L from the arguments given in this section. Indeed, the referee of [BG10b] has remarked that another candidate for the Artin motive attached to f which may allow this is the $\mathbb{Q}$ representation W, of dimension $2[E : \mathbb{Q}]$ over $\mathbb{Q}$, obtained from ρ_f by restricting scalars from E to $\mathbb{Q}$. Now W is a model of V over $\mathbb{Q}$, since $W \otimes_{\mathbb{Q}} L \cong V$. Note, $\mathrm{End}_{\mathbb{Q}[H]}(W)$ contains E and has same $\mathbb{Q}$ dimension as X, and the referee has suggested that they are isomorphic.

5.2 Brauer class of X

Theorem 5.1.1 shows the importance of the central simple algebra X in describing the structure of the endomorphism algebra X_f for forms of weight greater than one and Theorem 5.1.11 shows that X is important even for weight one modular forms. We will study the Brauer class of X for any weight greater or equal to 1. In this section we study the Brauer class of X as an element of the Brauer group of F.

5.2.1 Definition of α

In higher weights a crucial role is played by a certain map $\alpha : G_{\mathbb{Q}} \to E^*$.

The map α in higher weights is defined directly on the geometric object A_f or M_f, using the Skolem-Noether theorem (*cf.* [Ri81, section 2, p. 6] and [GGQ05, Theorem

4.0.10]). It seems difficult to extend this definition to weight one. However, a purely algebraic definition of α in higher weights was later given by Papier (*cf.* [Ri85, p. 192]), which extends nicely to all weights, as follows.

Recall, $f \in S_k(N, \epsilon)$ is a non-CM (non-dihedral for weight one) normalized newform, E is the Hecke field of f, and $\Gamma \subset \mathrm{Aut}(E)$ is the group of extra twists of f. For $\gamma \in \Gamma$, there is a unique Dirichlet character χ_γ such that $f^\gamma = f \otimes \chi_\gamma$, and hence $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$. For γ, $\delta \in \Gamma$, the identity

$$\chi_{\gamma \cdot \delta} = \chi_\gamma {\chi_\delta}^\gamma$$

shows that $\gamma \mapsto \chi_\gamma$ is a 1-cocycle. Specializing to $g \in G_\mathbb{Q}$, we see that $\gamma \mapsto \chi_\gamma(g)$ is a 1-cocycle as well. By Hilbert's theorem 90, $\mathrm{H}^1(\Gamma, E^*)$ is trivial, i.e., there is an element $\alpha(g) \in E^*$ such that

$$\alpha(g)^{\gamma - 1} = \chi_\gamma(g) \tag{5.2.1}$$

for all $\gamma \in \Gamma$. Clearly, $\alpha(g)$ is completely determined up to multiplication by elements of F^*. Varying $g \in G_\mathbb{Q}$, we obtain a well defined map

$$\tilde{\alpha} : G_\mathbb{Q} \to E^*/F^*.$$

Since each χ_γ is a character, $\tilde{\alpha}$ is a homomorphism. We mention some relevant properties of α for weight one, whose proofs work for higher weights also. The corresponding properties for weight greater than one are discussed in [Ri92]. We note that though [Ri92] only consider the case weight $k = 2$, the same proof works for all weights greater than or equal to two.

Lemma 5.2.1. *The homomorphism $\tilde{\alpha}$ satisfies:*

(i) $\tilde{\alpha}$ is unramified at all primes $p \nmid N$.

(ii) For all $g \in G_\mathbb{Q}$, we have $\alpha^2(g) \equiv \epsilon(g)$ mod F^*.

(iii) If $a_p \neq 0$, then $\alpha(\mathrm{Frob}_p) \equiv a_p$ mod F^*.

Proof. Noting $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$, for each $\gamma \in \Gamma$, we have:

(i) Since both ρ_{f^γ} and ρ_f are unramified at $p \nmid N$, we have $\chi_\gamma(g) = 1$ for all $g \in I_p$. The relation (5.2.1) shows that $\alpha(g) \in F^*$, for all $g \in I_p$, and thus $\tilde{\alpha}$ is unramified at $p \nmid N$.

(ii) Taking the determinant, we have $\chi_\gamma^2 = \epsilon^{\gamma-1}$. Again by (5.2.1) we conclude that $\alpha^2 \equiv \epsilon \bmod F^*$.

(iii) If the trace of $\rho_f(g)$ for $g \in G_{\mathbb{Q}}$ is non-zero, then a standard argument shows that

$$\alpha(g) \equiv \mathrm{trace}(\rho_f(g)) \bmod F^*.$$

In particular, if $p \nmid N$ and $a_p \neq 0$, then $\alpha(\mathrm{Frob}_p) \equiv a_p \bmod F^*$.

$\square$

5.2.2 The 2-cocycle underlying X

By [Ri81, Proposition 1, p. 9], whose proof also holds for all weights (including weight one), the class of X in $\mathrm{Br}(F) = \mathrm{H}^2(\mathrm{Gal}(\bar{\mathbb{Q}}/F), \bar{\mathbb{Q}}^*)$ is given by the 2-cocycle

$$(g, h) \mapsto \chi_g(h)$$

for g, $h \in \mathrm{Gal}(\bar{\mathbb{Q}}/F)$, where $\chi_g := \chi_\gamma$ for γ the image of g in Γ. By (5.2.1), this 2-cocycle is the same as the 2-cocycle given by

$$(g, h) \mapsto \frac{\alpha(h)^g}{\alpha(h)}$$

which in turn differs from the 2-cocycle

$$c(g, h) = \frac{\alpha(g) \cdot \alpha(h)}{\alpha(gh)} \tag{5.2.2}$$

by a coboundary. Hence, the class of X is given by the 2-cocycle $c(g, h)$ above.

Observe that the class of c is independent of the lift α of $\tilde{\alpha}$. Suppose α' is another lift of $\tilde{\alpha}$. Then $\alpha'(g) = \alpha(g) \cdot f(g)$, for some map $f : G_F \to F^*$. Let us denote the 2-cocycle obtained from α' by c'. Then c and c' differ by the map $(g, h) \mapsto \frac{f(g)f(h)}{f(gh)}$, which is clearly a 2-coboundary, as desired.

We also note that the class of c (hence X) is 2-torsion in the Brauer group of F. This follows immediately from part (2) of Lemma 5.2.1, noting that $c^2(g, h) = d(g)d(h)/d(gh)$ is a 2-coboundary, since $d(g) := \alpha^2(g)/\epsilon(g) \in F^*$.

5.2.3 Invariant map

To study the Brauer class of X, it suffices to study the Brauer class of $X_v := X \otimes_F F_v$ in $\mathrm{Br}(F_v)$, for each place v of F. It is well known that if v is finite then

$$\mathrm{inv}_v : \mathrm{Br}(F_v) \simeq \mathbb{Q}/\mathbb{Z}$$

via the invariant map inv_v at v. Since the class of X lies in 2-torsion in the Brauer group of F, we have that $\mathrm{inv}_v(X_v) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Identifying this group with $\mathbb{Z}/2$, we see that X_v is a matrix algebra over F_v if $\mathrm{inv}_v(X_v) = 0 \bmod 2$, and is a matrix algebra over a quaternion division algebra over F_v if $\mathrm{inv}_v(X_v) = 1 \bmod 2$.

To aid in the computation of $\mathrm{inv}_v(X_v)$, for finite places v, it is useful to recall the explicit definition of the invariant map, which we do now. Let I_v be the inertia subgroup of G_F at the prime v. Let $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$ be the Galois group of F_v^{nr}, the maximal unramified extension of F_v, over F_v. The inflation map

$$\mathrm{Inf} : \mathrm{H}^2(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), F_v^{\mathrm{nr}}) \to \mathrm{Br}(F_v)$$

is well-known to be an isomorphism. Now, the surjective valuation $v : F^* \to \mathbb{Z}$ can be extended uniquely to $(F_v^{\mathrm{nr}})^*$ which we continue to call v. This gives rise to a map

$$v : \mathrm{H}^2(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), F_v^{\mathrm{nr}}) \to \mathrm{H}^2(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathbb{Z})$$

which we again denote by v. Also, the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives rise to a long exact sequence of cohomology groups, with boundary map

$$\delta : \mathrm{H}^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathbb{Z})$$

which is an isomorphism since $\mathrm{H}^i(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathbb{Q}) = 0$ for $i = 1, 2$. We recall the definition of δ. If $\chi : \mathrm{Gal}(F_v^{\mathrm{nr}}/F_v) \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism, and $\tilde{\chi}$ is a lift of χ to $\mathbb{Q}$, then $\delta(\chi)$ is the $\mathbb{Z}$-valued 2-cocycle on $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$ given by

$$(g, h) \mapsto \frac{\tilde{\chi}(g)\tilde{\chi}(h)}{\tilde{\chi}(gh)}.$$

Finally, there is a map, say Ev (for evaluation)

$$\mathrm{Ev} : \mathrm{H}^1(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

obtained by evaluating a homomorphism at the arithmetic Frobenius at v. Then, by definition, the invariant map at v is given by

$$\mathrm{inv}_v = \mathrm{Ev} \circ \delta^{-1} \circ v \cdot \mathrm{Inf}^{-1} : \mathrm{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}.$$

5.2.4 Local 2-cocycle

Now let $K : G_v \to \bar{F}_v^*$ be any map. Then

$$c_K(g,h) = \frac{K(g)K(h)}{K(gh)}$$

defines a local 2-cocycle on G_v, if $c_K(g,h) \in F_v$, for all g, $h \in G_v$. We call it the local 2-cocycle defined by the function K. The following general lemma regarding the Brauer class of this local 2-cocycle will be very useful in computations.

Lemma 5.2.2. *Let $K : G_v \to \bar{F}_v^*$ be a map and let $t : G_v \to \bar{F}_v^*$ be an unramified homomorphism such that*

(i) $K(i) \in F_v^*$, for all $i \in I_v$,

(ii) $K(g)^2/t(g) \in F_v^*$, for all $g \in G_v$.

Then, for any arithmetic Frobenius Frob_v *at v, we have*

$$\mathrm{inv}_v(c_K) = \frac{1}{2} \cdot v\left(\frac{K(\mathrm{Frob}_v)^2}{t(\mathrm{Frob}_v)}\right) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z},$$

where $v : F_v^* \to \mathbb{Z}$ is the surjective valuation.

Proof. We will calculate $\mathrm{inv}_v(c_K)$, step by step, using the definition of inv_v just recalled.

Replacing the induced homomorphism $K : G_v \to \bar{F}_v^*/F_v^*$ with another lift $K : G_v \to \bar{F}_v^*$, which we again call K, does not change the cohomology class of c_K. By property (i) we may choose a lift K such that for $g \in G_v$, $K(gi) = K(g)$, for all $i \in I_v$. Denote the image of g under the projection map $G_v \to G_v/I_v = \hat{\mathbb{Z}}$ by $\bar{g}$. Define $c_{\bar{K}} : \hat{\mathbb{Z}} \times \hat{\mathbb{Z}} \to F_v^*$ by $c_{\bar{K}}(\bar{g},\bar{h}) = c_K(g,h)$. Then $c_{\bar{K}}$ is clearly a well-defined 2-cocycle on $\hat{\mathbb{Z}}$ whose image under the inflation map is c_K.

Now, by definition, $v(c_{\bar{K}})$ is the 2-cocycle defined by

$$(g,h) \mapsto v\left(\frac{K(g)K(h)}{K(gh)}\right) \in \mathbb{Z},$$

for g, $h \in G_v$.

By property (ii), $d(g) = K^2(g)/t(g) \in F_v^*$, for $g \in G_v$. The 2-cocycle above is the same as the 2-cocycle induced by

$$(g,h) \mapsto \frac{1}{2} \cdot v\left(\frac{d(g)d(h)}{d(gh)}\right) \in \mathbb{Z}.$$

Consider now the map $\chi : \mathrm{Gal}(F_v^{\mathrm{nr}}/F_v) \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\chi(g) = \frac{1}{2} \cdot v(d(g)) \bmod \mathbb{Z}.$$

Under the boundary map δ the 1-cocycle χ maps to the 2-cocycle above, so $(\delta^{-1} \circ v \circ \mathrm{Inf}^{-1})(c_K))$ is just χ. Hence

$$\mathrm{inv}_v(c_K) = (\mathrm{Ev} \circ \delta^{-1} \circ v \circ \mathrm{Inf}^{-1})(c_K) = \chi(\mathrm{Frob}_v) = \frac{1}{2} \cdot v\left(\frac{K(\mathrm{Frob}_v)^2}{t(\mathrm{Frob}_v)}\right) \bmod \mathbb{Z}.$$

$\square$

Chapter 6

Automorphic forms and Langlands principle of functoriality

In this chapter, we will only mention some basic results about automorphic forms and Langlands principle of functoriality, which appear in the main results proved in Chapter 7. We will try to avoid technicalities as much as possible. Instead, we will give suitable references.

6.1 Automorphic forms

In this section, we will closely follow [Ge75].

Notations

- $\mathbb{A}_k$= Adèle ring of any number field k.
- $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$= Adèle ring of $\mathbb{Q}$.
- $\mathbf{G}(S) = \mathrm{GL}_2(S)$, for any ring S.
- $K_p = \mathbf{G}(\mathbb{Z}_p)$.
- $\mathbf{G}(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$=restricted direct product of $\mathbf{G}(\mathbb{Q}_p)$ with respect to $\mathbf{G}(\mathbb{Z}_p)$.
- $Z_{\mathbb{A}} = \{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{A}_{\mathbb{Q}}^*\}$.

- $\mathrm{GL}_2^+(\mathbb{R})$ = the set of 2×2 matrices over $\mathbb{R}$ with determinant greater than zero.

- $K_0' = \prod_{p<\infty} K_p'$ and K_p' any choice of open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ with $K_p' = \mathrm{GL}_2(\mathbb{Z}_p)$ for almost all primes p.

- $O_2(\mathbb{R})$ = the set of 2×2 orthogonal matrices over $\mathbb{R}$.

By a *grossencharacter* we shall mean a unitary character of the idele class group, i.e., a character $\mathbb{A}_{\mathbb{Q}}^* \to \mathbb{C}^*$, which is trivial on the discrete subgroup $\mathbb{Q}^*$. Suppose ϵ is a character of $(\mathbb{Z}/N\mathbb{Z})^*$. It determines a character ϵ_p of $\mathbb{Z}_p^*$ for all primes $p \mid N$ by composing with the natural homomorphism of $\mathbb{Z}_p^*$ into $(\mathbb{Z}/N\mathbb{Z})^*$. The product $\prod_{p<\infty} \epsilon_p$ then determines a character of $\prod_{p<\infty} \mathbb{Z}_p^*$, which is trivial for almost all primes p. Hence it determines a character of $\mathbb{A}_{\mathbb{Q}}^*$.

Definition 6.1.1. An *automorphic* ψ cusp form on GL_2 is any function ϕ on $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ satisfying the following conditions:

- $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathbf{G}(\mathbb{Q})$,

- $\phi(gz) = \phi(zg) = \psi(z)\phi(g)$ for all $z \in Z_{\mathbb{A}}$,

- ϕ is right $K = O_2(\mathbb{R}) \prod_{p<\infty} K_p$ finite,

- Let $\mathfrak{z}$-denotes the centre of the universal enveloping algebra of G_∞. As a function of $G_\infty = \mathrm{GL}_2(\mathbb{R})$ alone, ϕ is smooth and $\mathfrak{z}$-finite. We refer to [Bu97, section 2.2, p. 145] for the definition of the universal enveloping algebra of G_∞ and p. 279 of the same book for the definition of $\mathfrak{z}$-finiteness.

- ϕ is slowly increasing, i.e., for every $c > 0$ and compact subset K of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$, there exist constants C and N such that
$$\phi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \leq C|a|^N,$$
for all $g \in K$ and $a \in \mathbb{A}_{\mathbb{Q}}^*$ with $|a| > c$.

- ϕ is cuspidal, i.e.,
$$\int_{\mathbb{Q}^* \backslash \mathbb{A}^*} \phi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) da = 0,$$
for almost every $g \in \mathbf{G}(\mathbb{A})$.

An automorphic form on GL_2 is an automorphic ψ cusp form for some *grossencharacter* ψ.

An automorphic form on a general reductive algebraic group can be defined in similar way. We choose not to define them in that generality in the present thesis, rather we refer to [BJ79] for details.

6.2 Modular forms and automorphic representations

We recall how to associate an automorphic form π_f on GL_2 corresponding to the modular form f. The principle of strong approximation for SL_2 over $\mathbb{Q}$ shows that

$$\mathbf{G}(\mathbb{A}_{\mathbb{Q}}) = \mathbf{G}(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0'.$$

In particular, the above principle is true with the choice of the subgroup

$$K_p' = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p : c \equiv 0 \pmod{N}\}.$$

Let $f \in S_k(N, \epsilon)$ be a newform. Now the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \epsilon_p(a)$$

defines a character on K_p'. We define a function ϕ_f on $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ by

$$\phi_f(g) = f(g_\infty(i))j(g_\infty, i)^{-k}\epsilon(k_0)$$

by decomposing g as $\gamma g_\infty k_0$ and $\epsilon = \prod_p \epsilon_p$.

