FRACTAL TRANSFORMATIONS

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Abstract. A strange game of soccer is used to introduce transformations and fractals. Low information content geometrical transformations of pictures are considered. Fractal transformations and a new way to render pictures of fractals are introduced. These ideas have applications in digital content creation.

1. A Fascinating Soccer Game

1.1. **See Debbie Kick!** Imagine that you are a great soccer player, with perfect ball control and perfect consistency. Wherever the ball is on the soccer pitch you can kick it so it lands half-way between where it was and a corner. And the ball comes to a dead stop right where it lands. You can always do this.

That is how Alf, Bert, Charlie and Debbie are. They play soccer on the soccer pitch ABCD in Figure 1. Debbie always kicks the ball when she gets to it first. She kicks it from X to the midpoint between D and X. See Debbie kick!

Alf acts in the same way, except that he kicks the ball half-way to A. And Bert kicks the ball half-way to B. You can guess where Charlie kicks the ball, when she gets to it first.

Who kicks the ball next is entirely random. It makes no difference where the players are on the pitch or who kicked it last. You can never reliably predict who is going to kick it next. The sequence of kickers might be determined by a random sequence of their initials: DABACBADAABCDCBACAADDBAC... .

The game goes on forever.

To watch this awful game of soccer is rather like watching four chickens in a farmyard chasing after a breadcrust. There is no team play, and no goals are ever scored. But at least no-one eats the ball.

What actually happens to the ball is fascinating. Almost certainly it jumps around all over the pitch forever, going incredibly close to all of the points on the pitch. If you mark any little circle on the pitch, eventually the ball will hit the ground inside the circle. Sometime later it will do so again. And again and again. The soccer ball marks out the pitch, going arbitrarily close to every point on it. We say that the ball travels "ergodically" about the pitch.

Alf, Bert, Charlie and Debbie represent "transformations" of the soccer pitch. Alf represents the transformation that takes the whole pitch into the bottom left quarter. Let ■ denote the soccer pitch. Then

$$Alf(\blacksquare) = \textit{Quarter A of Soccer Pitch},$$

the quarter at the bottom left. Think of Alf "kicking" the whole pitch into a quarter of the pitch.

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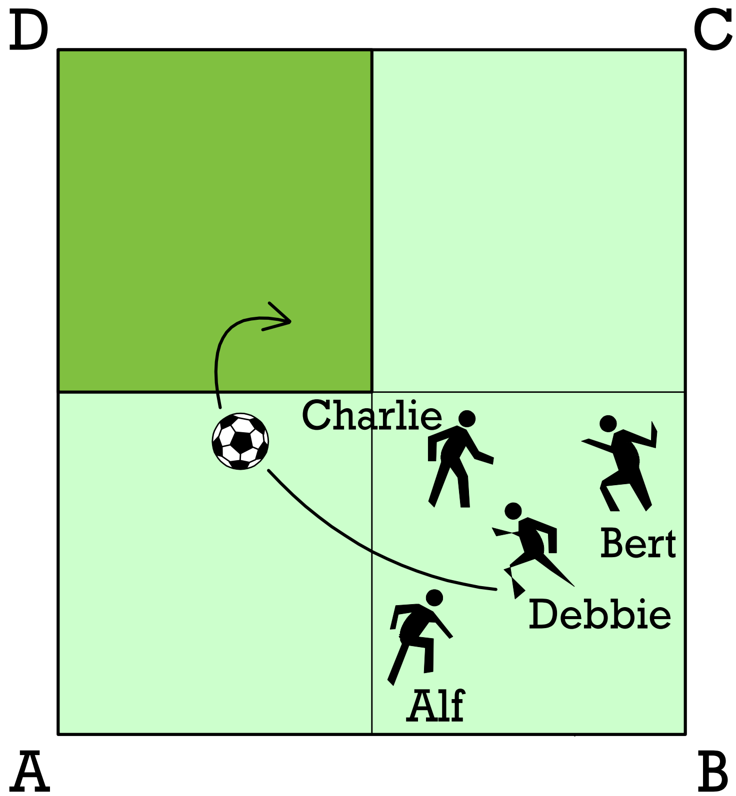

Figure 1. Debbie has just kicked the ball half-way towards D. If the ball was at X, then it lands at the midpoint of the line segment XD.

Similarly *Bert*(■) = *Quarter B of Soccer Pitch,* the quarter at the bottom right. Also *Charlie*(■) = *Quarter C of Soccer Pitch,* and *Debbie*(■) = *Quarter D of Soccer Pitch,* the quarter at the top left.

These transformations actually provide an "equation" for the soccer pitch:

$$\blacksquare = Alf(\blacksquare) \cup Bert(\blacksquare) \cup Charlie(\blacksquare) \cup Debbie(\blacksquare).$$

It says that that the pitch ■ is made of "four transformed copies of itself". It says that the pitch is the union of the four quarter pitches, just as the U.K. is the union of England, Northern Ireland, Scotland and Wales.

