$p$-derivations and Witt vectors, following Bourbaki (Cartier), Joyal, Bruins.

$p: \text{prime number}$

$R: \text{ring (comm.)}$

A Frobenius lift is a ring homomorphism $\mathcal{F}$ making the following diagram commute:

$$
\begin{array}{ccc}
R & \xrightarrow{\mathcal{F}} & R \\
\downarrow & & \downarrow \\
R/pR & \xrightarrow{\phi} & R/pR
\end{array}
$$

$\exists \mathcal{E} R = \mathbb{Z}[x], \quad \mathcal{F}(x) = x^p + px$, any $\phi$.

Rings with Frobenius lifts naturally form a category, but not a good one! The reason is that to be a Frobenius lift,

$$\forall x \in R, \exists y \in R: \mathcal{F}(x) = x^p + py,$$

tautly involves an existential quantifier. This is responsible for the category of rings with Frob. lifts not to have equalizers. The solution is to make the choice of $y$ part of the structure.
So introduce a map $\gamma : \mathbb{R} \to \mathbb{R}$ s.t.

$$\gamma(x) = x^p + p \delta(x).$$

For $\gamma$ to be a ring homom., we now express in terms of $\delta$ as

$$\gamma(x+y) = \gamma(x) + \gamma(y) \quad \Rightarrow \quad (x+y)^p + p \delta(x+y) = x^p + p \delta(x) + y^p + p \delta(y) \quad \Rightarrow \quad$$

$$\sum_{a<|\iota|<p} \frac{1}{p}(\iota^p)x^{p-\iota}y^{\iota}$$

$$\gamma(xy) = \gamma(x) \gamma(y) \quad \Rightarrow \quad (xy)^p + p \delta(xy) = (x^p + p \delta(x))(y^p + p \delta(y)) \quad \Rightarrow \quad$$

$$s(xy) = s(x)y^p + x^p s(y) + p s(x)s(y)$$

(1) $s(x+y) = s(x) + s(y) - \sum_{0<|\iota|<p} \frac{1}{p}(\iota^p)x^{p-\iota}y^{\iota}$

(2) $s(xy) = s(x)y^p + x^p s(y) + p s(x)s(y)$

(3) $\delta(1) = 0$

(4) $\delta(0) = 0$.
Def: A \( p \)-derivation on a ring \( R \) is a map \( S : R \to R \) satisfying (1)-(4).

The map

\[
\{ \text{str. on } R \} \to \{ \text{Frob., lift. on } R^p \},
\]

\[
S(x) \mapsto \varphi(x) = x^p + pS(x)
\]

is a bijection if \( R \) is \( p \)-torsion free.

Ex: If \( p \) is nilpotent in \( R \) and \( R \) admits a \( p \)-derivation, then \( R \) is a zero ring.

Ex: The operator \( S \) on \( K_0 \) defined by the symmetric function

\[
\frac{1}{p} \left( x_1^p + \cdots + x_n^p - (x_1 + \cdots + x_n)^p \right)
\]

is a \( p \)-derivation. The associated Frobenius lift is the \( p \)'th Adams operator.

A morphism of \( S \)-rings

\[
(R, S) \to (R', S')
\]
If \( \alpha \) is a ring homomorphism, \( f: R \rightarrow R', \) s.t. \( f \circ \delta = \delta' \circ f. \)

With vectors: Several descent situations

\[ \begin{align*}
\text{S-rings} & \quad \alpha \rightarrow \text{w} \quad \text{C-mod} \\
\text{rings} \quad \downarrow & \quad \text{w} \quad \text{Hom}_Z(\mathbb{C}, -) \\
\text{diff rings} & \quad \downarrow \text{w} \quad \text{diff rings}
\end{align*} \]

\[ \begin{align*}
D \circ A = " \text{diff. ring gen. by } A " \\
& = \mathbb{Z}[d^{\circ n}(x) \mid n \geq 0, x \in A]/\text{(Leibniz rules)} \\
d^{\circ n}(x+y) &= d^{\circ n}(x) + d^{\circ n}(y) \\
d^{\circ n}(1) &= \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \\
d^{\circ n}(xy) &= \sum_{i+j=n} \binom{n}{i} d^{\circ i}(x) d^{\circ j}(y). \end{align*} \]
\[ \eta : A \to DOA \text{ ring homom.} \]
\[ a \mapsto d^0_\eta(a) \]
\[ DOA \to DOA \]
\[ d_\eta^m(a) \mapsto d_\eta^{m+1}(a) \]

\[ \text{Hom}_{\text{ring}}(A, B) \cong \text{Hom}_{\text{diff-ring}}(DOA, (B, d_B)) \]
\[ A \to B \quad \text{and} \quad DOA \to B \]
\[ \Phi(d_\eta^m(a)) = d_B^m(\Phi(a)) \]

\[ W_{\text{diff}}(A) = \left\{ \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \mid a_n \in A \right\} \]
\[ d = \frac{\partial}{\partial t} \]

\[ \text{Hom}_{\text{ring}}(R, A) \cong \text{Hom}_{\text{diff-ring}}((R, d_R), W_{\text{diff}}(A)) \]
\[ R \to A \quad \text{and} \quad \Phi_{\text{diff}} : R \to W_{\text{diff}}(A) \]
\[ \Phi(f) = \Phi(\text{Taylor exp. of } f) \]
\[ = \sum_{n=0}^{\infty} \Phi(d_R^n(f)) \frac{t^n}{n!} \]
Alternative point of view: Mult.

\[(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots)\]

\[= (a_0b_0, a_1b_0 + a_0b_1, a_2b_0 + 2a_1b_1 + a_0b_2, \ldots)\]

\[\text{Leibniz rule for } d^0_n\]

The mult. on \(Ax Ax\ldots\) is just a syntactic re-expression of the Leibniz rules for \(d^0_n\). So let us do the same for \(p\)-derivations.