We note that ϕ_f is well defined, since

$$\mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_2^+(\mathbb{R}) \prod_p K_p' = \Gamma_0(N)$$

and $f \in S_k(N, \epsilon)$. Hence for all $\gamma \in \mathbf{G}(\mathbb{Q})$, we have $\phi_f(\gamma g) = \phi_f(g)$.

Proposition 6.2.1. *The map f goes to ϕ_f describes an isomorphism between the spaces $S_k(N, \epsilon)$ and the set of automorphic ϵ cusp forms ϕ on $G_{\mathbb{A}} = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, as defined in 6.1.1 and satisfying the following properties:*

- $\phi(g.k_0) = \phi(g)\epsilon(k_0)$ *for all* $k_0 \in \prod K'_p$.
- $\phi(gr(\theta)) = e^{-ki\theta}\phi(g)$ *if* $r(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.
- *The function* ϕ*, viewed as a function on* $\mathrm{GL}_2^+(\mathbb{R})$*, satisfies the differential equation*

$$\Delta\phi = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\phi.$$

Proof. [Ge75, Proposition 3.1, p. 42]. □

6.3 Admissible homomorphisms

Let k be a non-archimedean local field and let W_k be the Weil group of k. For $w \in \mathrm{W}_k$, let $||w||$ denote the power of q (the number of elements of the residue field) to which w raises elements of the residue field.

Definition 6.3.1 ([Ta79])**.** The Weil-Deligne group W'_k is the group scheme over $\mathbb{Q}$, which is the semi-direct product of W_k by $\mathbb{G}_a$, on which W_k acts by the rule $wxw^{-1} = ||w||x$.

Let G be an arbitrary reductive algebraic group over a field k.

Definition 6.3.2. The (Langlands) dual group or L-group of G is ${}^L(\mathrm{G}/k) = {}^L\mathrm{G}^0 \rtimes \mathrm{Gal}(\bar{k}/k)$.

We refer to [Bo79, p. 29] for the precise definition of ${}^L\mathrm{G}^0$. Instead, we list ${}^L\mathrm{G}^0$ for some reductive algebraic groups G in the following table:

G	${}^L\mathrm{G}^0$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C})$
PGL_2	$\mathrm{SL}_2(\mathbb{C})$
adjoint type	simply connected
simply connected	adjoint type

Definition 6.3.3 ([Bo79])**.** Let k be a local field and let W'_k be the associated Weil-Deligne group, a homomorphism $\phi : \mathrm{W}'_k \to {}^L\mathrm{G}$ is called admissible if

- ϕ is a homomorphism over $\mathrm{Gal}(\bar{k}/k)$, i.e., the following diagram commutes

$$\begin{array}{ccc} \mathrm{W}'_k & \longrightarrow & {}^L\mathrm{G} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\bar{k}/k) & = & \mathrm{Gal}(\bar{k}/k) \end{array} .$$

- ϕ is continuous, $\phi(\mathbb{G}_a)$ is unipotent in ${}^L\mathrm{G}^0$ and ϕ maps semisimple elements to semisimple elements in ${}^L\mathrm{G}$. We refer to [Bo79, section 8.1, p. 39], for the definition of semisimple elements of W'_k. In ${}^L\mathrm{G}$, an element is said to be semisimple if its image under any representation $r : {}^L\mathrm{G} \to \mathrm{GL}_n(\mathbb{C})$ is semisimple.

- If $\phi(\mathrm{W}'_k)$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^L\mathrm{G}$ then P is relevant. We refer to [Bo79, section 3.3, p. 32] for the definition of parabolic subgroups, Levi subgroups and "relevant".

Let $\Phi(\mathrm{G})$ denote the category of admissible homomorphisms $\phi : \mathrm{W}'_k \to {}^L\mathrm{G}$, modulo the inner automorphisms by elements of ${}^L\mathrm{G}^0$. We again refer to [Bo79, section 8.1, p. 39] for the precise definition.

6.4 Admissible representations

A locally profinite group is a topological group G such that every open neighbourhood of the identity in G contains a compact, open subgroup of G. Let k be a local field, then $\mathrm{GL}_n(k)$ is a locally profinite group. Let (π, V) be a representation of a locally profinite group G, i.e., V is a complex vector space (possibly of infinite dimension) and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism.

Definition 6.4.1 ([BH06])**.** The representation (π, V) is called *smooth*, if for every $v \in V$, there exists a compact, open subgroup K_v, depending on v, such that $\pi(g)v = v$ for all $g \in K_v$. For any compact open subgroup K of G, let V^K denotes the space of $\pi(K)$ fixed vectors of V. A representation is smooth if and only if

$$V = \cup_K V^K,$$

where K ranges over all compact, open subgroups of G.

Definition 6.4.2 ([BH06]). A smooth representation (π, V) is called *admissible*, if the space V^K is *finite* dimensional, for each compact, open subgroup K of G.

Let $\mathfrak{A}(\mathrm{G}) = \mathfrak{A}(\mathrm{G}(k))$ denote the equivalence class of irreducible, admissible complex representations of $\mathrm{G}(k)$.

6.5 Local Langlands conjecture

Let k be a local field. According to the local Langlands conjecture [Co01], there is a surjective map from $\mathfrak{A}(\mathrm{G})$ to $\Phi(\mathrm{G})$, with finite fibres which partitions the category $\mathfrak{A}(\mathrm{G})$ into disjoint union of finite sets $\mathfrak{A}_\phi = \mathfrak{A}_\phi(\mathrm{G})$, for $\phi \in \Phi(\mathrm{G})$. The sets $\mathfrak{A}_\phi$ for $\phi \in \Phi(\mathrm{G})$, are called L-packets.

In the case of $\mathrm{G} = \mathrm{GL}_n$, this map is actually known to be bijective [HT01].

6.6 Functoriality

In this section, we recall the definition of functoriality [Co01]. Let k denote either a local or global field and let H and G be two connected, reductive algebraic groups defined over k. A homomorphism $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$ is called an L-homomorphism if

- It is a homomorphism over $\mathrm{Gal}(\bar{k}/k)$, i.e., the following diagram commutes

$$\begin{array}{ccc} {}^L\mathrm{H} & \longrightarrow & {}^L\mathrm{G} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\bar{k}/k) & = & \mathrm{Gal}(\bar{k}/k) \end{array}.$$

- u is continuous.
- The restriction of u to ${}^L\mathrm{H}^0$ is a complex analytic homomorphism $u : {}^L\mathrm{H}^0 \to {}^L\mathrm{G}^0$.

6.6.1 Local functoriality

Let k be a local field and $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$ be a local L-homomorphism. If we take $\pi \in \mathfrak{A}(\mathrm{H})$ an irreducible representations of $\mathrm{H}(k)$, then this is parametrised (conjecturally) by an admissible homomorphism $\phi = \phi_\pi : \mathrm{W}'_k \to {}^L\mathrm{H}$. We compose ϕ with u and obtain $\phi' = \phi \circ u$, an admissible homomorphism of W'_k to ${}^L\mathrm{G}$. Then ϕ' parametrises a local L-packet $\mathfrak{A}_{\phi'}(\mathrm{G})$ and this L-packet is the functorial lift of π.

6.6.2 Global functoriality

Suppose now that k is a global field. Let H and G be two connected reductive k-groups and $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$ be an L-homomorphism. For each place v of k, we have associated local L-homomorphism $u_v : {}^L\mathrm{H}_v \to {}^L\mathrm{G}_v$. Let $\pi \in \mathfrak{A}(\mathrm{H})$, $\pi = \otimes' \pi_v$ be an irreducible automorphic representation of $\mathrm{H}(\mathbb{A}_k)$. We have a local parameter $\phi_v : \mathrm{W}'_{k_v} \to {}^L\mathrm{H}_v$ for π_v. We can form a local lift Π_v, as a representation of $\mathrm{G}(k_v)$ associated to the parameter $\phi'_v = u_v \circ \phi_v$. We call an automorphic representation $\Pi = \otimes' \Pi_v$ of $\mathrm{G}(\mathbb{A}_k)$ to be a weak functorial lift of π with respect to u if for all archimedean places and almost all finite places v where π_v is unramified, there exist a local functorial lift Π_v of π_v with respect to u_v. We call Π to be a *strong* functorial lift of π if Π_v is a local functorial lift of π_v for all places v of k.

6.7 Adjoint lift

Now suppose that $\mathrm{H} = \mathrm{GL}_2$ and $\mathrm{G} = \mathrm{GL}_3$ are defined over $\mathbb{Q}$. By definition, the connected parts of the corresponding L-groups are ${}^L\mathrm{H}^0 = \mathrm{GL}_2(\mathbb{C})$ and ${}^L\mathrm{G}^0 = \mathrm{GL}_3(\mathbb{C})$. The natural adjoint action of $\mathrm{GL}_2(\mathbb{C})$ on the three dimensional vector space consisting of the trace zero matrices of $\mathrm{M}_{2\times 2}(\mathbb{C})$, induces L-homomorphisms u and u_p, for each prime p. On diagonal elements (of the first factor) the map u_p is easily checked to be the map

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha}{\beta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\beta}{\alpha} \end{pmatrix}.$$

By a classical theorem of Gelbart and Jacquet [GJ78, Theorem 9.3, p. 534], every automorphic representation of H has a *strong* lift to G. If $\pi = \pi_f$ is the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponding to $f \in S_k(N,\epsilon)$, let $\mathrm{ad}(\pi)$ denote the automorphic lift to $\mathrm{G}(\mathbb{A}_\mathbb{Q})$. The image of arithmetic Frobenius Frob_p at p under ϕ_p is of the form

$$\left(\begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}, \mathrm{Frob}_p\right).$$

If $p \nmid N$ is an unramified prime, α_p and β_p are the Satake parameters of π_p. Then by definition of u_p it is clear that the image of Frob_p under ϕ'_p is a diagonal matrix with entries $\frac{\alpha_p}{\beta_p}, 1, \frac{\beta_p}{\alpha_p}$ (on the first factor, and just Frob_p on the second factor). It is more convenient to work with $\Pi = (\mathrm{ad}(\pi) \oplus 1)(k-1)$, the $(k-1)$-th twist of the automorphic representation on $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ obtained by adding the trivial representation to $\mathrm{ad}(\pi)$.

6.7.1 The slope m_v

We define the *slope* m_v of Π at $v \mid p$ to be

$$m_v := [F_v \cdot \mathbb{Q}_p] \cdot v(t_p),$$

where v is normalized so that $v(p) = 1$ and $t_p \in F$ is defined to be the sum of the four parameters of Π_p, namely

$$t_p = \left(\frac{\alpha_p}{\beta_p} + 1 + \frac{\beta_p}{\alpha_p} + 1\right) \cdot p^{k-1} = \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} \cdot p^{k-1}.$$

We note that t_p can be computed easily in various cases. When $p \nmid N$ an easy check shows

$$t_p = \frac{a_p^2}{\epsilon(p)}.$$

When $p \mid N$ and $N_p = 1$ and $C_p = 0$, it is known that $\alpha_p = a_p$ and $\beta_p = pa_p$ (up to multiplication by a constant), and so

$$t_p = p^{k-2}(p+1)^2.$$

Finally, if $N_p = C_p$, then a natural choice is $\alpha_p = a_p$ and $\beta_p = \bar{a}_p \epsilon'(p)$ (again up to multiplication by a constant), so

$$t_p = \frac{{a_p}^2}{\epsilon'(p)} + 2p^{k-1} + \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)},$$

noting $|a_p|^2 = p^{k-1}$. In fact the Weil Deligne parameter in this case is ramified at p, so there are other choices for α_p and β_p and hence for t_p. This causes some complications in the statements and the proofs of results in this case.

6.8 Galois representations

All the above formulas can be computed on the Galois side as well. We recall the precise form of the Galois representation ρ_f attached to f as described in Theorem 3.2.3.

Let $\pi = \pi_f$ be the automorphic representation corresponding to f. Then ρ_π, the Galois representation attached to π, differs a bit from ρ_f (e.g., the Satake parameters differ from the roots of the polynomial $x^2 - a_p x + \epsilon(p)p^{k-1}$ by a factor of $p^{(k-1)/2}$, and similarly the L-functions satisfy $L(s, f) = L(s - \frac{k-1}{2}, \pi, 1))$. However the resulting adjoint Galois representation obtained by making $G_\mathbb{Q}$ act by conjugation on $\mathrm{M}_{2\times 2}(E_\lambda)$ is the same, and we let

$$\rho_{\mathrm{Ad}(\pi)} : G_\mathbb{Q} \to \mathrm{GL}_4(E_\lambda)$$

be defined by $\rho_{\mathrm{Ad}(\pi)}(g)(X) = \rho_\pi(g)X\rho_\pi(g)^{-1}$, for all $X \in \mathrm{M}_{2\times 2}(E_\lambda)$ and $g \in G_\mathbb{Q}$. Finally, let

$$\rho_\Pi = \rho_{\mathrm{Ad}(\pi)} \otimes \chi_\ell^{k-1}$$

be the representation obtained by taking the $(k-1)$-fold twist of the adjoint representation by the ℓ-adic cyclotomic character.

Corollary 6.8.1. *Let $p \neq \ell$ be a prime. We have*

- *If $p \nmid N$, then* $\mathrm{trace}(\rho_\Pi(\mathrm{Frob}_p)) = a_p^2/\epsilon(p)$.
- *If $N_p = 1$ and $C_p = 0$, then* $\mathrm{trace}(\rho_\Pi(\mathrm{Frob}_p)) = p^{k-2}(p+1)^2$.
- *If $N_p = C_p \geq 1$, then in many cases there exists an arithmetic Frobenius* Frob_p *such that* $\mathrm{trace}(\rho_\Pi(\mathrm{Frob}_p)) = a_p^2/\epsilon'(p) + 2p^{k-1} + \bar{a}_p^2/\bar{\epsilon}'(p)$.

Proof. If

$$\rho_\pi(\mathrm{Frob}_p) \sim \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix},$$

then

$$\rho_{\mathrm{Ad}(\pi)}(\mathrm{Frob}_p) \sim \begin{pmatrix} \frac{\alpha_p}{\beta_p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\beta_p}{\alpha_p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\chi_\ell(\mathrm{Frob}_p)^{k-1} = p^{k-1}$. Taking the trace of $\rho_\Pi(\mathrm{Frob}_p)$ gives the corollary. □

Chapter 7

Adjoint lifts and modular endomorphism algebras

The aim of this chapter is to show the locally at the prime v, the ramification of the endomorphism algebra of the motive M_f, attached to a modular form f of weight greater or equal to 2, is controlled by the *slope* m_v of Π at the the prime v, if the slope is finite. In other words, we will prove Theorem 2.0.5 in this chapter.

7.1 Good primes

Theorem 7.1.1. *Assume* $\gcd(p, N) = 1$ *and assume* $a_p \neq 0$. *Let* v *be a place of* F *lying over* p. *Then* X_v *is a matrix algebra over* F_v, *if and only if the slope*

$$m_v = [F_v : \mathbb{Q}_p] \cdot v(a_p^2/\epsilon(p)) \in \mathbb{Z}$$

is even, where v *is normalized such that* $v(p) = 1$.

Proof. This follows from the Lemma 5.2.2 by taking $K = \alpha$ and $t = \epsilon$. Indeed, we have $\mathrm{inv}_v(c_\alpha) = \frac{1}{2}v(\alpha^2(\mathrm{Frob}_v)/\epsilon(\mathrm{Frob}_v)) \bmod \mathbb{Z}$, and it is known from Lemma 5.2.1 that $\alpha(\mathrm{Frob}_p) \equiv a_p \bmod F^*$. $\square$

For the cases where $a_p = 0$ we have the following criterion (which is not in terms of a slope). Let $p^\dagger \nmid N$ be a prime such that $p^\dagger \equiv p \bmod N$ and $a_{p^\dagger} \neq 0$. Let

$$m_v^\dagger := [F_v : \mathbb{Q}_{p^\dagger}] \cdot v(a_{p^\dagger}^2/\epsilon(p^\dagger)) \in \mathbb{Z},$$

where v is normalized so that $v(p) = 1$.

Theorem 7.1.2. *Let* $\gcd(p, N) = 1$ *and suppose* $a_p = 0$*. Let* v *be a place of* F *lying over* p*. Then* X_v *is a matrix algebra over* F_v *if and only if* $m_v^\dagger \in \mathbb{Z}$ *is even.*

Proof. The proof is similar to that of the previous theorem, with minor changes. Note that $p^\dagger \equiv p \bmod N$ implies $\chi_\gamma(p) = \chi_\gamma(p^\dagger)$, for all $\gamma \in \Gamma$. So, if Frob_p and $\mathrm{Frob}_{p^\dagger}$ denote the Frobenii at the prime p and $p^\dagger$, then by (5.2.1), we have $\alpha(\mathrm{Frob}_p) \equiv \alpha(\mathrm{Frob}_{p^\dagger}) \equiv a_{p^\dagger} \bmod F^*$. Hence

$$\mathrm{inv}_v(c_\alpha) = \frac{1}{2} v\left(\frac{\alpha^2(\mathrm{Frob}_v)}{\epsilon(\mathrm{Frob}_v)}\right) = \frac{1}{2} \cdot f_v \cdot v\left(\frac{\alpha^2(\mathrm{Frob}_p)}{\epsilon(p)}\right) = \frac{1}{2} \cdot f_v \cdot v\left(\frac{a_{p^\dagger}^2}{\epsilon(p^\dagger)}\right)$$

mod $\mathbb{Z}$. □

7.2 Steinberg primes

Let now turn to the cases where $p \mid N$. In this section we assume that $N_p = 1$ and $C_p = 0$. Thus $N = Mp$, where M is a positive integer with $(M, p) = 1$, and ϵ is a character mod M.