For us each player is a transformation or a function, providing a unique correspondence between each location on the pitch (from where the ball is kicked) and another location on the pitch (the point where the ball lands).

1.2. **Charlie Hurts Her Leg.** What has all this got to do with fractals? Lots, as we shall see.

Suppose Charlie gets kicked in the shin and cannot play. Only Alf, Bert and Debbie kick the ball. Their sequence of kicks is still random, for example starting out in the order DBAABADBADDABBAD... .

The game begins with the "kick-off", with the ball in the middle of the pitch.

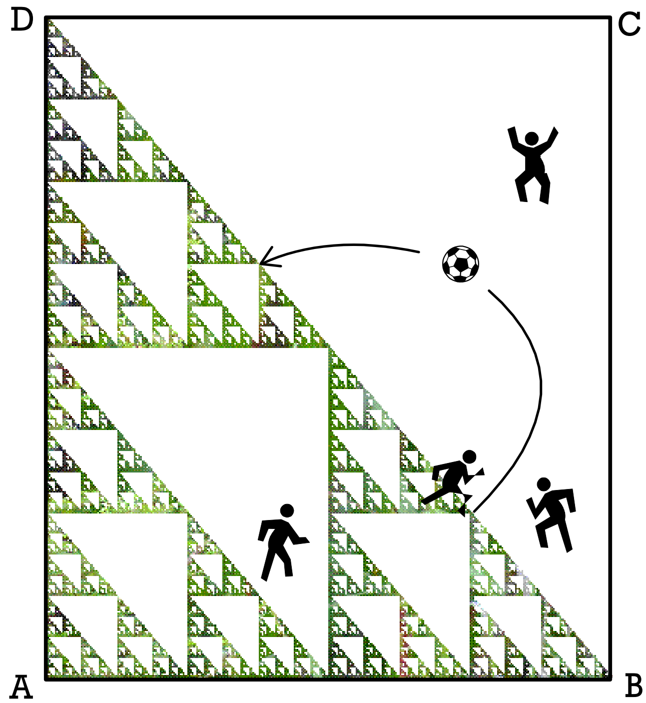

FIGURE 2. Charlie hurts her leg and can't play for a while. The ball travels "ergodically" on the Sierpinski Triangle ABD.

Where now does the ball go? To find out we cover it with greeny-black ink. Now the ball makes a dot on the white pitch every time it lands. It will make a picture while the game is played.

Amazingly, the picture it makes, almost always, looks like the one in Figure 2. This is called "The Sierpinski Triangle ABD". We denote it by ▲. With Charlie out of the game, the ball travels ergodically about ▲. Mark a small circle centered at any point on ▲. The ball will visit this circle over and over again.

The Sierpinski Triangle ▲ is a bona fide fractal. Notice how it is "made of three transformed copies of itself". One copy lies in the top left quadrant, one in the lower left quadrant, and one in the lower right quadrant. It appears now that the soccer players "kick" ▲ into smaller parts of ▲.

Our equation this time reads

$$\blacktriangle = Alf(\blacktriangle) \cup Bert(\blacktriangle) \cup Debbie(\blacktriangle).$$

This is the equation for ▲, the Sierpinski Triangle ABD.

1.3. **The Players Change How They Kick.** Charlie is back in the game.

Alf, Bert, Charlie and Debbie are fed up that they have not scored any goals. So they change the way they kick the ball.

Each player kicks in his or her own special way, methodically and reliably. Alf now kicks the ball so that it always lands in a certain quadrangle. Bert kicks the ball so that it lands in another quadrangle. Charlie and Debbie kick the ball into their own quadrangles. You can see the soccer pitch and the four quadrangles in Figure 3.

Alf kicks straight lines into straight lines in this way. Let P, Q and R be three points that lie on a straight line. He kicks the ball from P to $Alf(P)$, from Q to $Alf(Q)$ and from R to $Alf(R)$. Then $Alf(P)$, $Alf(Q)$, and $Alf(R)$ lie on a straight line! For example, if the ball is on one of the sidelines of the soccer pitch, Alf kicks it to land on one of the sides of his quadrangle. If the ball lies at the center of the soccer pitch, Alf kicks it so that it lands at the intersection of the two diagonals of his quadrangle. Alf is a precision kicker.

If Alf kicks the ball from two different points, it lands at two points closer together than the starting points. We say that Alf represents a "contractive" transformation.

The other players act similarly; the only differences are the quadrangles that they kick to.

Where does the ball go this time? The selection of the order in which the players kick the ball is again random. The game goes on for eons. The players never get bored or tired. They are immortal. And the ball marks green or black points on the white pitch wherever it lands, after the first year of play. The resulting pattern of dots forms the fern in Figure 3. We call this fern F. The soccer ball travels ergodically on F.

Our equation this time reads

$$F = Alf(F) \cup Bert(F) \cup Charlie(F) \cup Debbie(F).$$

The fern F is the union of "four transformed copies of itself".

What is amazing is that the transformations represented by the players, and this type of equation, define one and exactly one picture, the fern in this case, the Sierpinski Triangle in the previous case, and the whole soccer pitch in the first case. Change the way the players kick the ball and you will change the picture upon which eventually the ball "ergodically" travels.