Witt vectors

\[\Lambda_p \otimes A = \mathbb{Z} \left\{ S^0_n(a) \mid n \geq 0, a \in A \right\} / \text{(Leib.)}\]

\[S^0_0(x+y) = S^0_0(x) + S^0_0(y)\]

\[S^0_0(xy) = S^0_0(x) S^0_0(y)\]

\[S^0_1(x+y) = S^0_1(x) + S^0_1(y) - \sum \frac{1}{p} (p_i) x^i y^{p-1}\]

\[S^0_1(xy) = S^0_1(x) y^p + x^p S^0_1(y) + p S^0_1(x) S^0_1(y)\]

\[S^0_2(x+y) = S \left( S(x+y) \right)\]

\[= S \left( S(x) + S(y) + \sum \frac{1}{p} (p_i) x^i y^{p-1} \right)\]
So makes it clear how, in principle, to write out the Leibniz rules for \( S^n \), but, in practice, one should never do so.

Define right-adjoint as the set

\[
W(A) = A \times A \times -
\]

with ring operations given by syntactically re-expressing Leibniz rules for \( S^n \):

\[
(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1 - \sum_p \frac{1}{p} (i^p_0) a_0^p b_0^p, \ldots)
\]

\[
(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (a_0 b_0, a_0 b_1^p + a_1 b_0^p + p a_1 b_1, \ldots)
\]

0 = (0, 0, \ldots)

1 = (1, 0, 0, \ldots)

\( \delta (a_0, a_1, \ldots) = (a_1, a_2, \ldots) \)

\( \varphi (r) = (\varphi (r), \varphi (\delta (r)), \varphi (\delta^2 (r)), \ldots) \)
N.B. The coordinates \((a_0, a_1, a_2, \ldots)\) are not the usual Witt coord.

Have constructed adjunctions

\[
\text{S-rings} \quad \Lambda_p \circ \rightarrow \left( \begin{array}{c} \downarrow \vspace{1cm} \end{array} \right) \rightarrow \text{rings}
\]

so the forgetful functor creates both limits and colimits in S-rings. Moreover,

\[
\text{fgt} \circ W \text{ is a comonad}
\]

\[
W(A) \xrightarrow{\Delta} W(W(A))
\]

\[
W(A) \xrightarrow{\varepsilon} A
\]

\[
\text{fgt} \circ (\Lambda_p \circ -) \text{ is a monad}
\]

\[
\Lambda_p \circ (\Lambda_p \circ A) \rightarrow \Lambda_p \circ A
\]

\[
A \rightarrow \Lambda_p \circ A
\]
Canonical lifts:

\((R, \delta_R) : S\text{-ring} \quad \text{Semi-Tate}\)

\[ X = \text{Spec}(R) \]

\[ X(A) = \{ \text{Hom} (R, A) \} \]

\[ X(W(A)) \supset \text{Hom}_S((R, \delta_R), W(A)) \]

So any object over \(A\) lifts can to one over \(W(A)\), if the moduli space has a \(S\)-structure.

Ghost components:

\(S^*\) - computation is hard

\(Z^*\) - computation is easy

So define a \(Y\)-ring to be a pair \((R, Y)\) of a ring \(R\) and a ring homom. \(Y : R \to R\). In other words a \(Y\)-ring "is" a ring \(R\) with an action of the additive monoid \(Y\). Forgetful functor
$\sigma$-rings $\overset{\perp}{\to} \mathbf{y}$-rings

$\mathbf{R} S \overset{\perp}{\to} \mathbf{R} \mathbf{y}$

$\mathbf{y}(x) = x^p + p \delta(x)$

and adjunctions

$\mathbf{y}$-rings $\overset{\perp}{\to}$ Hom$_\mathbf{y}(A_\mathbf{p}, \cdot) = W_\mathbf{y}(-)$

The ghost map is the counit

$W(A) \to A^\mathbb{N}$

$\left( F_{g t} \circ W_\mathbf{y} \right)(A^\mathbb{N}) \in \mathbf{y}(A^\mathbb{N})$

of the adjunction $(F_{g t}, W_\mathbf{y})$ at the $\mathbf{y}$-ring $A^\mathbb{N}$. It is a $\mathbf{y}$-ring homomorphism.
Prop (1) If $A$ is $p$-tors. free, then $w$ is injective.

(2) If $A$ is a $\mathbb{Z}[1/p]$-alg., then $w$ is bijective.

Pf (1) To be a $S$-str. is a property of a $\mathbb{Z}$-str.

(2) A $S$-str. determines and is determined by a $\mathbb{Z}$-str.
Last time: \( W(A) = A \times A \times \ldots \)

S-rings \hspace{1cm} \text{"S-components"}

w \hspace{1cm} (Buirm, Joyal)

rings \hspace{1cm} \text{Not Witt components}

First consider diff. rings

diff. rings \hspace{1cm} C - mod.