Lemma 7.2.1. *If* (γ, χ_γ) *is an extra twist for* f*, then the conductor of* χ_γ *divides* M*.*

Proof. A general result due to Atkin-Li [ALi78, Thm. 3.1] allows one to calculate the exact level of the newform attached to a twisted form $f \otimes \chi$. We recall this now. Let $f \in S_k(N, \epsilon)$ be a newform of weight $k \geq 2$, and nenbentypus ϵ. In the notation of *loc. cit.*, let $q \mid N$ be a prime and let Q be the q-primary factor of N. So $N = QM$, with $(M, q) = 1$. Let the conductor of ϵ_Q, the q-part of ϵ, be q^α, for $\alpha \geq 0$. Let χ be a character of conductor q^β, with $\beta \geq 1$. Set

$$Q' = \mathrm{Max}\{Q, q^{\alpha+\beta}, q^{2\beta}\}.$$

According to the theorem, the level of the newform attached to $f \otimes \chi_\gamma$ is $Q'M$, provided that

- $\max\ \{q^{\alpha+\beta}, q^{2\beta}\} \leq Q$, if $Q' = Q$, or
- Conductor of $\epsilon_Q \chi =\ \max\ \{q^\alpha, q^\beta\}$, if $Q' > Q$.

In our case, taking $Q = q = p$, we have $\epsilon_Q = \epsilon_p = 1$. We let χ be the p-part of χ_γ. Suppose towards a contradiction that χ_γ has level divisible by p. Then $\alpha = 0$ and $\beta = 1$. Then $Q' = p^2 > Q = p$ and the Q-part of the conductor of $\epsilon_Q \chi_\gamma = \chi_\gamma$ is p. So the second condition above is satisfied and we get the p-part of the level of the newform attached to $f \otimes \chi_\gamma$ is p^2. On the other hand, $f \otimes \chi_\gamma = f^\gamma$ has the same level as f namely Mp, which is not divisible by p^2, a contradiction. Thus the p-part of the conductor of χ_γ must be trivial, as desired. $\square$

Recall that $a_\ell^\gamma = a_\ell \chi_\gamma(\ell)$ for all $\ell \nmid N$. We show that this also holds for $p||N$. We have:

Lemma 7.2.2. *$a_p^\gamma = \chi_\gamma(p) \cdot a_p$, for all $\gamma \in \Gamma$.*

Proof. We use the precise form of the local Galois representation at p from Langlands Theorem ([Hid00, Theorem 3.26, p. 109]). According to the theorem

$$\rho_f|_{G_p} \sim \begin{pmatrix} \lambda(pa_p) & * \\ 0 & \lambda(a_p) \end{pmatrix}.$$

where $\lambda(x) : G_p \longrightarrow \mathbb{Z}_l^*$ is the unramified character taking arithmetic Frobenius to x. Note that both characters make sense since both pa_p and a_p are ℓ-adic units. By the previous lemma, the conductor of χ_γ for $\gamma \in \Gamma$, is prime to p and so $\chi_\gamma(p)$ makes sense, and is an ℓ-adic unit, and locally we have $\chi_\gamma|_{G_p} = \lambda(\chi_\gamma(p))$. Applying Langlands' theorem for f^γ, we get

$$\rho_{f^\gamma}|_{G_p} \sim \begin{pmatrix} \lambda(pa_p^\gamma) & * \\ 0 & \lambda(a_p^\gamma) \end{pmatrix}.$$

Since $f^\gamma = f \otimes \chi_\gamma$, implies $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$, we have locally that

$$\begin{pmatrix} \lambda(pa_p^\gamma) & * \\ 0 & \lambda(a_p^\gamma) \end{pmatrix} \sim \begin{pmatrix} \lambda(pa_p)\lambda(\chi_\gamma(p)) & * \\ 0 & \lambda(a_p)\lambda(\chi_\gamma(p)) \end{pmatrix}.$$

An important part of Langlands' theorem (not mentioned explicitly above) is that $* \neq 0$, since the inertia groups I_p acts unipotently with infinite image. Thus comparing like diagonal entries, we see that $a_p^\gamma = \chi_\gamma(p) \cdot a_p$. $\square$

It is a result of Ribet that the map $\tilde{\alpha}$ is unramified and primes of semi-stable reduction (for $k = 2$). At primes of good reduction it is known that $\alpha(\mathrm{Frob}_p) = a_p \mod F^*$ (*cf.* 5.2.1). We show that this continues to hold for primes of semi-stable reduction.

Proposition 7.2.3. *Suppose p is a prime such that $N_p = 1$ and $C_p = 0$. Then $\alpha(\mathrm{Frob}_p) \equiv a_p$ mod F^*.*

Proof. Since for $\gamma \in \Gamma$, the conductor of χ_γ is prime to p, we have $\chi_\gamma(i) = 1$, for $i \in I_p$. By (5.2.1), we deduce $\alpha(i) \in F^*$, for all $i \in I_p$. Thus we recover the fact that $\tilde{\alpha}$ is unramified at the Steinberg primes for any $k \geq 2$. In any case, it makes sense to speak of $\alpha(\mathrm{Frob}_p)$ mod F^*.

By Lemma 7.2.2, we have $a_p^{\gamma-1} = \chi_\gamma(p)$, for $\gamma \in \Gamma$. By (5.2.1), $\alpha(\mathrm{Frob}_p)^{\gamma-1} = \chi_\gamma(p)$. Since these identities hold for all $\gamma \in \Gamma$, we deduce that $\alpha(\mathrm{Frob}_p) \equiv a_p$ mod F^*. $\square$

Theorem 7.2.4. *Let $N_p = 1$ and $C_p = 0$ and let $v \mid p$ be a prime of F. Then X_v is a matrix algebra if and only if $[F_v : \mathbb{Q}_p] \cdot (k-2)$ is even.*

Proof. Applying Lemma 5.2.2 with $K = \alpha$ and $t = \epsilon$, we get $\mathrm{inv}_v(c_\alpha) = \frac{1}{2} v(\frac{\alpha^2(\mathrm{Frob}_v)}{\epsilon(\mathrm{Frob}_v)})$ mod $\mathbb{Z}$. By the previous proposition, $\alpha(\mathrm{Frob}_v) \equiv a_p^{f_v}$ mod F^*. Thus $\mathrm{inv}_v(c_\alpha) = \frac{1}{2} \cdot f_v \cdot v(\frac{a_p^2}{\epsilon(p)})$. By Theorem 4.6.17 [Mi89], $\frac{a_p^2}{\epsilon_M(p)} = p^{k-2}$. Also we may replace the valuation v by $e_v \cdot v$, where the second v is normalized such that $v(p) = 1$. We obtain that $\mathrm{inv}_v(c_\alpha) = [F_v : \mathbb{Q}_p] \cdot (k-2)$ mod 2, as desired. $\square$

7.3 Ramified principal series primes

We now assume that $N_p = C_p \geq 1$. Let v be a place of F lying above p. Let e_v and f_v be the ramification degree and inertia degree of v over p. Recall that in this case π_p is in the ramified principal series.

Recall that $\epsilon = \epsilon' \cdot \epsilon_p$ is a decomposition of the nebentypus ϵ into its prime-to-p part and p part. We use repeatedly a fundamental theorem of Langlands, which states that the local Galois representation at the prime p is given by

$$\rho_f|_{G_p} \sim \begin{pmatrix} \lambda(\bar{a}_p \epsilon'(p)) \cdot \epsilon_p & 0 \\ 0 & \lambda(a_p) \end{pmatrix},$$

where $\lambda(x)$ is the usual local unramified character.

Lemma 7.3.1. *Let $\mu = \frac{a_p^2}{\epsilon'(p)}$ and $\nu = \bar{\mu} = \frac{\bar{a}_p^2}{\epsilon'(p)}$. Then $\mu^f + \nu^f \in F$, for all integers $f \geq 1$.*

Proof. Let (γ, χ_γ) be an extra twist for the form f. Thus we have $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$. Hence, by Langlands' theorem, locally on G_p we have

$$\begin{pmatrix} \lambda(\bar{a}_p^\gamma \epsilon'(p)^\gamma) \cdot \epsilon_p^\gamma & 0 \\ 0 & \lambda(a_p^\gamma) \end{pmatrix} \sim \begin{pmatrix} \lambda(\bar{a}_p \epsilon'(p)) \cdot \epsilon_p \cdot \chi_\gamma & 0 \\ 0 & \lambda(a_p) \cdot \chi_\gamma \end{pmatrix}.$$

One of the two characters on the left is unramified and the other one is ramified. Thus the same must be true on the right hand side. Moreover, the unramified characters on both sides must be equal and the ramified characters must also be equal.

We decompose χ_γ into its prime-to-p and p parts, namely $\chi_\gamma = \chi'_\gamma \cdot \chi_{\gamma,p}$. First, assume that χ_γ is unramified at p. Then, $\chi_\gamma = \chi'_\gamma = \lambda(\chi_\gamma(p))$, and comparing unramified characters, we get $a_p^\gamma = \chi_\gamma(p) a_p$. Using the fact that $\chi_\gamma^2 = \epsilon^{\gamma - 1}$, we have $\chi_\gamma^2(p) = \epsilon'(p)^{\gamma-1}$. Thus $(\mu^f)^\gamma = \mu^f$ and $(\nu^f)^\gamma = \nu^f$, since Γ is abelian, so complex conjugation commutes with γ. Hence, γ fixes $\mu^f + \nu^f$.

Now assume that χ_γ is ramified at p. Comparing ramified characters, we get, on I_p, that $\chi_{\gamma,p} = \epsilon_p^\gamma$ and $\epsilon_p \chi_{\gamma,p} = 1$. Thus $\bar{\epsilon}_p = \epsilon_p^\gamma = \chi_{\gamma,p}$. Now, comparing unramified characters, we get $a_p^\gamma = \bar{a}_p \cdot \epsilon'(p) \cdot \chi'_\gamma(p)$. Again, since $(\chi'_\gamma)^2 = (\epsilon')^{\gamma-1}$, we deduce that

$$\frac{(a_p^2)^\gamma}{\epsilon'(p)^\gamma} = \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)}.$$

In other words, $\mu^\gamma = \nu$, and hence $(\mu^f)^\gamma = \nu^f$, for all integers $f \geq 1$. Applying complex conjugation we see that similarly $(\nu^f)^\gamma = \mu^f$. Hence again γ fixes $\mu^f + \nu^f$.

In both cases $\gamma \in \Gamma = \mathrm{Gal}(E/F)$ is arbitrary, so $\mu^f + \nu^f$ must belong to F, for all integers $f \geq 1$. ☐

For later use we state the following generalization of Lemma 7.3.1 which can be proved in a similar manner, or directly by noting that $\alpha^2 \equiv \epsilon \bmod F^*$.

Lemma 7.3.2. *Let* Frob_v *be an arithmetic Frobenius at* v*, and let* $\zeta = \epsilon_p(\mathrm{Frob}_v)$*. Then* $\mu^{f_v} \cdot 1/\zeta + \nu^{f_v} \cdot \zeta \in F$*.*

We will end the section, by mentioning few lemmas which we will use later.

Lemma 7.3.3. *Let* $G_v = \mathrm{Gal}(\bar{\mathbb{Q}}_p/F_v)$ *be the decomposition group at the prime* v *and let* I_v *be the inertia group at the prime* v*. The kernel of* ϵ_p *is a totally ramified extension of* F_v*.*

Proof. Let the kernel of ϵ_p be denoted by T. For any field K, L, let us denote $f(L/K)$ to be the inertia degree of L over K. We will use the fact that, if K, L are two extensions of F, such that L/F is a Galois extension, then $\mathrm{Gal}(KL/K)$ is naturally a subgroup of $\mathrm{Gal}(L/F)$. Let us look at the map

$$\epsilon_p : G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathbb{C}^*.$$

It's kernel will be a cyclotomic extension. Let the kernel be $\mathbb{Q}_p(\zeta_{p^n})$. It is a totally ramified extension. Then, using the above mentioned facts for the residue fields, we get $f(F_v(\zeta_{p^n})/F_v) = 1$. Being a local field, $F_v(\zeta_{p^n})$ is a totally ramified extension of F_v. Since, T is a subextension of the local field $F_v(\zeta_{p^n})$, we conclude that T is a totally ramified extension. □

Lemma 7.3.4. *Let $G_v = \mathrm{Gal}(\bar{\mathbb{Q}}_p/F_v)$ be the decomposition group and I_v be the inertia group at the prime v. Then,*

$$\epsilon_p(G_v) = \epsilon_p(I_v).$$

Proof. Let T denote the extension of F_v, corresponding to the kernel of ϵ_p. Let us denote, $I(T/F_v)$ to be the inertia group of T over F_v. Since T is a totally ramified extension of F_v, $I(T/F_v) = \mathrm{Gal}(T/F_v)$. Hence we conclude that,

$$\epsilon_p(G_v) = \mathrm{Gal}(T/F_v) = I(T/F_v) = \epsilon_p(I_v).$$

□

Lemma 7.3.5. *We can choose $\sigma_v = \mathrm{Frob}_v$ such that $\epsilon_p(\sigma_v) = 1$.*

Proof. Let σ_v be any Frobenius. Let us assume $\epsilon_p(\sigma_v) = \zeta_r$ is a r-th root of unity. By previous Lemma, $\epsilon_p({\sigma_v}^{r-1}) = \epsilon_p(i)$ for some $i \in I_v$. Hence

$$\epsilon_p^r(\sigma_v) = 1 = \epsilon_p(\sigma_v) \cdot \epsilon_p(i) = \epsilon_p(\tilde{\sigma}_v).$$

But $\tilde{\sigma}_v$ is also a Frobenius. □

7.3.1 Unequal slope

In this section, we assume that

$$v\left(\frac{a_p^2}{\epsilon'(p)} + 2p^{(k-1)} + \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)}\right) < k-1.$$

Here v is a valuation such that $v(p) = 1$.

By an elementary calculation, it can be shown that the above assumption is equivalent to the assertion that for all place w of E lying over v, we have $w(a_p) \neq w(\bar{a}_p)$. Let O_v be the ring of integers of F_v. Let P_v be the prime ideal of O_v and let π_v be the prime element of O_v. Let $U_v^{(n)} = 1 + P_v^n$, for $n \geq 1$.

Lemma 7.3.6. *μ and ν belongs to F_v.*

Proof. By Lemma 7.3.1, $\mu + \nu$ belongs to F. Consider the quantity

$$\frac{(\mu - \nu)^2}{(\mu + \nu)^2} = 1 - 4\frac{\mu \cdot \nu}{(\mu + \nu)^2}.$$

Now $\mu\nu = p^{2(k-1)}$. Since the slopes of μ and ν are not the same, the expression on the right hand side belongs to $U_v^{(1)} = 1 + \pi_v O_v$, for p odd, and it belongs to $U_v^{(3e_v)} = 1 + \pi_v^{3e_v} O_v$, for $p = 2$. It therefore has a square root in $U_v^{(1)} = 1 + \pi_v O_v$, in both cases. Hence, $\frac{\mu-\nu}{\mu+\nu}$ belongs to F_v. Since we have already proved that $\mu + \nu$ belongs to F, we see $\mu - \nu$ belongs to F_v. Hence, individually, both μ and ν belong to F_v. $\square$

The case of odd primes

We now assume that p is an odd prime. We say that ϵ_p is tame if the order of ϵ_p divides $p - 1$.

Lemma 7.3.7. *If ϵ_p is tame on G_v, then for any arithmetic Frobenius* Frob_v *at v,*

$$\frac{(a_p^{f_v} + \epsilon_p(\mathrm{Frob}_v)(\bar{a}_p\epsilon'(p))^{f_v})^2}{a_p^{2f_v} + (\bar{a}_p\epsilon'(p))^{2f_v}} \in F_v^{*2}$$

is a square.