1.4. **Infernal Football Schemes (IFS).** Many different fractal pictures and other geometrical objects can be described using an IFS. The letters stand for Iterated Function System but here we pretend they mean Infernal Football Scheme.

An IFS is made up of a soccer pitch and some players each with their own special way of kicking. Each player must always kick the ball from the pitch to the pitch according to some consistent rule. And each player must represent a contractive transformation, must "kick" the pitch into a smaller pitch. We continue to call the players Alf, Bert, Charlie and Debbie, but there may be more or less players.

Then there will always be a unique special picture, a "fractal", a collection of dots on the white pitch, that obeys the equation

$$fractal = Alf(fractal) \cup Bert(fractal) \cup Charlie(fractal) \cup Debbie(fractal).$$

We call this picture a fractal, but it might be something as simple as a straight line, a parabola, or a rectangle. This picture can be revealed by playing random soccer as in the above examples.

An example of a fractal made using an IFS of three transformations is shown in Figure 4. Can you spot the transformations?

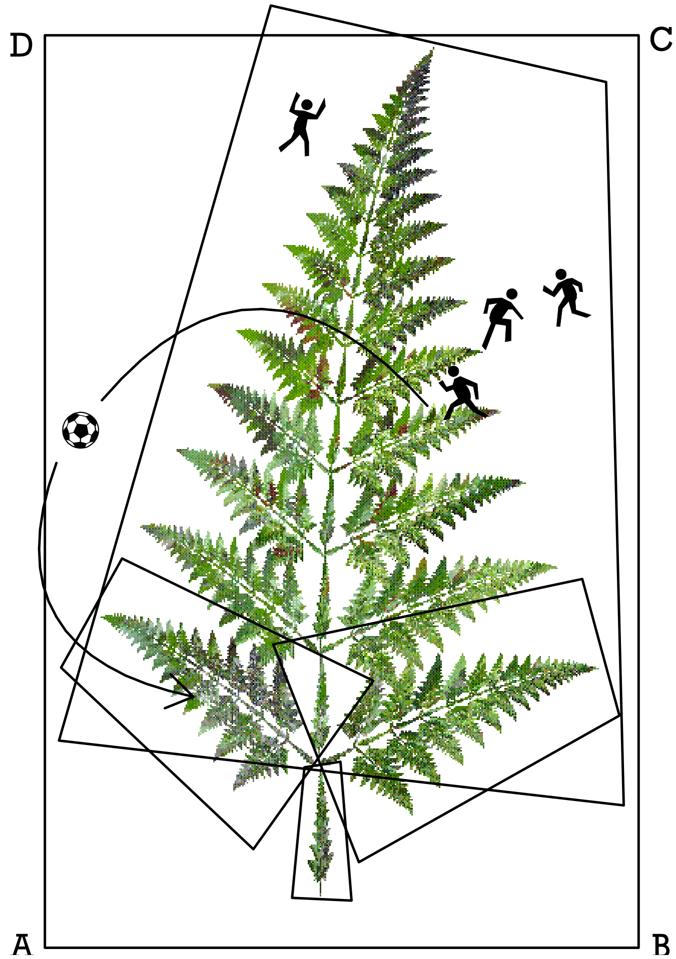

FIGURE 3. Each soccer player now represents a projective transformation. One transformation corresponds to each of the quadrangles inside the pitch ABCD. The places where the ball lands makes a picture of a fern. The ball travels "ergodically" upon the fern.

In this way many fractals and other geometrical pictures can be encoded using a few transformations. Once one knows an IFS for a particular fractal one knows its secret. One knows that despite its apparent visual complexity, it is really very simple. One can make it and variations of it, over and over again. One can describe it with infinite precision.

Given a picture of a natural object, such as a leaf, a feather, or a mollusc shell, it is interesting to see if one can find an IFS that describes it well. If so, then one would have an efficient way to model and compare some biological specimens.