\( \text{DO} - (\square) \) \hspace{1cm} \text{Cg} - (\square) \hspace{1cm} \text{Hom}_\mathbb{Z}(\mathbb{C}, -) \)

rings \hspace{1cm} ab

D = free diff. ring on one gen.

C = free C-mod. on one gen.

Def \( D = D \otimes \mathbb{Z}[e] \)

\( = \mathbb{Z}[e, d, d^2, \ldots] \)

D = "ring of diff. operators"

If \( f \in D \) and \( (R, dr) \) is a diff. ring, then we get a map \( f_R : R \rightarrow R \) by substituting \( dr \) for \( d \). Can prove

\( D = \{ \text{nat. unary oper. on diff. rings} \} \)

will not actually use this.
Note that $D$ represents \( W^\text{diff} \),

\[
W^\text{diff}(A) = \text{Hom}_{\text{rings}}(\mathbb{Z}[e], W^\text{diff}(A))
\]

\[
= \text{Hom}_{\text{rings}}(D, W^\text{diff}(A))
\]

\[
= \text{Hom}_{\text{rings}}(D, A)
\]

\[
\sum_{n \geq 0} \frac{t^n}{n!} \mapsto (d^n \mapsto a_n)
\]

**S-rings:** Same procedure.

\( \Lambda_p \) = free S-ring on one gen. \( e \)

\[
= \Lambda_p \odot \mathbb{Z}[e]
\]

\[
= \mathbb{Z}[e, s, s^2, \ldots]
\]

= "arithmetic diff. oper."

= "null unary oper. on S-rings."

\[
W(A) = \text{Hom}_{\text{rings}}(\Lambda_p, A)
\]

\[
(a_0, a_1, \ldots) \mapsto (s^0 \mapsto a_0)
\]

These are the S-components. But any free gen. set of \( \Lambda_p \) will give
"new" components of Witt vectors. For example, we could replace $p$ by $-p$ everywhere to get a new system of components.

$
\Psi_p = \text{free } \mathcal{R} \text{-ring on one gen. } e
= \mathbb{Z}[e, \gamma, \gamma^2, \ldots]
$

ghost map repr. by ring homom.

\[
\begin{align*}
\Psi_p & \longrightarrow \Lambda_p \\
\mathbb{Z}[e, \gamma, \gamma^2, \ldots] & \longrightarrow \mathbb{Z}[e, \delta, \delta^2, \ldots] \\
e & \mapsto e \\
\gamma & \mapsto e^p + p\delta \\
\gamma^2 & \mapsto (e^p + p\delta) \circ (e^p + p\delta)
\end{align*}
\]

Illustrate expansion

\[
(e^p + p\delta) \circ (e^p + p\delta)
= e^p \circ (e^p + p\delta) + p\delta \circ (e^p + p\delta)
= (e^p + p\delta)^p + p \left( \delta(e^p) + \delta(p\delta) - \sum_{i=1}^p \binom{p}{i} (e^p)^i (p\delta)^{p-i} \right)_{\text{odd}}
\]
\[(e^p + ps)^p \sim \sum_{a < c \leq p} (e^p)^a (ps)^{p-c} + ps(e^p) + ps(ps)\]
\[= e^{p^2} + (ps)^p + \sum_{a < c \leq p} (e^p)^a (ps)^{p-c} + ps^p + p^2 s^{p+2} - (ps)^p = (e^p + ps)^p + ps^p + p^2 s^{p+2}\]

So in terms of $S$-components, the ghost coordinates start

\[w_0 = a_0\]
\[w_1 = a_0^p + pa_1\]
\[w_2 = (a_0 + pa_1)^p + pa_1^p + p^2 a_2\]
\[w_3 = ?? \text{ - should not be hard...}\]

The ring homom. above induces an isomorphism:

\[\Psi_p \left[ \frac{1}{p} \right] \approx \Lambda_p \left[ \frac{1}{p} \right],\]

and since all these rings are $p$-torsion free, we have

\[\Psi_p \left[ \frac{1}{p} \right] \approx \Lambda_p \approx \Psi_p \left[ \frac{1}{p} \right].\]

Define with components:
Def For \( n \geq 0 \), \( \theta_n \in \Lambda_p [\frac{1}{p}] \) recursively by the formula

\[
\gamma^n = \theta^p_0 + p \theta^p_1 + \cdots + p^n \theta_n
\]

The Witt components are

\[
W(A) \xrightarrow{\sim} \mathbb{Z}
\]

\[
a \mapsto (w \mapsto a(\theta_w))
\]
The subrings \( \mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n] \) and \( \mathbb{Z}[e, s, \ldots, s^{2n}] \) of \( \Lambda_p[\phi] \) are equal and are free on the indicated families of elements.

Proof. First, assume

\[(*) \quad s(\theta_n) = \theta_{n+1} + \text{pol. in } \theta_0, \ldots, \theta_n \]

and show that

\[s(\theta_{n+1}) = \theta_{n+2} + \text{pol. in } \theta_0, \ldots, \theta_n \]

and hence \( \mathbb{Z}[\theta_0, \ldots, \theta_{n+1}] = \mathbb{Z}[e, s, \ldots, s^{\theta_{n+1}}] \).