Proof. We may rewrite this expression as

$$\frac{\mu^{f_v} + \epsilon_p^2(\mathrm{Frob}_v)\nu^{f_v}}{\mu^{f_v} + \nu^{f_v}} \cdot \left(1 + 2\epsilon_p(\mathrm{Frob}_v) \cdot \frac{p^{(k-1)f_v}}{\mu^{f_v} + \epsilon_p^2(\mathrm{Frob}_v)\nu^{f_v}}\right),$$

where μ and ν are as above. By the previous lemma, μ and ν belong to F_v. Since ϵ_p is tame, the image of ϵ_p belongs to $\mathbb{Q}_p$, and hence to F_v. Thus all terms in the display above are in F_v. Now, since p is odd, and the slopes are unequal, the second term (in parentheses) is in $U_v^{(1)}$, hence a square. If $w(a_p) > w(\bar{a}_p)$, the first term is of the form $\epsilon_p^2(\mathrm{Frob}_v)$ times an element of $U_v^{(1)}$, and if $w(a_p) < w(\bar{a}_p)$, then the first term is in $U_v^{(1)}$, so in both cases, the first term is also a square. $\square$

Lemma 7.3.8. *If ϵ_p is tame on G_v and Frob_v is an arithmetic Frobenius at v, then*

$$\alpha^2(\mathrm{Frob}_v) \equiv a_p^{2f_v} + (\bar{a}_p\epsilon'(p))^{2f_v} \bmod F_v^{*2}.$$

Proof. If the trace of $\rho_f(g)$ is non-zero, for $g \in G_{\mathbb{Q}}$, then (*cf.* 5.2.1)

$$\alpha^2(g) \equiv (\mathrm{trace}\ \rho_f(g))^2 \bmod F^{*2}.$$

Since $w(a_p) \neq w(\bar{a}_p)$, the trace of $\rho_f(\mathrm{Frob}_v)$ is non-zero. Using Langlands' theorem to compute the trace we obtain

$$\alpha^2(\mathrm{Frob}_v) \equiv (a_p^{f_v} + \epsilon_p(\mathrm{Frob}_v)(\bar{a}_p\epsilon'(p))^{f_v})^2 \bmod F^{*2}.$$

The lemma now follows from the previous lemma. $\square$

Lemma 7.3.9. *If ϵ_p is tame on G_v, then $\alpha(i)$ belongs to F_v^*, for $i \in I_v$.*

Proof. If $i \in I_v$, and σ_v is an arithmetic Frobenius at v, then $\sigma_v' = \sigma_v i$ is also an arithmetic Frobenius at v. By the lemma above, $\alpha(\sigma_v) \equiv \pm\alpha(\sigma_v') \bmod F_v^*$. Since

$$c_\alpha(\sigma, i) = \frac{\alpha(\sigma_v)\alpha(i)}{\alpha(\sigma_v')} \in F^*,$$

we see that $\alpha(i)$ belongs to F_v^*. $\square$

Theorem 7.3.10. *Let p be an odd prime such that $p \mid N$ and $N_p = C_p$. Let v be a place of F lying above p. Let w be an extension of v to a place of E. If $w(a_p) \neq w(\bar{a}_p)$, then X_v is a matrix algebra if and only if*

$$m_v = [F_v : \mathbb{Q}_p] \cdot v\left(\mu + 2p^{k-1} + \nu\right) \in \mathbb{Z}$$

is even, where v is normalized so that $v(p) = 1$.

Proof. Let L be the extension of F_v cut out by the wild part of ϵ_p. So ϵ_p, thought of as a character of G_L, is tame. Note that L/F_v is a totally ramified extension of odd (p-power) degree. By Lemma 7.3.9, $\tilde{\alpha} : G_L \to \bar{F}_v^*/F_v^*$ is an unramified character. On

G_L, we have $\alpha^2 \equiv \epsilon' \bmod F_v^*$, since this is true with ϵ' replaced with ϵ, and on G_L we have $\epsilon' \equiv \epsilon \bmod F_v^*$, since $\epsilon_p(G_L) \subset \mathbb{Q}_p^* \subset F_v^*$, since $\epsilon_p|_{G_L}$ is tame. We calculate $\mathrm{inv}_L(\mathrm{res}_{F_v/L} c_\alpha)$ using Lemma 5.2.2 applied to $K = \alpha|_{G_L}$ and $t = \epsilon'|_{G_L}$. Let u be the prime of L lying over v and let Frob_u be an arithmetic Frobenius at u. We obtain

$$\mathrm{inv}_L(\mathrm{res}_{F_v/L} c_\alpha) = \frac{1}{2} \cdot u\left(\frac{\alpha^2(\mathrm{Frob}_u)}{\epsilon'(\mathrm{Frob}_u)}\right) \bmod \mathbb{Z} \in {}_2\mathrm{Br}(L).$$

Since f_v is also the residue degree of $u \mid p$, by Lemma 7.3.8 we obtain

$$\alpha^2(\mathrm{Frob}_u) \equiv a_p^{2f_v} + (\bar{a}_p \epsilon'(p))^{2f_v} \bmod F_v^{*2}.$$

Hence

$$\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_u) \equiv \mu^{f_v} + \nu^{f_v} \bmod F_v^{*2}.$$

Now $[L : F_v] \cdot \mathrm{inv}_v c_\alpha = \mathrm{inv}_L(\mathrm{res}_{F_v/L} c_\alpha)$, and for $x \in F_v$, $u(x) = [L : F_v] \cdot v(x)$, where both u and v are the surjective valuations onto $\mathbb{Z}$. But $[L : F_v]$ is a power of p, so is odd, and so in both cases can be ignored. We obtain

$$\mathrm{inv}_v c_\alpha = \frac{1}{2} \cdot v\left(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_u)\right) = \frac{1}{2} \cdot v(\mu^{f_v} + \nu^{f_v}) = \frac{1}{2} \cdot v(\mu^{f_v} + \nu^{f_v} + 2p^{(k-1)f_v}) \bmod \mathbb{Z}.$$

Since the last three terms lie in F and have distinct valuations, replacing v with the valuation v satisfying $v(p) = 1$, we obtain the theorem. □

The case of $p = 2$

We now assume that $p = 2$, so that $N_2 = C_2 \geq 1$. We continue to assume that $w(a_2) \neq w(\bar{a}_2)$.

Lemma 7.3.11. *There exists an arithmetic Frobenius* Frob_v *such that* $\epsilon_p(\mathrm{Frob}_v) = 1$.

Proof. Let σ_v be an arithmetic Frobenius at v. Then $\epsilon_p(\sigma_v) = \zeta_{2^n}$, a 2^n-th root of unity, for $n \geq 0$. If $n = 0$, we are done. Otherwise, since $\epsilon_p(G_v) = \epsilon_p(I_v)$, there exists $i \in I_v$ such that $\epsilon_p(\sigma_v^{2^n - 1}) = \epsilon_p(i)$. Hence $\epsilon_p^{2n}(\sigma_v) = 1 = \epsilon_p(\sigma_v) \cdot \epsilon_p(i) = \epsilon_p(\tilde{\sigma}_v)$, where $\tilde{\sigma}_v = \sigma_v i$ is another arithmetic Frobenius at v. □

Lemma 7.3.12. *If* Frob_v *is an arithmetic Frobenius at* v*, then* $\epsilon_p(\mathrm{Frob}_v)$ *belongs to* F_v^*.

Proof. Let $\sigma_v = \mathrm{Frob}_v$. Assume $\epsilon_p(\sigma_v)$ is a primitive 2^m-th root of unity, for $m \geq 0$. Let $r \geq 1$ be such that F_v contains a primitive 2^r-th root of unity, but not a 2^{r+1}-th root of unity. It is enough to prove $m \leq r$.

Assume, towards a contradiction, that $m \geq r+1$. Then $\epsilon_p^{2^{m-r-1}}(\sigma_v)$ is a 2^{r+1}-th root of unity. Using the fact that $\epsilon_p(G_v) = \epsilon_p(I_v)$, we can find $i \in I_v$ such that $\epsilon_p^{2^{m-r-1}}(\sigma_v) = \epsilon_p(\sigma_v \cdot i)$ (see proof of previous lemma). For example, if $m = r+1$, we can take $i = 1$. Now $\sigma'_v = \sigma_v i$ is another arithmetic Frobenius at v. Using Langlands' theorem to compute the (non-zero) trace of $\rho_f(\sigma'_v)$ we obtain

$$\alpha(\sigma'_v) \equiv a_p^{f_v} + \epsilon_p(\sigma'_v)(\bar{a}_p \epsilon'(p))^{f_v} \bmod F^*.$$

Since $\alpha^2 \equiv \epsilon \bmod F^*$, we deduce that

$$\frac{\mu^{f_v} + \epsilon_p^2(\sigma'_v)\nu^{f_v}}{\epsilon_p(\sigma'_v)} \in F^*.$$

By Lemma 7.3.6, μ^{f_v} and ν^{f_v} belong to F_v. Also, $\epsilon_p^2(\sigma'_v)$ is a primitive 2^r-th root of unity, so belongs to F_v. We conclude that the primitive 2^{r+1}-th root of unity $\epsilon_p(\sigma'_v) = \epsilon_p^{2^{m-r-1}}(\sigma_v)$ belongs to F_v, a contradiction. $\square$

Lemma 7.3.13. *If $i \in I_v$, then $\alpha(i)$ belongs to F_v^*.*

Proof. If $\epsilon_p(G_v) = \pm 1$, then by Langlands' theorem

$$\alpha(\mathrm{Frob}_v) \equiv a_p^{f_v} \pm (\bar{a}_p \epsilon'(p))^{f_v} \bmod F^*.$$

Let i be an arbitrary element of I_v and let σ_v and $\sigma'_v = \sigma_v i$ be two arithmetic Frobenii at v. The above congruence for α (and a calculation similar to that in Lemma 7.3.6 and Lemma 7.3.8 in the case of unequal sign) guarantees that $\alpha(\sigma_v) \equiv \alpha(\sigma'_v) \bmod F_v^*$. Since $\alpha(\sigma_v)\alpha(i)/\alpha(\sigma'_v) \in F^*$, so $\alpha(i) \in F_v$.

Let us assume now that $\epsilon_p(G_v) \neq \pm 1$. We first show that if $\epsilon_p(i) \neq -1$, then $\alpha(i) \in F_v$. We first choose an arithmetic Frobenius σ_v such that $\epsilon_p(\sigma_v) = 1$, by Lemma 7.3.11. Then $\epsilon_p(i) = \epsilon_p(\sigma_v) \cdot \epsilon_p(i) = \epsilon_p(\sigma'_v)$, for $\sigma'_v = \sigma_v i$. Hence $\epsilon_p(i) \in F_v$, by Lemma 7.3.12. By Langlands' theorem, we know $\alpha(i) \equiv 1 + \epsilon_p(i) \bmod F^*$. Hence, $\alpha(i)$ belongs to F_v^*. If $\epsilon_p(i) = -1$, we choose $j \in I_v$ such that $\epsilon_p(j) \neq \pm 1$, using the fact that $\epsilon_p(G_v) = \epsilon_p(I_v)$. Since $\epsilon_p(j)$ and $\epsilon_p(ij) \neq -1$, the previous argument shows that $\alpha(j)$ and $\alpha(ij)$ belongs to F_v. Since $\alpha(i)\alpha(j)/\alpha(ij) \in F^*$, we see that $\alpha(i) \in F_v^*$. $\square$

Lemma 7.3.14. *Let* Frob_v *be an arithmetic Frobenius at* v. *Then*

$$\alpha^2(\mathrm{Frob}_v) \equiv a_p^{2f_v} + (\bar{a}_p\epsilon'(p))^{2f_v} + 2p^{(k-1)f_v}\epsilon'(p)^{f_v} \ \mathit{mod}\ F_v^{*2}.$$

Proof. Let σ_v be a Frobenius as in Lemma 7.3.11, and let $\tilde{\sigma}_v$ be any arithmetic Frobenius at v. Then σ_v and $\tilde{\sigma}_v$ will differ by an element of I_v. By Lemma 7.3.13,

$$\alpha(\tilde{\sigma}_v) \equiv \alpha(\sigma_v) \bmod F_v^*.$$

Since $\epsilon_p(\sigma_v) = 1$, we get by Langlands' theorem

$$\alpha^2(\sigma_v) \equiv a_p^{2f_v} + (\bar{a}_p\epsilon'(p))^{2f_v} + 2p^{(k-1)f_v}\epsilon'(p)^{f_v} \bmod F^{*2}.$$

Hence,

$$\alpha^2(\tilde{\sigma}_v) \equiv a_p^{2f_v} + (\bar{a}_p\epsilon'(p))^{2f_v} + 2p^{(k-1)f_v}\epsilon'(p)^{f_v} \bmod F_v^{*2}.$$

□

Theorem 7.3.15. *Let* $p = 2$ *and assume* $N_2 = C_2 \geq 1$. *Let* $v \mid 2$ *be a place of* F. *Assume that* $w(a_2) \neq w(\bar{a}_2)$. *Then* X_v *is a matrix algebra over* F_v *if and only if*

$$m_v = [F_v : \mathbb{Q}_2] \cdot v(\mu + 2p^{k-1} + \nu) \in \mathbb{Z}$$

is even, where v *is normalized such that* $v(p) = 1$.

Proof. By Lemma 7.3.13, the map $\alpha : G_v \to \bar{F}_v^*/F_v^*$ is unramified. Applying Lemma 5.2.2 with $K = \alpha|_{G_v}$ and $t = \epsilon'|_{G_v}$, we have

$$\mathrm{inv}_v c_\alpha = \frac{1}{2} \cdot v\left(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_v)\right) = \frac{1}{2} \cdot v(\mu^{f_v} + 2p^{(k-1)f_v} + \nu^{f_v}) \bmod \mathbb{Z},$$

where the last equality follows from Lemma 7.3.14. The theorem now follows replacing v by the valuation v normalized such that $v(p) = 1$. □

7.3.2 Equal slope

In this section, we assume that

$$v\left(\frac{a_p^2}{\epsilon'(p)} + 2p^{(k-1)} + \frac{\bar{a}_p^2}{\bar{\epsilon}'(p)}\right) \geq k - 1.$$

Here v is a valuation such that $v(p) = 1$. So, $w(a_p) = w(\bar{a}_p)$ for all place w of E lying above v. In this case it is possible for $m_v = \infty$. To avoid this we introduce m_v^ζ, for any root of unity ζ in the image of ϵ_p, defined by

$$m_v^\zeta := e_v \cdot v\left(\mu^{f_v} \cdot 1/\zeta + 2p^{(k-1)f_v} + \nu^{f_v} \cdot \zeta\right) \in \mathbb{Z} \cup \{\infty\},$$

where v is normalized such that $v(p) = 1$. By Lemma 7.3.2, the three term expression above is in F so the above expression is well defined. Moreover, for some ζ, the three term expression above is non-zero and $m_v^\zeta \in \mathbb{Z}$ is finite. When $\zeta \in F_v^*$, e.g., if ζ is the value of the tame part of ϵ_p, then we may rewrite

$$m_v^\zeta = e_v \cdot v\left(\mu^{f_v} + 2\zeta p^{(k-1)f_v} + \zeta^2 \nu^{f_v}\right) \ \in \mathbb{Z} \cup \{\infty\}.$$

Note that in the unequal slope case $m_v^\zeta = m_v$ if $\zeta \in F_v^*$, so the quantities m_v^ζ may be considered as generalizations of m_v in the equal slope case. In particular taking $\zeta = +1$ we have

$$m_v^+ = m_v^{+1} = e_v \cdot v\left(\mu^{f_v} + 2p^{(k-1)f_v} + \nu^{f_v}\right) \in \mathbb{Z} \cup \{\infty\},$$

and if -1 belongs to the image of the tame part of ϵ_v, then

$$m_v^- = m_v^{-1} = e_v \cdot v\left(\mu^{f_v} - 2p^{(k-1)f_v} + \nu^{f_v}\right) \in \mathbb{Z} \cup \{\infty\}.$$

We remark that m_v^+ is finite if and only if $a_p^{f_v} + (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, and m_v^- is finite if and only if $a_p^{f_v} - (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, so if -1 belongs to the image of the tame part of ϵ_v, then one of the two quantities $m_v^\pm$ is always finite.

The case of odd primes

We now assume that p is odd and work under a condition on the tame part of ϵ_p.

Theorem 7.3.16. *Let p be an odd prime with $N_p = C_p \geq 1$ and $v \mid p$ be a place of F. Assume that the tame part of ϵ_p on G_v is not quadratic. Let ζ be in the image of the tame part of ϵ_p. Then the parity of*

$$m_v^\zeta = e_v \cdot v\left(\mu^{f_v} + 2\zeta p^{(k-1)f_v} + \zeta^2 \nu^{f_v}\right) \in \mathbb{Z} \cup \{\infty\}$$

is independent of ζ when it is finite, and then X_v is a matrix algebra over F_v if and only if $m_v^\zeta \in \mathbb{Z}$ is even. In particular

- *if* $a_p^{f_v} + (\bar{a}_p \epsilon'(p))^{f_v} \neq 0$, *then* X_v *is a matrix algebra over* F_v *if and only if* $m_v^+ \in \mathbb{Z}$ *is even, and,*

- *if* -1 *belongs to the image of the tame part of* ϵ_v *and* $a_p^{f_v} - (\bar{a}_p \epsilon'(p))^{f_v} \neq 0$, *then* X_v *is a matrix algebra over* F_v *if and only* $m_v^- \in \mathbb{Z}$ *is even.*

Proof. The proof goes along the lines of the proof of Theorem 7.3.10, with a few modifications. We base change to L so that $\epsilon_p|_{G_L}$ is tame, compute the invariant there, and then descend back to F_v.