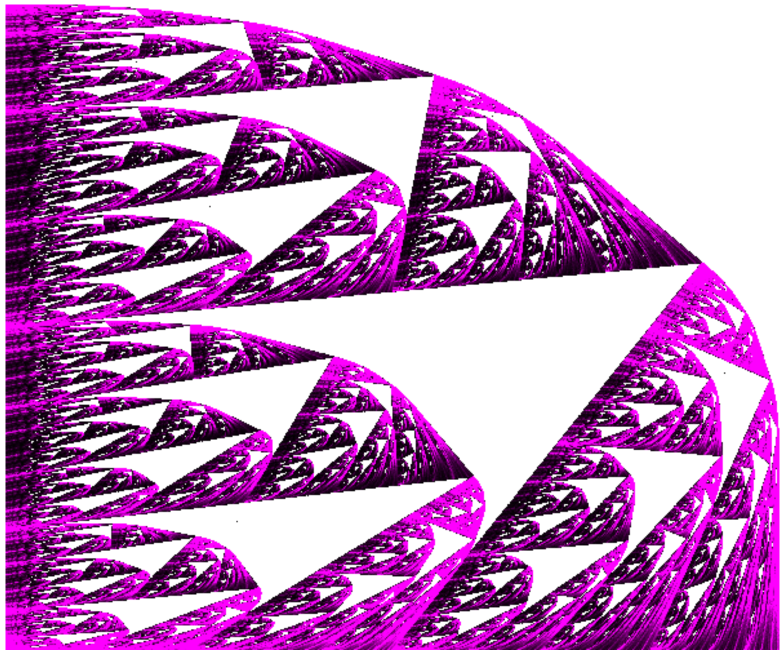

FIGURE 4. Fractal made with three projective transformations. It is rendered in black and shades of purple. Can you spot the transformations?

2. GEOMETRICAL TRANSFORMATIONS

2.1. **Simple Transformations.** So far we have shown that to understand fractals one needs to understand transformations. But which transformations?

Transformations can be very complicated. They may involve bending one part of space, squeezing another, and be expressed using elaborate formulas that take pages to write down.

But one of the goals of fractal geometry is to describe pictures of natural objects in an efficient manner. Clearly, if one makes a description of a fern using an IFS, and the transformations that are used are very complicated, then little is gained in the way of simplification. So we seek simple transformations – ones that are easy to write down, explain, and understand.

One source of simple transformations is classical Geometry, which involves the study of invariance properties of collections of transformations. For example, Euclidean Geometry studies properties of geometrical pictures that remain unchanged when elementary displacements and rotations are applied to them. The distance between a pair of points is invariant under a Euclidean transformation. So is the angle between a pair of straight lines.

Similarity Geometry involves the transformations of Euclidean Geometry as well as similarity transformations, so called because they magnify or shrink pictures by fixed factors. Many well know fractals may be expressed with similarity transformations, for example the Sierpinski Triangle ▲ and the soccer pitch ■. But Projective

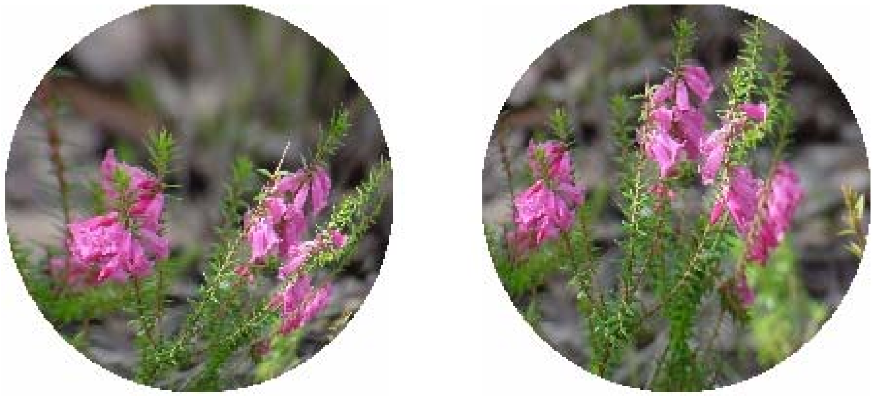

FIGURE 5. Two circle-preserving projective transformations of a picture of an Australian heath.

Geometry provides a much richer simple set of transformations for describing natural shapes and forms.

2.2. **Projective Transformations.** Projective transformations are of the type represented by Alf, Bert, Charlie and Debbie when they started kicking the soccer pitch into quadrangles. Given any pair of quadrangles, one can find always find a projective transformation that converts one into the other, even making the corners go to specified corners.

Projective transformations arise naturally in optics, in explaining perspective effects, and play an important role in modern Physics. They seem to appear naturally when one searches for order and pattern in the arrangements of matter and light in the natural physical world. They are indeed natural in the following way. Suppose you take a wonderfully sharp photo of a tree full of flat leaves, some bigger, some smaller, but all of the same shape. Then all of the whole leaves in the photo will be (almost) projective transformations of one another.

When you watch television from a difficult angle, the images that fall on your retinas are in effect projective transformations of what they would have been if you viewed face-on. But, within reasonable limits, the mind/eye system copes with the distortion. "Recognisability" is an invariance property of projective transformations.

Projective transformations have the property that they often transform images of plants and leaves into recognisable images of plants and leaves. This is illustrated in Figures 5 and 6. Note how the straight lines of the veins in the beech leaf are transformed into other straight lines in Figure 6.

Images of the real world contains much repetition. Often nearby leaves look similar for biological and physical reasons. And the local weather pattern seems to clump clouds into regions of similar looking ones. This similarity and repetition may be specified with projective transformations.

Projective transformations take points to points and straight lines to straight lines. Even more remarkable, they map conic sections into conic sections. That is, if you make a picture of circles, ellipses, parabolas, hyperbolas and straight lines, then apply a projective transformation, the resulting picture will also be made of

FIGURE 6. Two different ellipse-preserving projective transformations of a beech leaf. The straight lines along which the veins nearly lie are preserved.

these same shapes. That is *not* to say that circles are transformed to circles, ellipses to ellipses, parabolas to parabolas, or hyperbolas to hyperbolas.