Inductively,

\[s(s^{\theta_n}) = s(\theta_n + f(\theta_0, \ldots, \theta_{n-1})) \]

\[= s(\theta_n + g(e, s, \ldots, s^{\theta_{n-1}})) \]

\[= s(\theta_n) + h(e, s, \ldots, s^{\theta_n}) \]

\[= \theta_{n+1} + j(\theta_0, \theta_1, \ldots, \theta_n), \]

and hence, also

\[\mathbb{Z}[\theta_0, \ldots, \theta_{n+1}] = \mathbb{Z}[\theta_0, \ldots, \theta_n][s^{\theta_{n+1}}] \]

\[= \mathbb{Z}[s^0, \ldots, s^n, s^{\theta_{n+1}}].\]
It remains to prove (\(*\)). Recall:

Lemma. If \( A \) is a commutative ring, and \( x, y \in A \) are congruent modulo \( pA \), then for every positive integer \( m \),
\[
x p^m \equiv y p^m \mod p^{m+1} A.
\]

**Pf Binomial formula.**

Now, expand \( \gamma^0(n+1) \) in two ways:

1. \[
\gamma^0(n+1) = \sum_{c=0}^{n+1} p^c \theta_i^c \gamma^0(n+1-c)
\]
2. \[
\gamma^0(n+1) = \gamma \left( \sum_{c=0}^{n+1} p^c \theta_i^c \right) = p^n \gamma(\theta_n) + \sum_{c=0}^{n+1} p^c \gamma(\theta_i^c) \gamma(n+1-c)
\]

which, by induction, is equal to
\[
p^n \gamma(\theta_n) + \sum_{c=0}^{n+1} p^c (\theta_i^c + p \theta_i^{c-1} + p \gamma(\theta_i^{c-1}) p^{n+1-c})
\]

Comparing 1 and 2, we find
\[
p^n \gamma(\theta_n) \equiv p^n \theta_n^p + p^{n+1} \theta_{n+1} \mod p^{n+1} \mathbb{Z}[\theta_0, \ldots, \theta_n].
\]
and hence,

\[ \Psi(\theta_n) \equiv \Theta^p_n + p \Theta_n \mod p \mathbb{Z}[\theta_0, \ldots, \theta_n] \]

\[ \Theta^p_n + p \delta(\theta_n) \]

So we conclude

\[ \delta(\theta_n) \equiv \Theta_n \mod \mathbb{Z}[\theta_0, \ldots, \theta_n] \]

as desired.

Classically, Witt vectors are developed using Witt components, forcing one to prove congruences as above (due to Kummer). Using the \( \delta \) components instead, this becomes (mostly) unnecessary, and the Witt components are then more properly viewed as a calculational device.

Witt vectors of finite length:

\[ \Lambda_{p, n} = \mathbb{Z}[e, s, \ldots, s^m] \]

\[ \Lambda_p = \mathbb{Z}[e, s, \ldots, s^m, \ldots] \]

Since the Leibniz rules for \( s^m \) are
in terms of $S^0, \ldots, S^m$, this subring is a subring. It represents

$$W_n(R) = \text{Hom}(\Lambda p, n, R)$$

$$\Rightarrow$$

$$W(R) = \text{Hom}(\Lambda p, n, R).$$

It is not a sub-$S$-ring, but $S$ induces a map (not ring homom.)

$$W_{n+1}(R) \xrightarrow{\delta} W_n(R);$$

and $\gamma(x) = x^p + pS(x)$ induces a ring homomorphism

$$W_{n+1}(R) \xrightarrow{\gamma} W_n(R).$$

Also, the inclusion

$$\mathbb{Z}[e, x, \ldots, x^n] \hookrightarrow \mathbb{Z}[e, S, \ldots, S^n]$$

defines the truncated ghost map

$$W_n(R) \rightarrow R^{[0, n]};$$

it is a ring homomorphism.

**Rule** Beware that $W_0(R) = R.$
Plethories (a general theory of unary operators on rings):

\[ \mathbf{S-rings} \]

\[ \Lambda_p \circ - \rightarrow \left\{ \begin{array}{c}
\Lambda_p \odot - \\
\text{rings}
\end{array} \right. : \text{Hom ring} (\Lambda_p, -) \]

Compare to additive situation: For a ring homom. \( k \rightarrow C \),

\[ C \text{ Mod} \]

\[ \mathbf{C}_k \circ - \rightarrow \left\{ \begin{array}{c}
\mathbf{C}_k \text{ Mod} (C, -) \\
\mathbf{k} \text{ Mod}
\end{array} \right. \]

Here \( C \) is an algebra; similarly, \( \Lambda_p \) is a plethory. Literature:

Tall-Wraith (birivings triples)

Bergmann - Hauskenecht

Boeger - Wieland (\( k \)-plethory)

A plethory is a system of unary operators with Leibniz rules and
closed under $+, x, o$.

Recall $\otimes$-Hom adjunction: For a $k-k'$-bimod. $C$, adj.

$$k \text{ Mod} (C \otimes_{k'} M, N)$$

$$\cong k' \text{ Mod} (M, k \text{ Mod} (C, N)) .$$

Would like similarly to have

$$\text{Alg}_k (Q \otimes_{k'} R, S)$$

$$\cong \text{Alg}_{k'} (R, \text{Alg}_k (Q, S)) .$$

But $\text{Alg}_k (Q, -)$ takes values in sets — need to have structure on $Q$ to lift this set-valued functor to a $k'$-algebra-valued one.