We first show that $\tilde{\alpha} : G_L \to \bar{F}_v^*/F_v^*$ is unramified. If the trace of $\rho_f(g)$ is nonzero, then $\alpha(g) \equiv$ trace $\rho_f(g)$ mod F^*, for $g \in G_L$ (*cf.* 5.2.1). If the tame part of ϵ_p is trivial on G_v, then $\alpha(i) \equiv 1 + \epsilon_p(i) = 2$ mod F^*, for all $i \in I_L$. So, we may assume that ϵ_p is non trivial on G_v. We first prove that if $\epsilon_p(i) \neq -1$, for $i \in I_L$, then $\alpha(i)$ belongs to F_v^*. Indeed by Langlands' theorem, $\alpha(i) \equiv 1 + \epsilon_p(i)$ mod F^*, and since ϵ_p is tame on G_L, $\epsilon_p(i) \in \mathbb{Q}_p^* \subset F_v^*$. If $\epsilon_p(i) = -1$, for $i \in I_L$, we choose $j \in I_L$ such that $\epsilon_p(j) \neq \pm 1$. Such a choice is possible since by assumption the tame part of ϵ_p is not quadratic. The above argument shows that $\alpha(j)$ and $\alpha(ij)$ belong to F_v^*, and since $\alpha(i)\alpha(j)/\alpha(ij) \in F^*$, $\alpha(i) \in F_v^*$ as well.

Write u for the prime of L lying over v and Frob_u be an arithmetic Frobenius at u. We calculate $\mathrm{inv}_L(\mathrm{res}_{F_v/L} c_\alpha)$ using Lemma 5.2.2 applied to $K = \alpha|_{G_L}$ and $t = \epsilon'|_{G_L}$, and get

$$\mathrm{inv}_L(\mathrm{res}_{F_v/L} c_\alpha) = \frac{1}{2} \cdot u \left(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_u) \right) \mod \mathbb{Z}.$$

Since $[L : F_v]$ is odd (a power of p) we may descend to F_v as before to get

$$\mathrm{inv}_v c_\alpha = \frac{1}{2} \cdot v \left(\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_u) \right) \mod \mathbb{Z}.$$

Let $\zeta = \epsilon_p(\mathrm{Frob}_u) \in \mathbb{Q}_p^* \subset F_v^*$. Then the usual argument using Langlands' theorem shows that

$$\frac{\alpha^2}{\epsilon'}(\mathrm{Frob}_u) \equiv \mu^{f_v} + 2\zeta p^{(k-1)f_v} + \zeta^2 \nu^{f_v} \mod F_v^{*2}$$

and replacing v with the valuation v such that $v(p) = 1$ we obtain the theorem. We note that the parity of m_v^ζ is independent of ζ since $\tilde{\alpha}$ is unramified on G_L. $\square$

$p = 2$ and the remaining odd prime cases

We now show that if $p = 2$ and ϵ_2 is not quadratic on G_v, then the ramification of X_v is also determined by $m_v^{\pm}$, up to an error term n_v which depends purely on the nebentypus ϵ_2, and which we define now.

If ϵ_2 is trivial on G_v, set $n_v = 0$. If ϵ_2 has order 2^r on G_v, for $r > 1$, let $F_v(\sqrt{t})/F_v$, for $t \in F_v^*$ be the quadratic extension of F_v cut out by the quadratic character $\epsilon_2^{2^{r-1}}$ on G_v. Let ζ_{2^r} be a primitive 2^r-th root of unity and define

$$z = \frac{(1+\zeta_{2^r})^2}{\zeta_{2^r}} \in F^*,$$

noting that $z \in F^*$ by Langlands' theorem. Define n_v mod 2 by

$$(-1)^{n_v} = \epsilon_v(-1) \cdot (t, z)_v,$$

where ϵ_v is the restriction of ϵ_2 to G_v and $(t, z)_v$ is the Hilbert symbol of t and z at v .

Let c_ϵ be the cocycle

$$c_\epsilon(g, h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}},$$

for $g, h \in G_{\mathbb{Q}}$. Then $[c_\epsilon]$ is 2-torsion in the $\mathrm{Br}(\mathbb{Q})$.

Lemma 7.3.17. *Let $v \mid p$ be a prime of F and let ϵ_v denote the restriction of the character ϵ to G_v. Then $[c_\epsilon]_v = 1$ if and only if $\epsilon_v(-1) = 1$.*

Proof. Let $p : \bar{F}_v^* \to \bar{F}_v^*$ be the map $t \to t^2$. We have a short exact sequence of abelian groups, considered as a G_v module with a trivial action of G_v.

$$1 \to \{\pm 1\} \to \bar{F}_v^* \xrightarrow{p} \bar{F}_v^* \to 1.$$

This short exact sequence will induce a long exact sequence of cohomology groups namely,

$$... \to \mathrm{Hom}(G_v, \bar{F}_v^*) \longrightarrow \mathrm{Hom}(G_v, \bar{F}_v^*) \xrightarrow{\delta} H^2(G_v, \pm 1) \to .....$$

Now $\epsilon : G_v \to \bar{F}_v^*$ is a homomorphism. Hence we calculate $\delta(\epsilon)$ from [Se79] and get $\delta(\epsilon) = [c_\epsilon]_v$. Hence $[c_\epsilon]_v = 1$ if and only if $\delta(\epsilon) = 1$, which is true if and only if ϵ belongs to the image of the previous map, which in turn is true if and only if ϵ

is a square of a character. By class field theory, we look at ϵ_v as a character on F_v^*. We identify $F_v^*/\{\pm 1\}$ with F_v^{*2} by the map p. If $\epsilon_v(-1) = 1$, then ϵ_v will define a character on F_v^{*2} and hence it has a square root. On the other hand, if ϵ_v has a square root then $\epsilon_v(-1) = 1$. Hence, square root of ϵ_v exists if and only if $\epsilon_v(-1) = 1$. We get the desired result. $\square$

Lemma 7.3.18. *Assume ϵ_2 has order 2^r on G_v. Let h be the function on G_v defined by*

$$h(g) = \begin{cases} \frac{1+\epsilon_2(g)}{\sqrt{\epsilon_2(g)}} & \text{if } \epsilon_2(g) \neq -1 \\ 1 & \text{if } \epsilon_2(g) = -1, \end{cases}$$

and let c_h be the corresponding F-valued 2-cocycle on G_v. Then the class of c_h in ${}_2\mathrm{Br}(F_v)$ is given by the symbol $(t, z)_v$.

Proof. We first claim that if $-1 \neq \zeta = \epsilon_2(g)$ is not a primitive 2^r-th root of unity, then $\frac{1+\zeta}{\sqrt{\zeta}} \in F^*$. Indeed, choose $g \in G_v$ such that $\epsilon_2(g) = \zeta_{2^r}$, where ζ_{2^r} is a primitive 2^r-th root of unity. We may assume $g \in I_v$, and applying Langlands' theorem we obtain that $\frac{(1+\epsilon_2(g))^2}{\epsilon_2(g)} \in F^*$, and hence that $\frac{1+\zeta_{2^{r-1}}}{\sqrt{\zeta_{2^{r-1}}}} \in F^*$, where $\zeta_{2^{r-1}} = \epsilon_2(g^2)$ is a primitive 2^{r-1}-th root of unity. Now set $h = g^2 \in I_v$. Set $d = \frac{\alpha^2}{\epsilon_2}$ on I_v. Then by Langlands' theorem $d(h) \in F^{*2}$. Since $d : I_v \to F^*/F^{*2}$ is a homomorphism we see that $d(h^a) \in F^{*2}$, for all integers a. Hence by Langlands' theorem again we deduce that $\frac{(1+\zeta_{2^{r-1}}^a)^2}{\zeta_{2^{r-1}}^a} \in F^{*2}$, if it is non-zero. Hence $\frac{1+\zeta_{2^{r-1}}^a}{\sqrt{\zeta_{2^{r-1}}^a}} \in F^*$, for all integers a, if it is non-zero, proving the claim. We now claim that if $\epsilon_2(g^b)$ with b odd is any primitive 2^r-th root of unity then $h(g^b) \equiv h(g) \bmod F^*$. Indeed by the discussion above $h(g^{b-1}) \in F^*$ since $b-1$ is even.

The two claims above show that the 2-cocycle c_h is cohomologous to the 2-cocycle c_l where

$$l(g) = \begin{cases} 1 & \text{if } \epsilon_2^{2^{r-1}}(g) = 1, \\ \frac{1+\zeta_{2^r}}{\sqrt{\zeta_{2^r}}} & \text{if } \epsilon_2^{2^{r-1}}(g) = -1. \end{cases}$$

Let σ be the non-trivial element of the Galois group $\mathrm{Gal}(F_v(\sqrt{t})/F_v)$. Let $z = \left(\frac{1+\zeta_{2^r}}{\sqrt{\zeta_{2^r}}}\right)^2 \in F^*$. Then the class of c_l is completely determined by the table

	1	σ
1	1	1
σ	1	z

which is precisely the symbol $(t, z)_v$. $\square$

Theorem 7.3.19. *Let $p = 2$ and assume that ϵ_2 is not quadratic on G_v. If ϵ_2 is trivial on G_v and $m_v^+ < \infty$, then X_v is a matrix algebra if and only if m_v^+ is even. If ϵ_2 on G_v has order 4 or more and if*

- *$a_p^{f_v} + (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, then X_v is a matrix algebra if and only if $m_v^+ + n_v \in \mathbb{Z}$ is even, and if*
- *$a_p^{f_v} - (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, then X_v is a matrix algebra if and only if $m_v^- + n_v \in \mathbb{Z}$ is even,*

noting that if both $a_p^{f_v} \pm (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$ then $m_v^\pm$ have the same parity.

Proof. If ϵ_2 is trivial on G_v, then by Langlands' Theorem, $\alpha(i) \equiv 2 \mod F^*$, for all $i \in I_v$. We can directly apply Lemma 5.2.2 to prove the first statement. So we may assume ϵ_2 is not of order 1 or 2 on G_v. Hence, there exists $i \in I_v$ such that $\epsilon_2(i) = \sqrt{-1}$. If $\epsilon_2(j) = -1$ for $j \in I_v$, then a short computation using the fact that $c_\alpha(i,j) \in F^*$ shows that $\alpha(j) \equiv \sqrt{-1} \mod F^*$.

We define a function $f : G_v \to E^*$ by

$$f(g) = \begin{cases} 1 + \epsilon_2(g) & \text{if } \epsilon_2(g) \neq -1, \\ \sqrt{-1} & \text{if } \epsilon_2(g) = -1. \end{cases}$$

Now define $K : G_v \to E^*$ by $K(g) = \frac{\alpha(g)}{f(g)}$, for $g \in G_v$. Then the cocycle c_α can be decomposed as $c_\alpha = c_K c_f$, where c_K and c_f are the cocycles corresponding to K and f respectively. That these are indeed cocycles follows from the fact that they are F-valued, which can be proved using $\epsilon_2(G_v) = \epsilon_2(I_v)$ and Langlands' theorem.

We first calculate $\mathrm{inv}_v c_K$. By choice of f, $K(i)$ belongs to F^*, for all $i \in I_v$. Since $\epsilon_2(G_v) = \epsilon_2(I_v)$ a computation using Langlands' theorem shows that $\frac{K^2(g)}{\epsilon'(g)} \in F^*$, for all $g \in G_v$. Let σ_v be the Frobenius at the prime v. By Lemma 5.2.2 applied to K as above and $t = \epsilon'$ we have

$$\mathrm{inv}_v c_K = \frac{1}{2} \cdot v\left(\frac{K^2}{\epsilon'}(\sigma_v)\right) \mod \mathbb{Z}.$$

Assume $a_2^{f_v} \neq -(\bar{a}_2\epsilon'(2))^{f_v}$, then we choose σ_v in such a way that $\epsilon_2(\sigma_v) = 1$. Then $\alpha(\sigma_v) \equiv (a_2^{f_v} + (\bar{a}_2\epsilon'(2))^{f_v}) \mod F^*$, so that $\frac{K^2}{\epsilon'}(\sigma_v) \equiv \mu^{f_v} + \nu^{f_v} + 2p^{(k-1)f_v} \mod F^{*2}$. Finally, the valuation considered in the statement of the theorem is normalized so that

$v(2) = 1$, and differs from the valuation used in the proof by e. Noting $ef = [F_v : \mathbb{Q}_2]$, we obtain

$$\mathrm{inv}_v c_K = \frac{1}{2} \cdot m_v^+ \bmod \mathbb{Z}.$$

If $a_2^{f_v} = -(\bar{a}_2 \epsilon'(2))^{f_v}$, then we choose σ_v in such a way that $\epsilon_2(\sigma_v) = -1$. Then $\alpha(\sigma_v) \equiv (a_2^{f_v} - (\bar{a}_2\epsilon'(2))^{f_v}) \bmod F^*$, so that $\frac{K^2}{\epsilon'}(\sigma_v) \equiv \mu^{f_v} + \nu^{f_v} - 2p^{(k-1)f_v} \bmod F^{*2}$. We obtain

$$\mathrm{inv}_v c_K = \frac{1}{2} \cdot m_v^- \bmod \mathbb{Z}.$$

We also remark that since $K^2/\epsilon' : G_v \to F^*/F^{*2}$ is an unramified homomorphism, $\mathrm{inv}_v c_K$ does not depend on the choice of arithmetic Frobenius at v, and in particular $m_v^{\pm}$ have the same parity if both are simultaneously finite.

Now we will calculate $\mathrm{inv}_v(c_f)$. Recall c_ϵ is the cocycle

$$c_\epsilon(g,h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}},$$

for $g, h \in G_{\mathbb{Q}}$. We further decompose $\mathrm{inv}_v c_f = \mathrm{inv}_v c_h + \mathrm{inv}_v c_\epsilon$ where h is the function defined in the previous lemma. The theorem follows from the previous two lemmas. $\square$

Corollary 7.3.20. *Assume that $p = 2$ and ϵ_2 is not quadratic on G_v. Assume also that $F = \mathbb{Q}$. Then,*

- *If $\epsilon_2(-1) = 1$, then X_v is a matrix algebra if and only if m_v is even.*
- *If $\epsilon_2(-1) = -1$, then X_v is a matrix algebra if and only if m_v is odd.*

Proof. The quadratic field $\mathbb{Q}(\sqrt{t})$ is contained in the kernel of ϵ_2, which is a cyclotomic extension of discriminant a power of 2 , so considering the discriminant, we get t can only be $2, -1$ and -2. A calculation using the fact that minimal polynomial of ζ_{2^r} is $X^{2^{r-1}} + 1 = 0$, we get $N_{\mathbb{Q}(\zeta_{2^r})/\mathbb{Q}}(z) = 4$. Since, $\mathbb{Q}(\zeta_{2^r})/\mathbb{Q}$ is a totally ramified extension, so $N_{\mathbb{Q}_2(\zeta_{2^r})/\mathbb{Q}_2}(z) = 4$. By assumption $z \in \mathbb{Q}_2$, so we get $z^{2^{r-1}} = 4$, that forces $r = 3$, by writing $z = 2^m.u$. Hence, the only possible value of z is 2 and using the formulas from [Se79, p. 212], we get $(t, z)_2 = 1$. $\square$

Remaining quadratic cases:- If the tame part of ϵ_p is quadratic on G_v for an odd prime p or if $p = 2$ and ϵ_2 is quadratic on G_v, we again show that X_v is determined completely by $m_v^\pm$ up to an extra Hilbert symbol. The following results are quite general and hold for the unequal slope case also. In the case of unequal slope the extra symbol is trivial.

We need some notations.

Assume that the quadratic extension cut out by the tame part of ϵ_p if p is odd, or by ϵ_p if $p=2$, is $F_v(\sqrt{t})$, for some $t \in F_v^*$.

Define

$$a = \frac{\mu^{f_v} + \nu^{f_v} + 2p^{(k-1)f_v}}{\mu^{f_v} + \nu^{f_v} - 2p^{(k-1)f_v}} \in F^* \cup \{0, \infty\}.$$

Note $a \in F^*$ if and only if $a_p^{f_v} \neq \pm(\bar{a}_p \epsilon'(p))^{f_v}$. In this case define the integer n_v mod 2 by $(-1)^{n_v} = (t,a)_v$. Let $p^\dagger \nmid N$ be a prime such that $a_{p^\dagger} \neq 0$ and such that

$$\chi_\gamma(p^\dagger) = \begin{cases} \text{-1} & \text{if } \chi_\gamma \text{ is ramified,} \\ 1 & \text{if } \chi_\gamma \text{ is unramified.} \end{cases}$$

We can always choose $p^\dagger$ as above, since f is a non-CM form. Since ϵ^{-1} is an extra twist, we have $\epsilon(p^\dagger) = -1$. Let

$$b = a_{p^\dagger}^2 = -\frac{a_{p^\dagger}^2}{\epsilon(p^\dagger)} \in F^*.$$

If $a_p^{f_v} = (\bar{a}_p \epsilon'(p))^{f_v}$, define n_v by $(-1)^{n_v} = (t,b)_v$, and if $a_p^{f_v} + (\bar{a}_p \epsilon'(p))^{f_v} = 0$, define n_v by $(-1)^{n_v} = (t,b)_v \cdot (-1)^{e_v v(b)}$.

Theorem 7.3.21. *Assume that the tame part of ϵ_p is quadratic for an odd prime p, or $p=2$ and ϵ_2 is quadratic.*

(i) Assume that $a \in F^*$. Then, X_v is a matrix algebra over F_v if and only if

$$m_v^+ + n_v$$

is even.