Are the coloured circular and elliptical cells on the wings of some butterflies more easily recognised by other butterflies, or predator species, because of this invariance?

2.3. **Mobius Transformations.** Mobius transformations are another type of transformation that is "simple". They are often used to describe fractals and, in a different way than projective transformations, seem to have some natural affinity with real world images. They have the remarkable property that they transform any circle into either a circle or a straight line. This is illustrated in Figure 7. They also preserve the angles at which lines in pictures cross, as can be seen by examining the bike frames in Figure 7.

In certain situations they transform patterns of fluid motion, represented by streamlines, into other possible fluid motion patterns. They also transform pictures of fish into other pictures of fish, as illustrated Figure 8.

Mobius transformations are the basic elements of Hyperbolic Geometry. They were used by Esher in some of his graphic designs, including ones with natural elements such as fish.

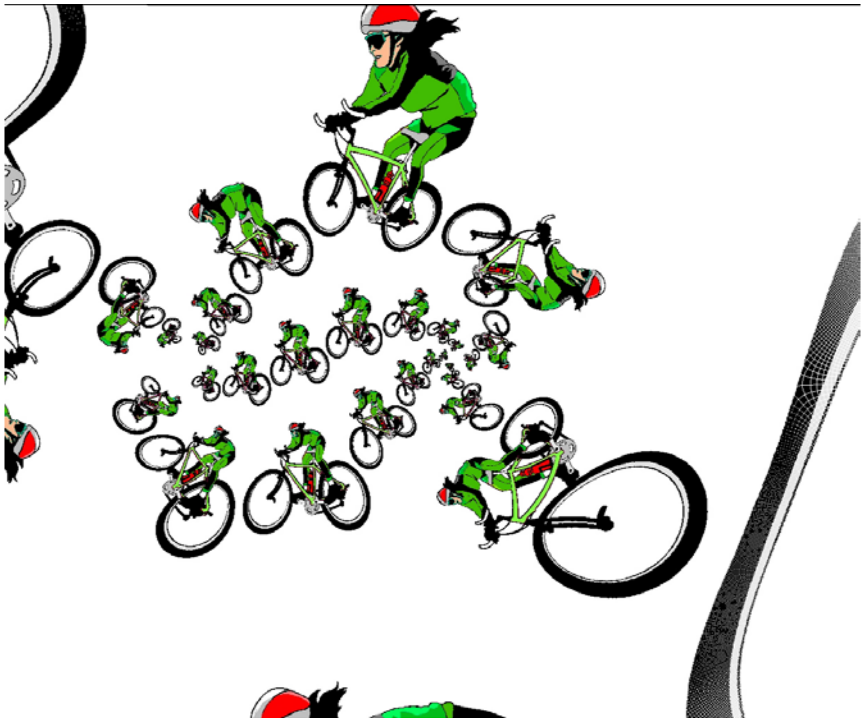

FIGURE 7. A single Mobius transformation is applied over and over again to a picture of a person on a bike. The images are massively distorted one from another, but the wheels are all round, except near the edges of the picture, where some precision has been lost. Also angles are preserved. Each bicycle frame is a curvilinear triangle with the same three angles.

In Figure 9 we illustrate the Circumscribed Fish Theorem. This is one of many such observations. It illustrates that Geometry applies not only to triangles, circles, and straight lines, but to all sorts of other pictures as well.

2.4. **The Cost of Describing Transformations.** Even "simple" transformations can be complicated if they involve "constants" which require lots of digits to express them accurately. To explain this point, let us look briefly at some "formulas" for simple transformations. The details of these formulas, other than the fact that they contain "constants", need not concern us.

Transformations in two-dimensional space may be represented using Cartesian coordinates (x, y) to represent points. A projective transformation can be expressed with a formula such as

$$Alf(x,y) = \left(\frac{ax+by+c}{gx+hy+1}, \frac{dx+ey+f}{gx+hy+1}\right)$$

where $a, b, c, d, e, f, g, h,$ and k, are numbers, the "constants", such as $a = 1.023$, $b = 7.1$, $c = -0.00035$, $d = 100$, $f = 9.1$, $g = 34.9$,and $h = 17.3$. Similarly, a

FIGURE 8. The same Mobius transformations is applied over and over again to a single fish, to produce this double spiral of fish. Notice that although the fish are massively distorted, they all look fish-like.

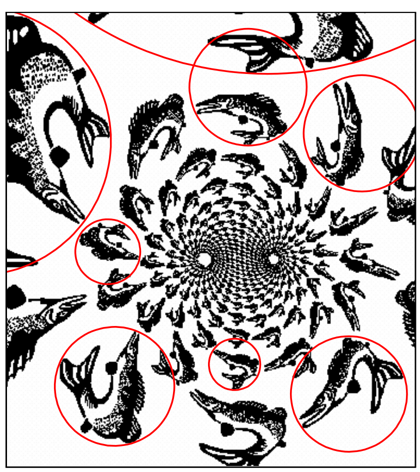

FIGURE 9. Illustration of the Circumscribed Fish Theorem. Although the fish in Figure 8 look quite various, they have the following property. Draw the smallest circle around each fish, such that the circle touches the fish in at least three points. Then each fish touches its circle with the same parts of its body.