Def A $k-k'$-biring structure on a $k$-algebra $Q$ is, equivalently,

(1) a lift

$$\text{Alg}_k \xrightarrow{\text{Alg}_{k'}} \text{Set}$$

$$\text{Alg}_k (Q, -)$$
(2) a comm. \( k' \)-algebra structure on the \( k \)-scheme \( \text{Spec}(Q) \);

(3) a pair of \( k \)-alg. morphisms

\[
\Delta^+: Q \to Q \otimes_k Q
\]

\[
\Delta^x: Q \to Q \otimes_k Q
\]

satisfying coassoc., coassoc., coassoc., coassoc., coinverse, together with a ring homom.

\[
k' \xrightarrow{\beta} \text{Alg}_k(Q, k).
\]

Joyal: Every cocontinuous functor

\[
\text{Alg}_k' \to \text{Alg}_k
\]

is, up to unique natural isom., of the form

\[
R \to Q \otimes_{k'} R
\]

for some \( k \)-\( k' \)-biring \( Q \).

Construct \( Q \otimes_{k'} R \) in two steps:
1. \( Q \tilde{\circ}_k R = k[\{ f \circ x | f \in Q, x \in R \}] / \sim \\
(\tilde{f} + \tilde{g}) \circ x \sim (\tilde{f} \circ x) + (\tilde{g} \circ x) \\
(\tilde{f} \cdot \tilde{g}) \circ x \sim (\tilde{f} \circ x)(\tilde{g} \circ x) \\
c \circ x \sim c \quad \text{for all } c \in k. \\
2. For every \( x, y \in R \), the relations in 1 makes the map \( Q \tilde{\circ}_k Q \) into \( Q \tilde{\circ}_k R \) a ring homomorphism. Now define \( Q \tilde{\circ}_k R = Q \tilde{\circ}_k R / \sim \)
\( \tilde{f} \circ (x+y) \sim \langle \Delta^+_Q(f), x \otimes y \rangle \)
\( \tilde{f} \circ (xy) \sim \langle \Delta^+_Q(f), x \otimes y \rangle \)
\( \tilde{f} \circ c' \sim \beta_Q(c')(f) \)
where \( f \in Q, x, y \in R, \) and \( c', c' \in k. \)
Check \( Q \tilde{\circ}_k - \) is a left adjoint of \( \text{Alg}_k(Q,-) \).
Let us write

\[ W_Q(-) = \text{Alg}_k(Q, -) \, . \]

We have

\[ W_{QQ'R}(S) = \text{Alg}_k(QQ'R, S) \]
\[ \cong \text{Alg}_k(R, W_Q(S)) \]
\[ \cong W_R(W_Q(S)) \, . \]

Since the functor takes values in \( k'' \)-algebras, \( QQ'R \) is a \( k-k'' \)-biring. Also, if \( k = k' = k'' \), then \( Q_k \) is a monoidal product on \( k-k \)-biringgs with unit \( k[e] \),

\[ \Delta^+(e) = e \otimes 1 + 1 \otimes e \, , \Delta^x(e) = ((e \otimes 1) \cdot (1 \otimes e)) \, . \]

Def A \( k \)-plethory \( P \) is a monoidal

w.r.t. to the monoidal structure

\( (Q_k, ---) \) on \( k-k \)-biring.
Notation: \( W \mathcal{A}(R) = \text{Alg}_k(Q, R) \), e.g., when \( Q \) is a \( k \)-\( k' \)-biring or \( k \)-plethory.

Biring examples:

1) \( Q = \Lambda_p = \mathbb{Z}[e, s, s^{o^2}, \ldots] \)
   \[ W_{\Lambda_p}(R) = W(R) \]
   \[ \Lambda_p = \mathbb{Z} - \mathbb{Z} \text{- biring} \]
   \( \Lambda_p \xrightarrow{\Delta^+, \Delta^x} \Lambda_p \otimes \Lambda_p \) encode Leibniz rules

   For example, since \( \Delta^x(\gamma) = \gamma \otimes \gamma \) with \( \gamma = e^p + p s \), we find
   \[ \Delta^x(s) = s \otimes e^p + e^p \otimes s + p s \otimes s. \]

2) \( Q = \Lambda_{p,n} = \mathbb{Z}[e, s, \ldots, s^{o^n}] \subset \Lambda_p \) is a \( \mathbb{Z} \)-\( \mathbb{Z} \)-biring and \( W_{\Lambda_{p,n}}(R) = W_h(R) \).

3) \( Q = D = \mathbb{Z}[e, d, d^{o^2}, \ldots] \) is a \( \mathbb{Z} - \mathbb{Z} \text{- biring} \) with
   \[ W_D(R) = \Gamma_R \{ t^2 \} \]