(ii) If $a_p^{f_v} = (\bar{a}_p \epsilon'(p))^{f_v}$, then X_v is a matrix algebra over F_v if and only if

$$m_v^+ + n_v$$

is even.

(iii) If $a_p^{f_v} + (\bar{a}_p\epsilon'(p))^{f_v} = 0$, then X_v is a matrix algebra over F_v if and only if

$$m_v^- + n_v$$

is even.

Proof. If $\epsilon_p(i) = -1$, for $i \in I_v$, then $\alpha^2(i) \in F^*$. We claim that the image of $\alpha(i)^2$ in F_v^*/F_v^{*2} is constant, i.e., there exists $d \in F_v^*$ such that $\alpha^2(i) \equiv d \bmod F_v^{*2}$. Indeed, a priori $\alpha(i) = \sqrt{t(i)}d(i)$, for some $t(i)$, $d(i) \in F_v^*$. If $j \in I_v$ with $\epsilon_p(j) = -1$, then by Langlands' theorem, since $\epsilon_p(ij) = 1$, $\alpha(ij) \in F^*$. Since $c_\alpha(i,j) \in F^*$, we get $\sqrt{t(i)} \equiv \sqrt{t(j)} \bmod F_v^*$, as desired. Thus $\sqrt{t(i)} \equiv \sqrt{d} \bmod F_v^*$ for all $i \in I_v$ such that $\epsilon_p(i) = -1$. We compute d and show that the ramification of X_v is controlled by $m_v^\pm$, and an extra Hilbert symbol involving d. In case (1) we show we can take $d = a$, whereas in case (2) and (3) we can take $d = b$.

For p odd, we do a base change as in Theorem 7.3.10 and assume without loss of generality that ϵ_p is tame (and quadratic).

Assume we are in case (1), so that $a \in F^*$. Let σ_v be an arithmetic Frobenius at v, such that $\epsilon_p(\sigma_v) = 1$. Let $i \in I_v$ be such that $\epsilon_p(i) = -1$. By Langlands' theorem,

$$\frac{\alpha(\sigma_v)}{\alpha(\sigma_v i)} \equiv \sqrt{a} \bmod F^*.$$

Since $c_\alpha(\sigma_v, i) \in F^*$, and a belongs to F^*, we have $\alpha(i) \equiv \sqrt{a} \bmod F^*$. We define a function f on G_v by

$$f(g) = \begin{cases} 1 & \text{if } \epsilon_p(g) = 1, \\ \sqrt{a} & \text{if } \epsilon_p(g) = -1. \end{cases}$$

Let $K(g) = \frac{\alpha(g)}{f(g)}$ on G_v. Then the cocycle c_α can be decomposed as $c_\alpha = c_K c_f$. Clearly $K(i)$ belongs to F_v^*, for all $i \in I_v$. Using Lemma 5.2.2 applied to K and $t = \epsilon'$, we have $\mathrm{inv}_v c_K = \frac{1}{2} \cdot v\left(\frac{K^2}{\epsilon'}(\sigma_v)\right) = \frac{1}{2} \cdot m_v^+ \bmod \mathbb{Z}$. To compute $\mathrm{inv}_v c_f$, let σ be the nontrivial element of $\mathrm{Gal}(F_v(\sqrt{t})/F_v)$. Then the cocycle table of the cocycle c_f is given by

	1	σ
1	1	1
σ	1	a

which gives the symbol $(t, a)_v$. This proves (1).

We now turn to parts (2) and (3). We wish to find $d \in F^*$, such that $\alpha(i) \equiv \sqrt{d} \mod F_v^*$, if $\epsilon_p(i) = -1$. We cannot take $d = a$ in parts (2) and (3) since $a = 0$ or ∞. So we argue a bit differently.

Let $i \in I_v$ with $\epsilon_p(i) = -1$. We claim that $\alpha(i) \equiv a_{p^\dagger} \mod F^*$. By (5.2.1) and the proof of Theorem 7.3.1, if χ_γ is unramified at p, then $\alpha(i)^\gamma = \alpha(i)$. Similarly, if χ_γ is ramified at p, then $\alpha(i)^\gamma = \chi_\gamma(i)\alpha(i) = \epsilon_p(i)\alpha(i) = -\alpha(i)$. Thus, if $\mathrm{Frob}_{p^\dagger}$ is an arithmetic Frobenius at the prime $p^\dagger$, then $\alpha(i) \equiv \alpha(\mathrm{Frob}_{p^\dagger}) \equiv a_{p^\dagger} \mod F^*$, as claimed. Define f on G_v by

$$f(g) = \begin{cases} 1 & \text{if } \epsilon_p(g) = 1, \\ a_{p^\dagger} & \text{if } \epsilon_p(g) = -1. \end{cases}$$

Let $K(g) = \frac{\alpha(g)}{f(g)}$ on G_v. Then the cocycle c_α can be decomposed as, $c_\alpha = c_K c_f$. We now proceed as in the proof of part (1). If $a_p^{f_v} + (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, the cocycle c_K has invariant $\mathrm{inv}_v c_K = \frac{1}{2} \cdot m_v^+ \mod \mathbb{Z}$. If $a_p^{f_v} - (\bar{a}_p\epsilon'(p))^{f_v} \neq 0$, then we get an extra term on evaluating α at an arithmetic Frobenius Frob_v for which $\epsilon_p(\mathrm{Frob}_v) = -1$, and get $\mathrm{inv}_v(c_K) = \frac{1}{2} \cdot (m_v^- - e_v \cdot v(b))$. It remains to calculate $\mathrm{inv}_v c_f$. Let σ be the nontrivial element of the Galois group of the quadratic field cut out by ϵ_p. The table for the cocycle c_f is given by

	1	σ
1	1	1
σ	1	b

which is clearly the symbol $(t, b)_v$. This proves (2) and (3). $\square$

The above theorem shows that the ramification of X at the place v is determined by $m_v^\pm$ and one extra Hilbert symbol. We can calculate those symbols using the formulas of page 211-212 of [Se79], except if $p = 2$ and $F_v \neq \mathbb{Q}_2$, in which case we can use the formulas stated in [Se81] and [FV93].

7.4 Supercuspidal primes

We assume in this section that $p > 2$, $N_p > C_p$ and $N_p \geq 2$. In this case, we prove a weaker result about the ramification of X_v. Since the slopes we study in this case tend to be infinite, the results so far we have studied, relating ramification to the valuations of expressions involving the Fourier coefficients are no longer possible. The cases for which the local Galois representation is a twist of the cases that have already been treated previously, can be dealt with easily since the ramification of the Brauer class is invariant under twist. Thus we may assume that the local Galois representation is supercuspidal and the local representation is induced by a character of χ of an index two subgroup G_K of G_p, i.e.,

$$\rho_f|G_p \sim \mathrm{Ind}_{G_K}^{G_p}\chi.$$

We manage to sometimes predict the ramification of X_v in terms of the character χ. Let σ be a non trivial automorphism of $K/\mathbb{Q}_p$. We define a suitable extension L of F_v and for an arithmetic Frobenius Frob_u of L, define

$$m_v = e_v \cdot v\left(\frac{(\chi(\mathrm{Frob}_u) + \chi^\sigma(\mathrm{Frob}_u))^2}{\epsilon'(p)^{f_v}}\right) \in \mathbb{Z} \cup \{\infty\},$$

where v is normalised such that $v(p) = 1$. We note that $m_v < \infty$ if and only if $\chi(\mathrm{Frob}_u) + \chi^\sigma(\mathrm{Frob}_u) \neq 0$.

Lemma 7.4.1. *Assume F_v contains K. If m_v is finite, then, X_v is a matrix algebra over F_v if and only if m_v is even.*

Proof. Since F_v contains K, so the local Galois representation is of the form

$$\rho|_{I_v} \sim \begin{pmatrix} \chi & 0 \\ 0 & \chi^\sigma \end{pmatrix}.$$

If $i \in I_v$ and $\chi(i) \neq -\chi^\sigma(i)$ then by Lemma 5.2.1, $\alpha(i) \equiv (\chi(i) + \chi^\sigma(i)) \bmod F^*$. If $K = \mathbb{Q}_{p^2}$ is unramified, then we write $\chi|_{I_K} = \omega_2^j\chi_1\chi_2$, following the notation of [GM09] , where ω_2 is the fundamental character of level two and χ_i's are characters of p power order. On the other hand if $K/\mathbb{Q}_p$ is ramified, then again in the notation of [GM09] we get $\chi|_{I_K} = \omega^j\chi_1\chi_2$, where ω is the Teichmüller character and χ_i's are the characters of p power order .

We can always choose an extension L of odd degree over F_v, such that χ_i's are trivial and ϵ_p is tame when restricted to the inertia subgroup of L.

If $K = \mathbb{Q}_{p^2}$ is an unramified extension, since $\omega^\sigma = \omega^p$ so $\alpha(i) \equiv (\omega_2^j(i) + \omega_2^{pj}(i)) \bmod F^*$ if i belongs to the inertia group I_L of L and $\omega_2^j(i) \neq -\omega_2^{pj}(i)$. The character ω_2 takes value in the p^2-1-th root of unity and F_v contains $\mathbb{Q}_{p^2}$, so $\alpha(i) \in F_v$

under the nonvanishing assumption. Since, $\tilde{\alpha}$ is a homomorphism, so $\alpha(i)$ belongs to F_v even if $i \in I_L$ and $\omega_2^j(i) = -\omega_2^{pj}(i)$.

On the other hand, if K is an ramified extension of $\mathbb{Q}_p$, since, $\omega = \omega^\sigma$, so $\alpha(i) \equiv 2\omega^j(i)$ if i belongs to the inertia group I_L of L. the ω takes value in the $p-1$-th root of unity, so $\alpha(i) \in F_v$. We use Lemma 5.2.2 applied to $K = \alpha$ and $t = \epsilon'$, both restricted to G_L, we have

$$\operatorname{inv}(\operatorname{res}_{F_v|L})c = \frac{1}{2}w(\frac{\alpha^2}{\epsilon}(\sigma_L)) = \frac{1}{2}w(\frac{\alpha^2}{\epsilon'}(\sigma_L)) \bmod \mathbb{Z}.$$

Since ϵ_p is tame when restricted to L. Since $\frac{\alpha^2}{\epsilon}(\sigma_L) \in F$ and $\epsilon_p(\sigma_L) \in \mathbb{Q}_p$, so $\frac{\alpha^2}{\epsilon'}(\sigma_L) \in F_v$. Let $[L : F_v] = p^t$, then using the fact that $\operatorname{inv}_L(\operatorname{res}_{F_v|L})c = p^t \cdot \operatorname{inv}_v c$, we get X_v is a matrix algebra if and only if $e_{L/F_v} v(\frac{\alpha^2}{\epsilon'}(\sigma_L)$ is even, where e_{L/F_v} is the degree of the totally ramified extension L/F_v and v is a surjective valuation of F_v onto $\mathbb{Z}$. We calculate that

$$\alpha(\sigma_L) \equiv (\chi(\sigma_u) + \chi^\sigma(\sigma_u)) \bmod F^*.$$

If we choose the valuation such that $v(p) = 1$, then X_v is a matrix algebra if and only if m_v is even, assuming that it is finite. Since, the inertia degree of $L/\mathbb{Q}_p$ also f_v, we get the desired result. □

Chapter 8

Brauer class of X for weight one forms

8.1 Good primes

We can now calculate the local Brauer class X_v at a prime $v|p$ with $p \nmid N$ (good prime), for modular forms of weight one. For each such v, define

$$m_v := [F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon^{-1}(p)) \in \mathbb{Z} \cup \{\infty\}$$

where v is now normalized so that $v(p) = 1$. This theorem is the analogue of the Theorem 7.1.1. We have:

Theorem 8.1.1. *Let $p \nmid N$ be a prime with $a_p \neq 0$, and let v be a place of F lying over p. Then $\mathrm{inv}_v(X_v) = m_v$ mod 2. Thus X_v is a matrix algebra over F_v if and only if the normalized slope $[F_v : \mathbb{Q}_p] \cdot v(a_p^2 \epsilon^{-1}(p))$ is even.*

Proof. The proof goes exactly as the proof of Theorem 7.1.1. □

8.1.1 Adjoint representation

In this section we will compute the quantity m_v appearing in Theorem 8.1.1 using the adjoint representation. This will allow us to essentially completely determine the

Brauer class of the algebra X at the good places $v|p$ with $a_p \neq 0$. The remaining good places v for which a_p vanishes are treated in the next section.

Recall that the adjoint representation

$$\mathrm{Ad}(\rho_f) : G_{\mathbb{Q}} \to \mathrm{GL}_4(E)$$

attached to the representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ is the 4-dimensional representation defined by the adjoint action:

$$\mathrm{Ad}(\rho_f)(g)(X) = \rho_f(g) X \rho_f(g)^{-1},$$

for $X \in \mathrm{M}_{2\times 2}(E)$. It is important for us because of the following elementary but useful fact.

Lemma 8.1.2. *Say $p \nmid N$. Then*

$$\mathrm{trace}(\mathrm{Ad}(\rho_f)(\mathrm{Frob}_p)) = \frac{{a_p}^2}{\epsilon(p)}.$$

Proof. If

$$\rho_f(\mathrm{Frob}_p) \sim \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix},$$

then a small computation shows that

$$\mathrm{Ad}(\rho_f)(\mathrm{Frob}_p) \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_p}{\beta_p} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\beta_p}{\alpha_p} \end{pmatrix}.$$

So

$$\mathrm{trace}(\mathrm{Ad}(\rho_f)(\mathrm{Frob}_p)) = 2 + \frac{\alpha_p}{\beta_p} + \frac{\beta_p}{\alpha_p} = \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} = \frac{{a_p}^2}{\epsilon(p)}.$$

□

Theorem 8.1.3 (Good primes of finite slope)**.** *Let f be a non-dihedral cuspidal newform in $S_1(N, \epsilon)$. Then*

X is unramified at v for all good primes $v|p$ with $p \nmid N$ and $a_p \neq 0$,

except possibly in the case when $p = 2$ and the projective image of ρ_f is S_4. Moreover, if the projective image of ρ_f is

(i) A_4*, then* $F = \mathbb{Q}$*,*

(ii) S_4*, then* $F = \mathbb{Q}$*,*

(iii) A_5*, then* $F = \mathbb{Q}(\sqrt{5})$*.*

Proof. We have $\mathrm{Ad}(\rho_f) = \mathrm{Ad}^0(\rho_f) \oplus 1$, where $\mathrm{Ad}^0(\rho_f)$ is the irreducible 3-dimensional representation, afforded by the trace zero matrices.

Say the projective image of ρ_f is A_4. From the character table of A_4, we see that A_4 has one 3-dimensional irreducible representation V, and three 1-dimensional representations (*cf.* [FH91, p. 20]). Thus

$$\mathrm{Ad}(\rho_f) = V \oplus U$$

where U is the trivial representation. Computing the character of $V \oplus U$ on the four conjugacy classes of A_4, and using Lemma 8.1.2 above, we obtain

$$\frac{{a_p}^2}{\epsilon(p)} = 4, 1, 1 \text{ or } 0.$$

Thus $F = \mathbb{Q}$ and by Theorem 8.1.1, X is unramified at all primes $v = p$, with $a_p \neq 0$ (note the 2-adic valuation of 4 is even!).

Suppose now that the projective image of ρ_f is S_4. From the character table of S_4 (*cf.* [FH91, p. 19]), we see that S_4 has two 1-dimensional representations, U (trivial) and U' (sign), one 2-dimensional irreducible representation, and two 3-dimensional irreducible representations V and $V \otimes U'$. A small check shows that $\mathrm{Ad}^0(\rho_f) = V \otimes U'$ (and not V), so that $\mathrm{Ad}(\rho_f) = (V \otimes U') \oplus U$. Computing traces on the 5 conjugacy classes as above, we get this time that

$$\frac{{a_p}^2}{\epsilon(p)} = 4, 0, 1, 2 \text{ or } 0.$$

So again, $F = \mathbb{Q}$, and if $v = p$ is a good odd prime with $a_p \neq 0$, then by Theorem 8.1.1, X is unramified at v. Note that since the 2-adic valuation of 2 is odd, X may be ramified at the prime 2 (we shall later give examples where in fact X_2 is ramified).

Finally, let the projective image of ρ_f be A_5. This time, there are two irreducible 3-dimensional representations Y and Z, and so $\mathrm{Ad}(\rho_f) = Y \oplus U$ or $Z \oplus U$, where U is the trivial representation. Again, computing traces on the 5 conjugacy classes using the character table of A_5 (*cf.* [FH91, p. 29]) we get

$$\frac{{a_p}^2}{\epsilon(p)} = 4, 1, 0, \frac{3 \pm \sqrt{5}}{2} \text{ or } \frac{3 \mp \sqrt{5}}{2}.$$

Since $F = \mathbb{Q}(\frac{{a_p}^2}{\epsilon(p)})$, we conclude that $F = \mathbb{Q}(\sqrt{5})$. Now $\mathrm{Norm}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\frac{3\pm\sqrt{5}}{2}) = 1$, so that $\frac{3\pm\sqrt{5}}{2}$ are units in F, and $v(\frac{3\pm\sqrt{5}}{2}) = 0$ for all primes v of F. Again, by Theorem 8.1.1, we conclude that X_v is unramified for all good places v of F lying above p with $a_p \neq 0$. $\square$

8.2 Symbols

In this section we give formulas for the Brauer class of X in terms of symbols, which help us determine the Brauer class completely.