Mobius transformation can take the form

$$Bert(x,y) = \frac{(a+\sqrt{-1}b)+(c+\sqrt{-1}d)(x+\sqrt{-1}y)}{(e+\sqrt{-1}f)+(g+\sqrt{-1}h)(x+\sqrt{-1}y)},$$

which uses complex arithmetic and also uses eight constants.

If we know that each constant is an integer between -127 and 128, which can be expressed using one byte of data (since $2^8 = 256$) then each of these transformations requires 8 bytes of information to express it, one byte for each constant. These are in, an obvious way, “simpler” transformations than ones in which each constant requires two bytes of information. And both of these possibilities are much simpler, that is, able to be expressed much more succinctly, than if each constant were a decimal number with random digits, such as $a = 1.79201434953...$, going on forever.

Now one might say that all of these extra digits are without significance. But in fractal geometry they are very significant, because fractal geometry is about details! Tiny changes in the constants will usually lead to tiny changes in a fractal built using the transformations. But when the fractal is under the microscope, so to speak, and one is zoomed in to look at fine detail, and a tiny change is made in a coefficient, the part of the fractal one is looking at may completely disappear – not only has its form changed, but it has moved out of the field of view.

For the application of fractals to image compression, for example, it is important that the transformations can be expressed succinctly, and that the constants involved do not require lots of digits. We say that such transformations have “low information content”.

One of the important features of fractals and other geometrical pictures is that they are simple to describe. Thus it is appealing to use low information content transformations, quite generally.

3. More Soccer : Fractal Transformations Are Discovered

3.1. **Alan, Brenda, Celia and Doug Start a Second Game.** We can use fractal soccer, using simple projective “kicks”, to make a new kind of transformation. We call these new transformations "fractal transformations". They too are of low information content. But they can transform pictures in very surprising ways, very differently from projective and Mobius transformations.

In Figure 10 two games of soccer are played at the same time. The game on the left is the same as in Figure 1, discussed at the start of this article. But in the game on the right, Doug kicks the pitch into the small rectangle at the top left, but Brenda kicks into the large rectangle at the bottom right. Similarly Celia kicks towards C and Alan kicks towards A, but the quadrangles that they kick to are of different dimensions than in the first game.

Alan, Brenda, Celia and Doug are copy cats. They watch the game on the left. When Alf kicks the ball, Alan kicks the ball in his game; when Bert kicks the ball, Brenda kicks the ball; when Charlie kicks the ball, then so does Celia; and when Debbie kicks the ball, so does Doug – he’s been watching her closely. But of course Alf, Bert, Charlie and Debbie stay on the pitch on the left, while Alan, Brenda, Celia and Doug stay to their soccer pitch on the right.

Now put a picture on the soccer pitch on the left, a great big one. This is the “Before” picture. To illustrate this, there is a big red and green fish painted on the left-hand pitch in Figure 11. Let the game begin. Then after each pair of kicks, one on each pitch, a dot is painted on the right-hand pitch at the spot where the ball

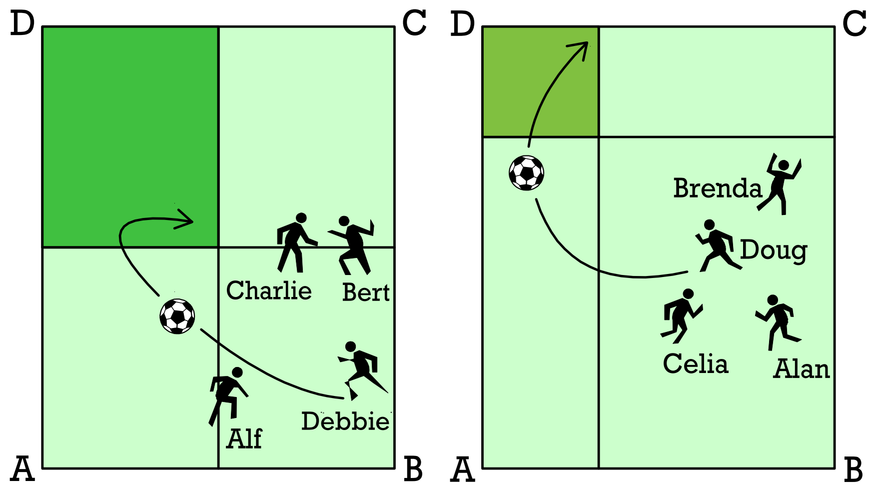

FIGURE 10. Alan, Brenda, Celia and Doug start up a second game. They are copy cats: Doug kicks the ball whenever Debbie does, Alan kicks the ball whenever Alf does, Celia kicks when Charlie does, and Bert copies Brenda. But they kick the ball a bit differently!

has landed, in the same colour as the point on the left-hand pitch where the ball on that pitch has landed. The result, after thousands and thousands of kicks, is shown in Figure 11, on the right-hand pitch. This is the "After" picture. The After picture is an amazingly deformed version of the Before picture, it is stretched greatly in some places and only a little in others. We call this a fractal transformation.