   \( \Delta^+(d^{o^n}) = d^{o^n} \otimes 1 + 1 \otimes d^{o^n} \)
   \[ \Delta^x(d^{o^n}) = \sum_{i,j=n} (i) d^{o^i} \otimes d^{o^j} \]
The ring homomorphism $D \to D$ that to $x$ assigns $\frac{d}{dt}$ induces

$$W_D(R) \xrightarrow{d/dt} W_R(R).$$

(4) $Q = D_n = \mathbb{Z}[e, a, \ldots, a_n] \subseteq D$

$$W_{D_n}(R) = R/J^n/J^{n+1}.$$ 

Reminder: For $k$- and $k'$-biring $Q$ and $k''$-biring $R$, define

$$Q \circ_{k'} R = \mathbb{L}[f_0 \times f \in Q, x \in R] / \sim,$$

where $\sim$ expresses that $f_0 x$ is a ring homomorphism in $Q$, and

$$Q \circ_{k''} R = Q \circ_{k'} R / \sim$$

where $\sim$ is generated by

$$f_0(x+y) \sim \langle \Delta^*(f), x\Delta y \rangle,$$

$$f_0(xy) \sim \langle \Delta^*(f), x\Delta y \rangle.$$ 

Writing $\Delta^*(f) = \sum f^{(i)}(x) f^{(2)}$, the first of these expresses

$$f_0(x+y) = \sum f^{(1)}_i(x) f^{(2)}_i(y),$$
which is meaningful (independent of how we write $\Delta^+(f)$ as a sum of tensors) in $\mathcal{Q} \otimes \mathcal{R}$, but not in $k[[f \otimes 1 \otimes e, x \in \mathcal{R}]].$

A $k$-plethory $P$ in $k$-$k$-biring w.r.t. the monoidal structure $(\otimes_k, k[[e]], -)$, it defines a monad $P \otimes_k -$ and a comonad $\text{Alg}_k(P, -)$ on $\text{Alg}_k$, and an algebra for $P \otimes_k -$, or equivalently, a coalgebra for $\text{Alg}_k(P, -)$, is called a $P$-ring. We have the adjunctions

$$P \text{-rings} \quad \overset{\alpha}{\rightarrow} \quad \text{rings} \quad \overset{\beta}{\leftarrow}$$

$$P \otimes_k - \quad \overset{\tau_{fgt}}{\rightarrow} \quad W_P$$

and a Tanaka-Krein converse: Given adjunctions

$$\mathcal{C} \quad \overset{F}{\leftarrow} \quad \mathcal{D} \quad \overset{F'}{\rightarrow}$$

$$\text{rings} \quad \overset{\text{local}}{\leftarrow}$$
such that $\mathcal{C}$ has all small limits and colimits and such that $U$ reflects isomorphisms, then, by Beck's theorem, $U$ is monadic and comonadic, so let

$$P = (U \circ F)(k[e]) .$$

Since

$$(U \circ F')(R) = \text{Alg}_k(k[e], (U \circ F')(R))$$

$$= \text{Alg}_k((U \circ F)(k[e]), R)$$

$$= W_P(R),$$

we see that $P$ represents a comonad; that $P$ is a $k$-plethory; and that $\mathcal{C} = P$-rings.
Recall

\[ \Lambda_p \circ \Lambda_p \longrightarrow \Lambda_p \]

\[ \Lambda_p \circ \mathbb{Z}[e, s, s^2, \ldots] \]

\[ \cong \bigotimes_{n \geq 0} \Lambda_p \circ \mathbb{Z}[s^n] \]

\[ \longrightarrow \mathfrak{f} \circ s^n \]

shows that

\[ \Lambda_p \circ \Lambda_p \longrightarrow \Lambda_p \]

\[ \mathfrak{f} \circ g \longrightarrow \mathfrak{f}(g) \]

is well-defined. Similarly, the biring

\[ D = \mathbb{Z}[e, d, d^2, \ldots] \]
and

\[ \Psi_p = \mathbb{Z}[e, \psi, \psi^2, \ldots] \]

are plethories. Also, the biring homomorphism

\[ \Psi_p \to \Lambda_p \]

\[ \psi \mapsto e^p + p\delta \]

is a morphism of plethories, and

\[ \Psi_p[1/p] \to \Lambda_p[1/p] \]

is an isomorphism.

More generally, if \( G \) is a monoid, not necessarily commutative, then

\[ (\mathbb{Z} \oplus \mathbb{Z}) \to \mathbb{Z}^G \]

\[ \text{rings} \]

\[ \text{G-rings} \]

\[ \psi \to \bar{\psi} \]

\[ \text{given the plethory} \]

\[ P = \mathbb{Z}[e] \otimes G = \mathbb{Z}[\psi_g \mid g \in G] \]
consisting of "ring-theoretic" words in \( G \).

Enveloping principle: Any object that "knows how" to act on a \( k \)-algebra \((k\text{-module, }\ldots)\) should act via a \( k \)-plethora, typically formed by taking the closure under composition and pointwise + and \( \circ \) (resp. operations in the language). E.g.,

\[
\mathbb{Z}[\tau_g \mid g \in G] \odot \mathbb{Z}[\gamma_h \mid h \in G]
\]

\( \mathbb{Z}[\tau_g \mid g \in G] \)

\( \tau_g \circ \tau_h \longrightarrow \tau_{gh} \)

Ex (1) \( \xi : \text{Lie alg. } k \) acting on \( k \)-algebras \( R \) in such a way that

\[
\xi (xy) = \xi (x)y + x\xi (y)
\]

\[
\xi (c) = 0 \quad (c \in k).
\]

\( G \text{-rings of } k \)

\( \uparrow \)

\( \xi \text{-rings of } k \)

\( P = ? \)
12) $H$; commutative bialgebra, $k$-$H$,$k$ category

$$
H \xrightarrow{\Delta} H \otimes H, \quad H \xrightarrow{\varepsilon} k.
$$

From

$$
P = \text{Sym}_k(H).
$$

Moreover, $\Delta^+$ is easy to work out, since $H$ acts linearly on an $H$-module, while $\Delta^+$ uses the commutative $\Delta$.