Let $\tilde{\rho}_f$ be the projectivization of ρ_f. Let D_K be the discriminant of the unique quadratic extension $K/\mathbb{Q}$ in the S_4-number field cut out by $\tilde{\rho}_f$, when $\tilde{\rho}_f(G_{\mathbb{Q}}) = S_4$. Define the 2-cocycle c_ϵ on $G_{\mathbb{Q}}$ by

$$c_\epsilon(g, h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}},$$

for g, $h \in G_{\mathbb{Q}}$. Then $[c_\epsilon]$ is 2-torsion in $\mathrm{Br}(\mathbb{Q})$.

Theorem 8.2.1. *The class of X in $\mathrm{Br}(F)$ is as follows. If the image of $\tilde{\rho}_f$ is*

(i) A_4, *then* $[X] = [c_\epsilon]$,

(ii) S_4, *then* $[X] = [c_\epsilon] \cdot (2, D_K)$, *where* (a, b) *is the symbol for* $F = \mathbb{Q}$,

(iii) A_5, *then* $[X] = [c_\epsilon]$.

Proof. We use a general formula from [Qu98] which works for weight one forms as well:

$$[X] = [c_\epsilon] \cdot [c_d],$$

where $[c_\epsilon] \in \mathrm{Br(F)}$ is as above, and $[c_d] \in \mathrm{Br}(F)$ is the product of the symbols

$$[c_d] = (t_1, d_1) \cdots (t_n, d_n),$$

where t_i and d_i are determined as follows. Note that $d = \alpha^2/\epsilon$ induces a continuous map

$$d : G_{\mathbb{Q}} \to F^*/F^{*2},$$

where F^*/F^{*2} has the discrete topology. Thus $G_{\mathbb{Q}}/\ker(d) \simeq \mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2)^m$, for some elementary 2-extension K of $\mathbb{Q}$. Now, for each i between 1 and m, let $\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})$ denote the element corresponding to $(0, 0.., 1, .., 0) \in (\mathbb{Z}/2)^m$, with a 1 in the i-th spot. Then $t_j \in \mathbb{Q}^*$ is defined by $\sigma_i(\sqrt{t_j}) = (-1)^{\delta_{ij}}\sqrt{t_j}$. We lift each σ_i to an element in $G_{\mathbb{Q}}$ which we continue to call σ_i, and set $d_i := d(\sigma_i) \in F^*/F^{*2}$.

We now claim that $\ker(\tilde{\rho}_f) \subset \ker d$. This is immediate, since if $\tilde{\rho}_f(g) = 1$, for some $g \in G_{\mathbb{Q}}$, then $\rho_f(g)$ is a scalar, so $\mathrm{trace}(\rho_f(g))^2/\det(\rho_f(g)) = 4$. Since the trace is non-zero, this last expression is also equal to $\alpha^2(g)/\epsilon(g)$ up to an element of F^{*2}, and hence $d(g) = 1 \bmod F^{*2}$, proving the claim. Thus, there is a surjection

$$G_{\mathbb{Q}}/\ker(\tilde{\rho}_f) \twoheadrightarrow G_{\mathbb{Q}}/\ker(d).$$

The group on the left is A_4, S_4 or A_5, and the group on the right is an elementary 2-group. We conclude that in the A_4 and A_5 cases, the 2-group is trivial, and hence d is trivial. This finishes the proof in these cases. In the S_4 case, we see that the 2-group must be $\mathbb{Z}/2$, and the quadratic field K cut out by d must be the fixed field of A_4 in the S_4-extension cut out by $\tilde{\rho}_f$. Thus $m = 1$ and $t_1 = D_K$. To compute d_1 we recall some general facts. One knows that each extra twist χ_γ satisfies $\chi_\gamma(g) = \psi_\gamma(g) \cdot \sqrt{\epsilon(g)}^{\gamma - 1}$, for $g \in G_{\mathbb{Q}}$, where ψ_γ is a quadratic character, and γ is here also thought of as an element of G_F. Hence, by (5.2.1), for $\gamma \in G_F$,

$$(\alpha(g)/\sqrt{\epsilon(g)})^\gamma = \psi_\gamma(g) \cdot (\alpha(g)/\sqrt{\epsilon(g)}),$$

for all $g \in G_{\mathbb{Q}}$. It follows that $\ker(d)$ is the intersection of all the $\ker(\psi_\gamma)$. We conclude that there is only one γ for which ψ_γ is non-trivial, and $d = \psi_\gamma$, for this γ. By part (3) of Lemma 5.2.1 we have:

$$(8.2.1) \qquad (a_p/\sqrt{\epsilon(p)})^\gamma = \psi_\gamma(p) \cdot (a_p/\sqrt{\epsilon(p)}),$$

for $\gamma \in G_F$, and each prime $p \nmid N$. Using the character of the adjoint representation, we had computed that $a_p/\sqrt{\epsilon(p)}$ is equal to the square-root of 4, 2, 1, or 0, and in particular lies in $F = \mathbb{Q}$, except in one case. By the Chebotarev density theorem, there is a prime $p \nmid N$ such that $a_p/\sqrt{\epsilon(p)} = \pm\sqrt{2}$. By (8.3.2) we must have $\psi_\gamma(p) = -1$, for the unique non-trivial quadratic character ψ_γ. But $d = \psi_\gamma$, and so taking $\sigma_1 = \mathrm{Frob}_p$

we see that $d_1 = d(\sigma_1) = 2$ up to a square in $F^* = \mathbb{Q}^*$. This shows that the class of c_d is given by the symbol $(2, D_K)$, completing the proof in the S_4 case as well. $\square$

Remark 8.2.2. The above proof is purely algebraic. One can give a slight variation of the above proof using the Chebotarev density theorem. For instance, in the A_4 and A_5 cases it suffices to show that all the ψ_γ are trivial. In the A_4 case, computations using the adjoint representation showed that for $p \nmid N$, $a_p/\sqrt{\epsilon(p)} = \sqrt{4}$, $\sqrt{1}$ or $\sqrt{0}$, so is always in $F = \mathbb{Q}$. By (8.3.2) we see that for each γ, we must have

$$\{p \nmid N \mid \psi_\gamma(p) = -1\} \subset \{p \nmid N \mid a_p^2/\epsilon(p) = 0\}.$$

By the Chebotarev density theorem applied to the A_4-number field cut out by $\tilde{\rho}_f$, the density of the larger set is $3/12 = 1/4$. If ψ_γ is quadratic the density of the smaller set would be $1/2$, which is not possible, and so ψ_γ must be trivial. A similar argument applies in the A_5 case, noting that for $p \nmid N$, $a_p/\sqrt{\epsilon(p)} = 2, 1, 0, \frac{\sqrt{5}\pm 1}{2} \in F = \mathbb{Q}(\sqrt{5})$, and the density of the larger set is again $15/60 = 1/4$. In the S_4 case this naïve argument fails, as it should, since this time we have

$$\{p \nmid N \mid \psi_\gamma(p) = -1\} \subset \{p \nmid N \mid a_p^2/\epsilon(p) = 0 \text{ or } 2\}$$

and the larger set has density $9/24 + 6/24 > 1/2$.

8.3 Prime level

In this section, we will give an alternative method of calculating the algebra X completely, at least when the modular form has prime level.

We start by recalling the following result [Se77a, Theorem 7] which shows that only the octahedral and icosahedral cases occur in prime level.

Theorem 8.3.1. *Say $f \in S_1(p, \epsilon)$ is a non-dihedral cuspidal newform of odd prime level p. Then*

(a) $p \not\equiv 1$ *mod* 8.

(b) *If $p \equiv 5$ mod 8, then ϵ is of order 4 and the projective image of ρ_f is S_4.*

(c) *If $p \equiv 3$ mod 4, then ϵ is quadratic and the projective image of ρ_f is either S_4 or A_5.*

Following Serre we break case (c) into two cases (c1) and (c2), depending on whether the projective image or ρ_f is S_4 or A_5. The following theorem completely determines the Brauer class of X in prime level.

Theorem 8.3.2. *Say $f \in S_1(p, \epsilon)$ is a form of prime level as above. Then*

(b) $X = (-1, -2)$

(c1) $X = (-2, -p)$

(c2) $X = (-1, -p)$

in $\mathrm{Br}(F)$.

Proof. We start with case (c2). So $p \equiv 3 \bmod 4$, $\epsilon(n) = \left(\frac{n}{p}\right)$, and the projective image of the Galois representation attached to f be A_5. In this case it is known (*cf.* [Se77a, p. 250]) that

$$E = \mathbb{Q}(\sqrt{-1}, \sqrt{5}).$$

We have $F = \mathbb{Q}(\sqrt{5})$ (by Theorem 8.1.3). In fact the order 2 group Γ is generated by complex conjugation c, and $f^c = f \otimes \epsilon^{-1}$ is the only extra twist of f.

We now use the following result from [BG04, p. 1665], valid for real-valued nebentypus characters, and which one can easily check is valid in weight one as well. Let $S(N)$ be the set containing all the odd primes dividing the level N as well as the formal symbol ± 2. Then the Brauer class of X is given by

$$X = \bigotimes_{q \in S(N)} (z_q, q^*),$$

where $q^* = (-1)^{\frac{q-1}{2}}$ if q is odd and $q^* = \pm 2$ if $q = \pm 2$, and $z_q \in F^*$ is determined as in [BG04, p. 1664].

Since the level is an (odd) prime p we need to only find z_p. We use some notation from [BG04, p. 1664]. Consider the homomorphism $\Gamma \to \mathbb{Z}/2$ given by mapping c to 1. It is clearly injective, so its kernel $\Gamma(p)$ is trivial, and $F(p)$, the fixed field of $\Gamma(p)$, is E. Since z_p is a quantity in F whose square root generates $F(p)$ we may take $z_p = -1$. It follows from the formula above that

$$X = (-1, -p).$$

Suppose now we are in case (c1). Again $p \equiv 3(4)$, $\epsilon(n) = \left(\frac{n}{p}\right)$. and the projective image of the Galois representation attached to f is S_4. In this $E = \mathbb{Q}(\sqrt{-2})$ ([Se77a, p. 250]), and $F = \mathbb{Q}$ (Theorem 8.1.3). By an argument similar to the one above we get

$$X = (-2, -p).$$

Finally, suppose we are in case (b). Thus $p \equiv 5 \bmod 8$, ϵ is of order 4, and the projective image of the Galois representation attached to f is S_4. This time $E = \mathbb{Q}(\sqrt{-1})$ ([Se77a, p. 250], and $F = \mathbb{Q}$ (Theorem 8.1.3). Again $f^c = f \otimes \epsilon^{-1}$ is the only non-trivial extra twist. In this case, since the twists are not quadratic we cannot use the formula from [BG04] quoted above. We will use instead a formula in [Qu98] which also works in weight one:

$$[X] = [c_\epsilon] \cdot [c_d]$$

where c_ϵ is the 2-cocycle given by $c_\epsilon(g,h) = \frac{\sqrt{\epsilon(g)}\sqrt{\epsilon(h)}}{\sqrt{\epsilon(gh)}}$, and $[c_d] \in \mathrm{Br}(F)$ is the product of the classes

$$[c_d] = (t_1, d_1) \cdots (t_n, d_n),$$

where t_i and d_i are determined as follows. Note that $d = \alpha^2/\epsilon$ induces a continuous map

$$d : G_{\mathbb{Q}} \to F^*/F^{*2}, \tag{8.3.1}$$

where F^*/F^{*2} has the discrete topology. Thus $G_{\mathbb{Q}}/\ker(d) = \mathrm{Gal}(K/\mathbb{Q}) = (\mathbb{Z}/2)^m$, for some elementary 2-extension K of $\mathbb{Q}$. Now, for each i between 1 and m, let $\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})$ denote the element corresponding to $(0, 0.., 1, .., 0) \in (\mathbb{Z}/2)^m$, with a 1 in the i-th spot. Then $t_j \in \mathbb{Q}^*$ is defined by $\sigma_i(\sqrt{t_j}) = (-1)^{\delta_{ij}}\sqrt{t_j}$. We lift each σ_i to an element in $G_{\mathbb{Q}}$ which we continue to call σ_i, and set $d_i := d(\sigma_i) \in F^*/F^{*2}$. In case (b), the class of c_ϵ is ramified exactly at p (since ϵ is odd) and at ∞, and a small check shows it is equal to the symbol $(-2, -p)$, since $p \equiv 5 \bmod 8$. We now compute the class of c_d. Recall $E = \mathbb{Q}(i)$ and $F = \mathbb{Q}$, and ϵ has order 4. We prove in the next lemma that the map d satisfies $\ker(d) = \ker(\epsilon^2)$. We use the fact and compute that

$$[c_d] = (2, p).$$

Hence by Quer's formula we obtain $[X] = [c_\epsilon] \cdot [c_d] = (-2, -p) \cdot (2, p) = (-1, -2)$. $\square$

Lemma 8.3.3. *The map d satisfies* $\ker(d) = \ker(\epsilon^2)$.

Proof. Recall $E = \mathbb{Q}(i)$ and $F = \mathbb{Q}$, and ϵ has order 4. Let $\sigma \in \ker(d)$. Then $d(\sigma) \in F^{*2}$, hence $\sqrt{d(\sigma)} \in F^*$. We conclude that

$$g\left(\frac{\alpha(\sigma)}{\sqrt{\epsilon(\sigma)}}\right) = \frac{\alpha(\sigma)}{\sqrt{\epsilon(\sigma)}}, \tag{8.3.2}$$

for all $g \in G_F$. Since $\alpha(\sigma) \in E$ we see that $g(\sqrt{\epsilon(\sigma)}) = \sqrt{\epsilon(\sigma)}$, for all $g \in G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$. In particular $\sqrt{\epsilon(\sigma)}$ has to be a fourth root of unity and $\epsilon^2(\sigma) = 1$.

Conversely, suppose σ belongs to $\ker(\epsilon^2)$. Let g be an element of G_F. Assume g induces the identity map on E. Then $\sqrt{\epsilon(\sigma)}^{g-1} = 1$, since $\sqrt{\epsilon(\sigma)} \in E$. Thus (8.3.2) holds vacuously, since $\alpha(\sigma) \in E$. Assume now that g induces complex conjugation γ on E. Then $\alpha(\sigma)^{g-1} = \chi_\gamma(\sigma) = \frac{1}{\epsilon(\sigma)} = \sqrt{\epsilon(\sigma)}^{\gamma-1}$, and again we obtain (8.3.2). We conclude $d(\sigma) \in F^{*2}$. ☐

Lemma 8.3.4. *$c_d = (2, p)$.*

Proof. By the previous lemma, $\ker(d) = \ker(\epsilon^2)$. Hence to apply Quer's formula, it is enough to look at the field cut out by $\ker(\epsilon^2)$. There is only one symbol in the expression for $[c_d]$ above. We compute t_1 and d_1. Since $p \equiv 1 \bmod 4$, the character ϵ^2 is real quadratic of conductor p and so $t_1 = p$. We now compute d_1. Since σ_1 acts non-trivial on $\sqrt{p}$, any preimage of σ_1 in $\mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$ is of order 4, so $\epsilon(\sigma_1) = \pm i$. Thus using the relation $\bar{\alpha}(\sigma_1) = \epsilon^{-1}(\sigma_1) \cdot \alpha(\sigma_0)$, and the fact that $\alpha(\sigma_1)$ is a Gaussian number, we conclude that $\alpha(\sigma_1) = a(1 \pm i)$ for some $a \in \mathbb{Q}^*$. Hence $d_1 = d(\sigma_1) = \alpha^2(\sigma_1)/\epsilon(\sigma_1) = 2 \bmod (F^*)^2$. ☐

Corollary 8.3.5. *Let $f \in S_1(p, \epsilon)$ be as above. Then X is ramified exactly at*

(b) *2 and ∞.*

(c1) *one, but not both of 2 or p, and at ∞.*

(c2) *the two primes lying above p when p splits in $F = \mathbb{Q}(\sqrt{5})$, and the two ∞ places.*

Proof. Cases (b) and (c1) are easy to check. Suppose we are in case (c2). If v is the (inert) place of $F = \mathbb{Q}(\sqrt{5})$ lying above 2, then $(-1, -p)_v = (\mathrm{Norm}_{F_v/\mathbb{Q}_p}(-1), p)_2 = (1, p)_2 = 1$. If p splits in $\mathbb{Q}(\sqrt{5})$ and $v|p$, then $(-1, -p)_v = -1$, since -1 is not a square in $F_v = \mathbb{Q}_p$, whereas if $v|p$ is inert, then $(-1, -p)_v = 1$, since -1 is a square in F_v, the unramified quadratic extension of $\mathbb{Q}_p$. ☐

Chapter 9

Numerical Examples

We will now discuss some numerical examples, which helped us to prove the various theorems proved in this thesis. These examples were generated by the program `Endohecke` due to Brown and Ghate which was made by suitably modifying the C++ program `Hecke` created by W. Stein. I wish to thank Prof. E. González-Jiménez for helping me to find an example supporting Theorem 7.3.19 and Prof. Jordi Quer for sharing with me [Qu06].