But the transformation between the Before and After pictures is fundamentally no more complicated than the transformations that are used to make it, the transformations represented by the players. Although the player transformations are quite smooth and regular, the fractal transformation is non-uniform and irregular.

In Figure 12 we show a prettier fish, before applying a fractal transformation to it. In Figure 13 we show the same fish after transformation. Another before and after pair is shown in Figure 14. Such effects clearly have applications in digital content creation.

In Figure 15 we show a before and after pair of pictures of Australian heath flowers. It is interesting to compare this Figure with Figure 5, where the two images are related by a circle-preserving projective transformation. In the present case the images are related by a rectangle-preserving fractal transformation (the rectangular picture frame is preserved). Under projective transformation, points that are collinear are mapped into collinear points. Under the present fractal transformations collinear points *parallel to the picture frames* are preserved.

3.2. **Colour Stealing.** Essentially the same algorithm to the one we have described in the previous section may be applied to render rich colouring to diverse IFS fractals. Here we show how a fern is coloured by this new algorithm. See

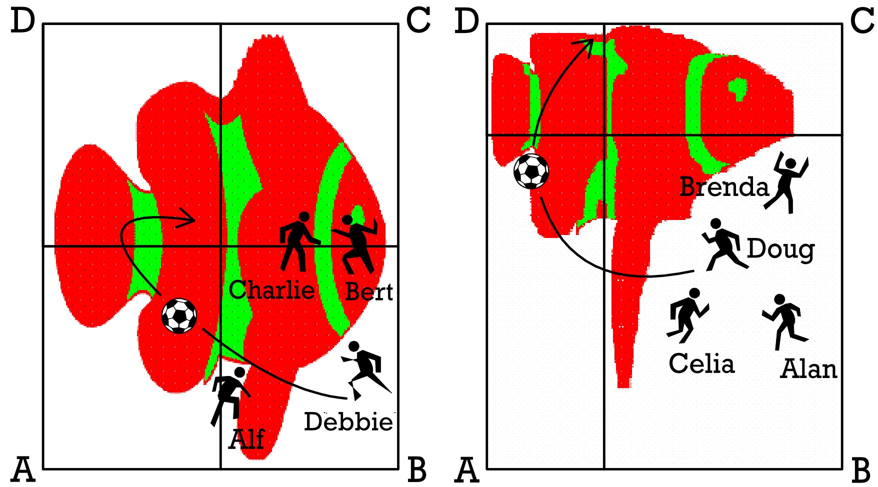

FIGURE 11. The fish is transformed by the two soccer games.

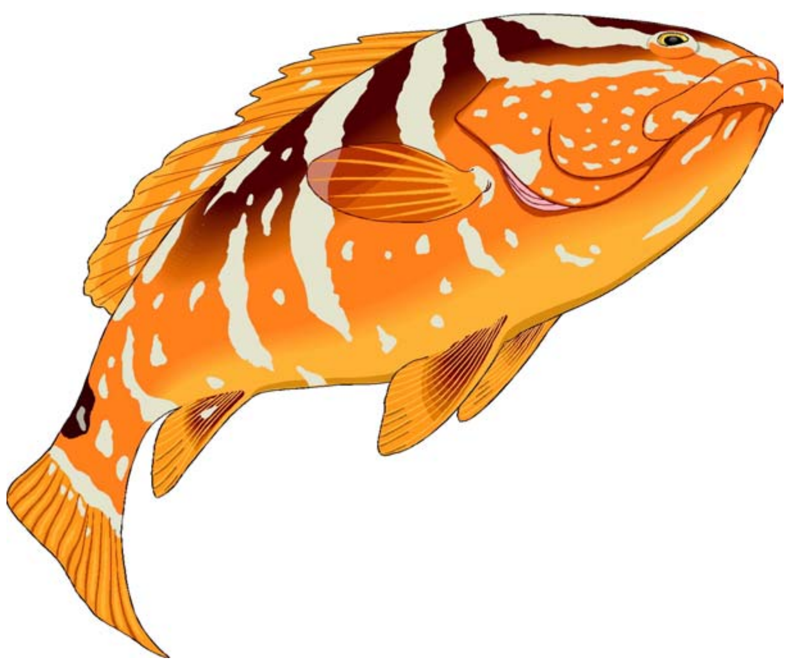

FIGURE 12. Before. Compare with Figure 13.

Figure 16. The main difference is that on the right-hand pitch the IFS that makes the fractal fern is used.

On the left-hand soccer pitch with the colourful photo on it, Alf, Bert, Charlie and Debbie play a game of random soccer as in Figure 1. Each player simply kicks

FIGURE 13. After.