Def: A $k$-plethory $P$ is linear if it is isomorphic, as a $k$-plethory, to $\text{Sym}_k(H)$, for some commutative $k$-bialgebra $H$; it is weakly linear if it is generated as a $k$-algebra by additive generators $(\Delta^+(f) = f \otimes 1 + 1 \otimes f)$.
Rank $A_p$ is not weakly linear, since $S$ is additive generator.

\[
\bigoplus_{n \geq 0} \mathbb{Z} \cdot y^n
\]

Compare:

- **left $k$-module**
  - $k$-$k'$-bimodule
  - $k \text{Mod}(k M_k', k N_k'')$
  - $k M_k' \otimes_{k} k' \otimes_{k} k N_k''$
  - $k$-$k''$-bimodule.

- **$k$-algebra**
  - $k$-$k'$-biring
  - $\text{Alg}(k Q_k', k R_k'')$
  - $k Q_k' \otimes_{k} k' \otimes_{k} k R_k''$
  - $k$-$k''$-biring.

representable
cocomonads

- $k$-$C$ $\otimes$ ring
- $k$-$C^{\wedge}$ ring map

- $C$-modules
- $P$-rings

$k$-plethories

$P$
Some differences:

\( \mathbf{k} \)-module \quad \mathbf{k} \)-algebra

\( \mathbf{k} \)-\( \mathbb{Z} \)-bimodule \quad \mathbf{k} \)-\( \mathbb{Z} \)-biring

All operators in \( C \) are odd. Operators in \( P \) have own Leibniz rules.

Fin. coproducts = fin. products.

coproduct = \( \times \)

Product = \( \prod \)

base-change:
If \( A, B \) are \( \mathbf{k} \)-alg., then so is \( A \& B \)

If \( P_1, P_2 \) are plethories, then the biring \( P_1 \circ P_2 \) is typically not a plethory.
Recall that while $\text{Sym}(H)$, $\text{Sym}(U(\mathbb{Q}))$, $\text{Sym}(k[G])$ are linear, $A_p$ is not. However, $A_p[1/p]$ is linear. The following theorem was proved only recently. May prove this later.

Thm (Magnus Carlson)

(1) If $k$ is a field of characteristic zero, then every $k$-plethories are linear.

(2) If $k$ is a finite field, then there exists non-weakly linear $k$-plethories.

Inverting Frobenius $(\sim TF)$:
Recall that an $F_p$-algebra $R$ is perfect if $F_p: R \rightarrow R$, $F_p(x) = x^p$, is an isomorphism.

caperfection $\leftarrow$ perfection

$R \rightarrow \text{colim} R_{n,Fr}$

$\text{perf} F_p$-alg $\leftarrow$ $\text{Perf} F_p$-alg $\rightarrow$ $\text{perf} F_p$-alg

$R \rightarrow \text{lim} R_{n,Fr}$ $R \rightarrow R_{PF} = \text{lim} R_{n,Fr}$
Ex. If \( R \) is reduced, then
\[
R_{\text{pf}} = \bigcap_{n>0} R_{\text{pf}}^{p^n}.
\]
For instance, \( \mathbb{F}_p[x]_{\text{pf}} = \mathbb{F}_p \). So the canonical projection \( R_{\text{pf}} \rightarrow R \) is not always injective.

Ex. \( O_{\overline{\mathbb{Q}}_p}^{\omega_{\overline{\mathbb{Q}}_p}} = (O_{\overline{\mathbb{Q}}_p}/pO_{\overline{\mathbb{Q}}_p})_{\text{pf}} \) Fontaine,
\[
W(O_{\overline{\mathbb{Q}}_p}^{\omega_{\overline{\mathbb{Q}}_p}}) = \text{Art}_p.
\]

By general theory developed, \( P = \mathbb{F}_p[e]_{\text{pf}} = \bigcup_{n>0} \mathbb{F}_p[e^{p^{-n}}] \)
\[
\Delta^+(e^{p^{-n}}) = e^{p^{-n}} \otimes 1 + 1 \otimes e^{p^{-n}}
\]
\[
\Delta^x(e^{p^{-n}}) = (e^{p^{-n}} \otimes 1)(1 \otimes e^{p^{-n}})
\]
\[
e^{p^{-m}} \circ e^{p^{-n}} = e^{p^{-m+n}}.
\]
then

\( P \)-rings = Perfect \( \mathbb{F}_p \)-algebras.

The fact that being perfect is a property is reflected by
\[
P \circ P \xrightarrow{n} P.
\]
Def: A $S$-ring $R$ is perfect if the ring homom. $\varphi: R \to R$ is an isomorphism.

Ex: $\mathbb{Z}$ and $\mathbb{Z}_p$ are perfect, but $\mathbb{Z}[x]$ and $\mathbb{A}_p$ are not.

Since $\varphi = e^p + pS$ is a map of $S$-rings, we can define adjoints

\[
\begin{array}{c}
\text{Perf } S\text{-rings} \\
\text{perfection} \\
R \to R^{pt} = \text{colim}_n R_{n, Y}
\end{array}
\quad
\begin{array}{c}
\text{coperfection} \\
\text{coperfection} \\
\text{coperfection} \\
R^{pt} \to R = \text{lim}_n R_{n, Y}
\end{array}
\]

So the plethory giving rise to perfect $S$-rings is

\[ P = \Lambda^{pt} = \text{colim}_n \Lambda_p \]

Write $\Lambda_p < Y^{0-1} >$ for this plethory.