9.1 Steinberg primes

Let $f \in S_5(15, [2,1])$ be the unique newform. It is Steinberg at the prime 5 since $N_5 = 1$ and $C_5 = 0$. F is a cubic extension of $\mathbb{Q}$. Now, 5 decomposes into two distinct primes v_1, v_2 in F with ramification index and inertia degree $(1,1)$ and $(2,1)$ respectively. It turns out that X_{v_1} is ramified but X_{v_2} is not ramified as predicted by Theorem 7.2.4.

9.2 RPS primes

9.2.1 Example of Theorem 7.3.10

Let $f \in S_3(35, [2, 2])$ be the unique newform of orbit size 4. `Endohecke` will give $F = \mathbb{Q}$ and X is ramified at the ramified principal series prime 5. `Hecke` will give $v(\mu + \nu) = v(\mu + \nu + 10) = 1$, corroborating Theorem 7.3.10.

9.2.2 Examples of Theorem 7.3.15

- Let $f \in S_2(88, [2, 2, 2])$ be the unique newform of orbit size 8. `Endohecke` will give $F = \mathbb{Q}(\sqrt{2})$. In F, there is a unique prime v lying above 2, with inertia degree 1 and ramification degree 2. Using `Endohecke` and `Pari`, we get X is ramified at the ramified principal series prime v. A calculation using `Hecke` and `Pari` will give $a_2^2 + \bar{a}_2^2 = \sqrt{2}$. Hence $v(\mu + \nu) = \frac{1}{2}$ (v is a valuation such that $v(2) = 1$), supporting Theorem 7.3.15.
- Let $f \in S_4(12, [2, 2])$ be the unique newform of orbit size 4. `Endohecke` will give $F = \mathbb{Q}$ and X is unramified at the ramified principal series prime 2. Hecke will give $v(\mu + \nu) = v(\mu + \nu + 10) = 2$, corroborating Theorem 7.3.15.

9.2.3 Example of Theorem 7.3.16

Let $f \in S_2(35, [4, 2])$ be the unique newform of orbit size 4. Here $[4, 2]$ is the character ϵ, such that ϵ_5 is of order 4 and ϵ_7 is quadratic. [Qu06] will give $F = \mathbb{Q}$ and X is ramified at the ramified principal series prime 5. `Sage` will give $\mu = -5i$ and hence $m_v^+ = v(\mu + \nu + 10) = 1$, corroborating Theorem 7.3.16.

9.2.4 Example of Theorem 7.3.19

Let $f \in S_2(112, [2, 4, 2])$ be the newform with $a_2 = 1 + i$. In this case $F = \mathbb{Q}$ and $p = 2$ is a prime with RPS reduction and ϵ_2 is a character of conductor 16 with tame part of order 2 and wild part of order 4. We also note that $v_2(\mu + \nu + 4) = v_2(4) = 2$

and $\epsilon_2(-1) = -1$. [Qu06] will show that X_2 is ramified at 2. This will support Theorem 7.3.19, using the corollary after the Theorem.

9.2.5 Example of Theorem 7.3.21 for $p = 2$

Let $f \in S_2(196, [2,2])$ be the unique non-CM newform of orbit size 4. Then 2 is an RPS prime. `Endohecke` gives X is ramified at 2, $F = \mathbb{Q}$ and $a_2^2 = 2i$ and hence, $w(a_2) = w(\bar{a}_2)$. We conclude that, $v_2(\mu + \nu + 2.2) = v_2(4) = 2$ is an even integer. So m_v^+ is even. According to the notations of Theorem 7.3.21, $a = -1$ and $t = -1$. We know $(-1,-1)_2 = -1$. The result is consistent with part (1) of Theorem 7.3.21.

9.2.6 Example of Theorem 7.3.21 for p odd

Let $f \in S_3(91, [2,2])$ be the eigenform of orbit size 4. In this case we can't use part (1) of Theorem 7.3.21, because $a = \infty$. We use part (2) of Theorem 7.3.21. `Endohecke` will give there is no ramification at 7, the same program will give $p^\dagger = 3$ and Hecke will give $b = a_3^2 = -26$. According to the notations of Theorem 7.3.21, we get $t = -7$ and $m_v^+ = 2$. We calculate and get $(-26,-7)_7 = 1$, which corroborates Theorem 7.3.21.

Chapter 10

Appendix

In this section, we will give an alternative proof of Theorem 7.1.1 and Theorem 7.2.4, based on the original approach in [BG04] and [GGQ05].

Remark 10.0.1. We first remark that we can easily remove the technical condition of [BG04] and [GGQ05]. Note that in this case also the above theorems are true. We observe that if $\psi_\gamma(p) = 1$ for all $\gamma \in \Gamma_0$, then $\frac{a_p}{\sqrt{\epsilon(p)}}$ lies in F. This is because, for all $\gamma \in \Gamma$,

$$\gamma\left(\frac{a_p}{\sqrt{\epsilon(p)}}\right) = \frac{\chi_\gamma(p) a_p}{\sqrt{\epsilon(p)}^{\gamma}}.$$

But $\psi_\gamma(p) = 1$ implies that $\chi_\gamma(p) = \sqrt{\epsilon(p)}^{\gamma - 1}$, so

$$\gamma\left(\frac{a_p}{\sqrt{\epsilon(p)}}\right) = \frac{a_p}{\sqrt{\epsilon(p)}}.$$

Hence if $\psi_\gamma(p) = 1$, for all $\gamma \in \Gamma_0$, then $\frac{a_p}{\sqrt{\epsilon(p)}}$ lies in F and hence m_v is even. Since, each $t_\gamma = 1$, X_v is trivially a matrix algebra. Hence, we can remove the condition in [BG04] and [GGQ05].

10.0.7 Good primes

Lemma 10.0.2. *If L/K is a finite extension of local fields of degree $n = [L : K]$, then the restriction map below is a surjective homomorphism and we have a commutative* *diagram*

$$\begin{array}{ccc} \mathrm{Br}(K) & \xrightarrow{\mathrm{inv}_K} & \mathbb{Q}/\mathbb{Z} \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow{\scriptstyle \times n} \\ \mathrm{Br}(L) & \xrightarrow{\mathrm{inv}_L} & \mathbb{Q}/\mathbb{Z} \end{array}$$

Proof. [Se79, p. 193]. □

Lemma 10.0.3. *If L/K is a finite extension of local fields, then for all $a, b \in L$,*

$$\mathrm{inv}_K \circ \mathrm{cores}(a,b)_L = \mathrm{inv}_L(a,b)_L,$$

where $\mathrm{cores} : \mathrm{Br}(L) \to \mathrm{Br}(K)$ *is the corestriction map.*

Proof. Since the restriction map is surjective, we may write $(a,b)_L = \mathrm{res}(D)$, for $D \in \mathrm{Br}(K)$. Hence,

$$\mathrm{inv}_K \circ \mathrm{cores}(a,b)_L = \mathrm{inv}_K \circ \mathrm{cores} \circ \mathrm{res}(D) = \mathrm{inv}_K(nD) = n \cdot \mathrm{inv}_K(D).$$

By the lemma above this is equal to

$$\mathrm{inv}_L \circ \mathrm{res}(D) = \mathrm{inv}_L(a,b)_L.$$

□

Now if $b \in K^*$ and $a \in L^*$, then by [Se79, p. 209],

$$\mathrm{cores}(a,b)_L = (N_{L/K}(a), b)_K.$$

We will use this to give an alternative proof of Theorem 7.1.1. The alternative proof works for non-CM modular forms of weight $k \geq 2$.

Theorem 10.0.4 (Good reduction)**.** *Let p be a prime with* $\gcd(p, N) = 1$. *Assume $a_p \neq 0$ and $k \geq 2$. Let v be a place of F lying over p. Then X_v is a matrix algebra over F_v if and only if the finite slope m_v of* Π *at v is even.*

Proof. We will give an alternative proof only for $p = 2$; for p odd see [GGQ05].

By [GGQ05, Theorem 4.1.3] we know

$$X_v = \bigotimes_{\gamma \in \Gamma_0} (z_{n_\gamma}, t_\gamma)_v.$$

We have

$$\mathrm{cores}(z_{n_\gamma}, t_\gamma)_v = (N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}), t_\gamma)_p.$$

Write $N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}) = p^{v_p(N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}))} z'_{n_\gamma}$. Now

$$(N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}), t_\gamma)_p = (p, t_\gamma)_p^{v_p(N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}))}(z'_{n_\gamma}, t_\gamma)_p = \left(\frac{t_\gamma}{p}\right)^{v_p(N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}))},$$

since both z'_{n_γ} and t_γ are prime to $p = 2$ and so $(z'_{n_\gamma}, t_\gamma) = 1$. Indeed, since t_γ's are discriminants of number fields, so by Stickelberger's criteria, t_γ's are congruent to 0 or 1 mod 4. Since the conductor of χ_γ divides N, so the conductor of ψ_γ divides N. So t_γ divides N, hence is prime to 2, since N is odd. Hence if $p = 2$, then $t_\gamma \equiv 1 \pmod 4$, for all $\gamma \in \Gamma_0$. Hence by the formulas in [Se79, p. 212] we get $(z'_{n_\gamma}, t_\gamma) = 1$. Also, by [Se79, Corollary 4, p. 29],

$$\left(\frac{t_\gamma}{p}\right)^{v_p(N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}))} = \left(\frac{t_\gamma}{p}\right)^{f_v v_\pi(z_{n_\gamma})}.$$

Here π is a uniformiser of F_v such that $v_\pi(\pi) = 1$. So

$$\mathrm{cores}(z_{n_\gamma}, t_\gamma)_v = (N_{F_v/\mathbb{Q}_p}(z_{n_\gamma}), t_\gamma)_p = \left(\frac{t_\gamma}{p}\right)^{f_v v_\pi(z_{n_\gamma})}.$$

By Lemma 10.0.3,

$$\mathrm{inv}_L(z_{n_\gamma}, t_\gamma)_v = \mathrm{inv}_K \mathrm{cores}(z_{n_\gamma}, t_\gamma)_v = \left(\frac{t_\gamma}{p}\right)^{f_v v_\pi(z_{n_\gamma})}. \tag{10.0.1}$$

Now the theorem can be proved exactly as [GGQ05, Theorem 4.1.11].

□

10.0.8 Steinberg primes

Lemma 10.0.5. *Suppose q is a prime such that $q \nmid M$, $a_q \neq 0$ and $z_q = \frac{a_q^2}{\epsilon(q)}$. For each $\gamma \in G_F$, there exists a unique quadratic character ψ_γ such that*

$$\gamma(\sqrt{z_q}) = \psi_\gamma(q)\sqrt{z_q}.$$

In addition, $\{\psi_\gamma | \gamma \in G_\mathbb{Q}\}$ is the group of characters of $G_\mathbb{Q}$ which factorise through $\mathrm{Gal}(K/\mathbb{Q})$. *Here d is the homomorphism as defined in 5.2.2 and K is the field cut out by kernel of d.*

Proof. For each $\gamma \in G_F$, there exists a unique character χ_γ such that $(\gamma|_E, \chi_\gamma)$ is an extra twists of f.

By Lemma 7.2.2, we have $\gamma(a_p) = \chi_\gamma(p)a_p$. Hence $\gamma(a_q) = \chi_\gamma(q)a_q$, for all $q \nmid M$ and $a_q \neq 0$. In other words for all $q \nmid M$ and $a_q \neq 0$,

$$\gamma(\sqrt{z_q}) = \frac{\chi_\gamma(q)\sqrt{z_q}\sqrt{\epsilon(q)}}{\gamma(\sqrt{\epsilon(q)})}.$$

The conductor of ϵ is C which divides M. So for all n such that $(n, C) = 1$, the function $n \mapsto \frac{\sqrt{\epsilon(n)}}{\gamma(\sqrt{\epsilon(n)})}$ defines a Dirichlet character modulo C. The map defined by

$$\psi_\gamma(n) = \chi_\gamma(n)\frac{\gamma(\sqrt{\epsilon(n)})}{\sqrt{\epsilon(n)}}$$

is a quadratic character. Uniqueness of ψ_γ follows from the uniqueness of extra twists corresponding to γ. Now according to Proposition 7.2.3 and Lemma 5.2.1, $\alpha(\mathrm{Frob}_q) \equiv a_q \pmod{F^*}$ for all primes $q \nmid M$ with $a_q \neq 0$. So

$$d(\mathrm{Frob}_q) \equiv \frac{\alpha^2(\mathrm{Frob}_q)}{\epsilon(\mathrm{Frob}_q)} \equiv z_q \mod F^{*2}.$$

Suppose $q \nmid M$ with $a_q \neq 0$. Let $\sigma_q \in G_{\mathbb{Q}}$ be the Frobenius at the prime q. If $\gamma_q \in G_F$ with $z_q \in F^{*2}$ then $\psi_\gamma(q) = 1$. We deduce that ψ_γ factorise through $\mathrm{Gal}(K/\mathbb{Q})$. By Chebotarev density theorem $z_q \neq 0$ generate the group $d(G_{\mathbb{Q}})$ in F^*/F^{*2}. So the number of different ψ_γ is equal to the order of the group $d(G_{\mathbb{Q}})$ which in turn is equal to the order of the group $\mathrm{Gal}(K/\mathbb{Q})$. ☐

Theorem 10.0.6. *Suppose $q \nmid M$ and $\{\psi_i\}_{i=1}^m$ is a basis of the group of quadratic characters $\{\psi_\gamma, \gamma \in G_{\mathbb{Q}}\}$. Let $\mathbb{Q}(\sqrt{t_i})$ be the quadratic fields cut out by $\ker \psi_i$. Suppose $p_1, \ldots, p_m$ are prime numbers such that $p_i \nmid M$ and $a_{p_i} \neq 0$ with $\psi_i(p_j) = \delta_{ij}$. Then the Brauer class of X is*

$$[X] = [c_\epsilon](t_1, z_{p_1}) \cdots (t_n, z_{p_n}).$$

Proof. The existence of the prime number satisfying the property is a consequence of the Chebotarev density theorem and the fact that f is a non-CM form. Suppose $\gamma_1, \ldots, \gamma_n \in G_{\mathbb{Q}}$ are the Frobenius elements at the primes $p_1, \ldots, p_n$. By Lemma 10.0.5, the elements of $\mathrm{Gal}(K/\mathbb{Q})$ obtained by restricting γ_i's to K, form a basis of the the group $\mathrm{Gal}(K/\mathbb{Q})$. By definition

$$\sigma_i(\sqrt{t_j}) = \psi_j(\gamma_i)\sqrt{t_j} = \delta_{ij}\sqrt{t_j}.$$

We use the form of X deduced in Lemma 1 of [Qu98]. Since $d(\gamma_i) \equiv u_{p_i} \mod F^{*2}$, we obtain the formula as in the theorem. $\square$

We define $z_n = \frac{a_n^2}{\epsilon(n)}$ for all n such that $(n, M) = 1$. Then

Theorem 10.0.7. *For $k \geq 2$*

$$X = [c_\epsilon] \otimes \bigotimes_{\gamma \in \Gamma_0} (z_{n_\gamma}, t_\gamma),$$

up to Brauer equivalence. Here z_{n_γ}, t_γ are as defined in [GGQ05].

Proof. The proof follows exactly in the same way as [GGQ05, Theorem 4.1] by applying Theorem 10.0.6 and replacing N by M. $\square$

Using the above theorem we will give an alternative proof of the Theorem 7.2.4.

Theorem 10.0.8 (Steinberg). *Let $N_p = 1$, $C_p = 0$ and let v be a prime of F lying above p. Then X_v is a matrix algebra over F_v if and only if $m_v = [F_v : \mathbb{Q}_p](k-2)$ is even.*

Proof. Note that $[c_\epsilon]_v = 1$ if and only if $\epsilon_v(-1) = 1$. Since f has a Steinberg reduction at the prime p, so $\epsilon_v(-1) = 1$ and hence $[c_\epsilon]_v = 1$. So using Theorem 10.0.7,

$$X_v = \bigotimes_{\gamma \in \Gamma_0} (z_{n_\gamma}, t_\gamma)_v.$$

Now

$$(z_{n_\gamma}, t_\gamma)_v = \left(\frac{t_\gamma}{p}\right)^{f_v \cdot v(z_{n_\gamma})}$$

by equation (9.0.1). So if $\psi_\gamma(p) = \left(\frac{t_\gamma}{p}\right) = 1$ for all $\gamma \in \Gamma_0$ then $X_v = 1$ as desired. Suppose on the other hand the subset T of the set $\{t_\gamma | \gamma \in \Gamma_0\}$ consisting of those t_γ for which $(\frac{t_\gamma}{p}) = -1$ is nonempty. Write the elements of T as $t_1, t_2, \ldots, t_m$ with $m \geq 1$. Define distinct prime r_j for $j = 0, \ldots, r_{m-1}$ as follows: set $r_0 = p$ and define r_j as in [GGQ05]. Now corresponding to $t_i \in T$ we choose n_i just as in [GGQ05]. Notice that the only difference is that now the n_i's are prime to M, instead of N. So the proof will follow exactly as in [GGQ05] and noting that $a_p^2 = \epsilon_M(p)p^{k-2}$ by [Mi89, Theorem 4.6.17]. $\square$

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