FIGURE 14. These two pictures of leaves and sky are related via a fractal transformation.

the ball to the quarter pitch labelled with his or her initial. The players in the game on the right are Alan, Brenda, Celia and Doug. They kick the ball into quadrangles, as in Figure 3. Alan kicks the ball when Alf does, Brenda kicks it when Bert does, Celia kicks when Charlie does and Doug kicks when Debbie does. A while after kick-off, each time after both balls have been kicked, the spot where the ball lands in the right-hand game is marked with a dot the same color as the point where the ball lands in the left-hand game. The result is a fractal fern, painted with the colours of the picture on the left.

Figure 17 contrasts two copies of the same fern coloured by a fractal transformation of two different pictures, samples of which are shown at left. Notice that there need be no particular relationship between the size of the picture from which

FIGURE 15. These images of Australian heath are related by a rectangle-preserving fractal transformation. Compare with Figure 5.

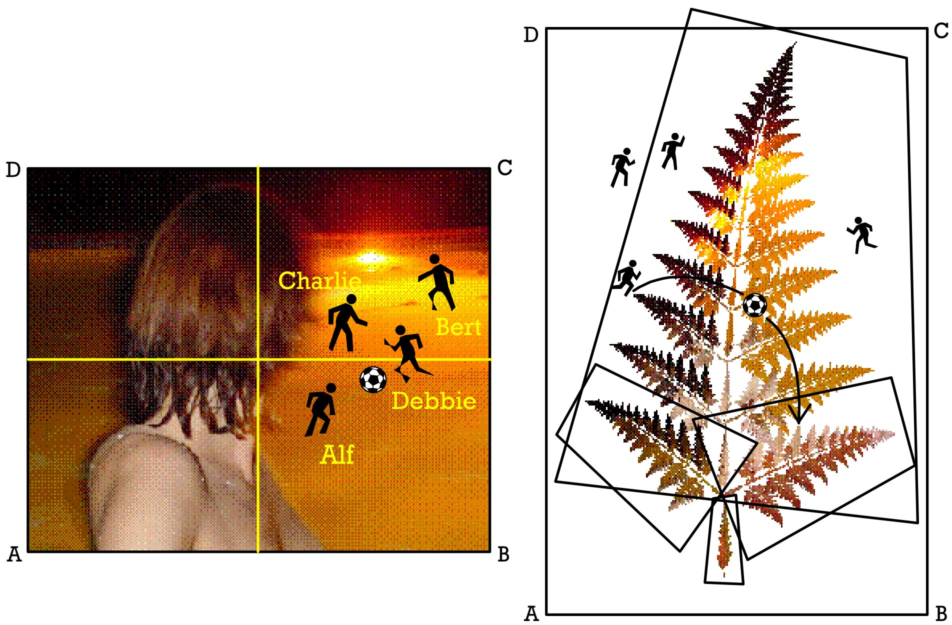

FIGURE 16. On the left-hand soccer pitch with the colourful photo on it, Alf, Bert, Charlie and Debbie play a game of random soccer. The players in the game on the right are Alan, Brenda, Celia and Doug. They kick the ball into quadrangles, as in Figure 3. **A**lan kicks the ball when **A**lf does, **B**renda kicks it when **B**ert does, and so on. Each time after both balls have been kicked, the spot where the ball lands in the right-hand game is marked with a dot the same color as the point where the ball lands in the left-hand game. The result is a painted fractal fern.

Figure 17. The same fractal fern is rendered using two different input images, parts of which shown.

the colour is stolen, and the target image, the fern in this case, that is painted with the stolen colours.

4. Comments, Background References and Further Reading

The ideas of fractal transformations and colour stealing using random iteration, the main goals of this article, are, so far as we know, entirely new and are presented for the first time here. What is actually going on in both cases is that a mapping is set up between two IFS attractors using the underlying code space, which is the same for both IFSs. This means that a fractal transformation between two "just-touching" IFS attractors is very nearly continuous, which explains why the colourings of the fern, for example, have a nice consistency from one frond to the next and do not vary too abruptly.

The random soccer game is a novel way of presenting geometrical transformations, the random iteration algorithm and IFS theory. Our goal has been to minimize the use of formulas and to try and rely on geometrical intuition and non-mathematical wording. The random iteration algorithm was first described formally, in the context of fractal imaging, in [1], although the seeds of this idea are mentioned in the early work of Mandelbrot, [9] p.198. This algorithm is also known as the "Chaos Game", but we think it may attract a wider audience if it is explained in terms of soccer.

The mathematical theory of IFS was originally formulated by John Hutchinson [6]. It was popularized and developed by one of us and coworkers as well as many others, see for example [5] and [8]. You can read about the application of IFS to image modelling, how to make fractal ferns and leaves, and about the underlying code space, in [2]. The application of IFS to image compression is described in [3] and in [7]. A lovely book about fractals made with Mobius transformations is [10].

The future holds another exciting discovery which you may read about in [4] and also hopefully in 2004 in a book entitled *Superfractals*.

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