The corresponding Witt vectors are

\[ W_{\Lambda_p < Y^{0-1} >} (R) = \text{Hom} (\Lambda_p < Y^{0-1} > , R) \]
\[ = \text{Hom} ( \text{colim}_n \Lambda_p , R) \]
\[ = \text{lim}_n \text{Hom} (\Lambda_p , R) \]
\[ \lim_{n \to \infty} W_n(R) = W(R) \]

\[ E_{\text{Art}} = W_{\Delta p}(\gamma^{p-1}) \left( 0 \rightarrow_{\Delta p} 0 \right) \]

Recall that \( \gamma \) induces
\[ W_n(R) \xrightarrow{\gamma} W_{n-1}(R) \]

So we also consider
\[ W'(R) = \lim_{n \to \infty} W_n(R) \]

Is this the Witt vectors corresponding to a plethory \( P \)? The functor \( W \) is represented by colim \( \Delta p_n \).

\[ \mathbb{Z}[e] \xrightarrow{\gamma} \mathbb{Z}[e, s] \xrightarrow{\gamma} \mathbb{Z}[e, s, s^{p^2}] \]

\[ \Delta p, 0 \quad \Delta p, 1 \quad \Delta p, 2 \]

\[ \uparrow \text{id} \quad \uparrow \gamma \quad \uparrow \gamma^{p^2} \]

\[ \mathbb{Z}[e] \xleftarrow{\gamma^{-1}} \mathbb{Z}[e, s] \xleftarrow{\gamma^{-p^2}} \mathbb{Z}[e, s, s^{p^2}] \]

Composition? We imagine
\[ (\gamma^{m-n} \circ \gamma^{n-m}) \circ g = \gamma^{p-(m+n)} \circ f \circ g \]

and rewrite this
\[ \Delta p, m \circ \Delta p, n \longrightarrow \Delta p, m+n \]
\[ \gamma \circ \gamma \quad \gamma^2 \]
\[ \Delta p, m+1 \circ \Delta p, n+1 \longrightarrow \Delta p, m+n+2 \]

which gives

\[
\left( \text{colim} \ \Delta p, m \right) \circ \left( \text{colim} \ \Delta p, n \right)
\]

\[ \xrightarrow{\gamma} \] \[ \text{colim} \ \Delta p, m \circ \Delta p, n \]

\[ \xrightarrow{\gamma} \text{colim} \ \Delta p, 2k \]

\[ \xrightarrow{\gamma^2} \text{colim} \ \Delta p, k \]

Write this

\[ \Delta p \circ \Delta p \longrightarrow \Delta p \]

So what is a $\Delta p$-ring explicitly? Define $a, b \in \Delta p$ by

\[ \text{colim} \ 1 \longrightarrow \Delta p \]

\[ e, -s \longrightarrow a \circ b \]

Formally,

\[ a = \gamma^{a-1}, \quad b = -\gamma^{b-1} \circ s \]
Leibniz rules for $a$ and $b$:

1. Let $\theta$ be a ring homomorphism:
   \[ \Delta^+(a) = a \theta 1 + 1 \theta a \]
   \[ \Delta^x(a) = (a \theta 1), (1 \theta a) \]

2. Let $\beta = \delta$, but must express Leibniz rules using $a$ and $b$:
   \[ \delta(x + y) = \delta(x) + \delta(y) + \frac{1}{p} \sum_{\alpha < p} (\delta(x)^\alpha \delta(y)^{p-\alpha}) \]
   so
   \[ b(x + y) = b(x) + b(y) + \frac{1}{p} \sum_{\alpha < p} a(x)^\alpha a(y)^{p-\alpha} \]

Similarly,
\[ \delta(xy) = \delta(x)^{y^p} + x^p \delta(y) + p \delta(x) \delta(y) \]
and applying $\gamma^{0-1}$, we get
\[ b(xy) = b(x)a(y)^p + a(x)^p b(y) + pb(x) b(y) \]
Also, $b(1) = 0$. 
We understand the relation
\[ a^p = e + pb \]
as \( a \) is a lift of the inverse Frobenius and \( b \) is a "witness" to this lifting.

So any \( \mathbb{Z}_p \)-ring has two operators \( a \) and \( b \), where \( a \) is a ring homomorphism; \( b \) satisfies

\[
\begin{align*}
    b(x+y) &= b(x) + b(y) + \frac{1}{p} \sum_{\sigma \in \mathbb{F}_p} \sigma(a(x)^p) \\
    b(xy) &= b(x) a(y)^p + a(x)^p b(y) + pb(x)b(y) \\
    b(1) &= 0
\end{align*}
\]

and where \( a^p = e + pb \), i.e.
\[ a(x)^p = x^p + pb(x) \quad \forall x. \]

Exercise: Show that the converse is true.
Summary:

\[ \Lambda_p \langle \mathbb{Q}^{o-1} \rangle \leftarrow \Lambda_p \]

\[ \uparrow \text{push out} \uparrow \]

\[ \Lambda'_p \leftarrow \mathbb{Z}[e] \]

\[ W(R)_{pf} \quad \rightarrow \quad W(R) \]

\[ \downarrow \text{pull back} \downarrow \]

\[ W'(R) \quad \rightarrow \quad R \]

Ghost component:

\[ \mathbb{R} \rightarrow \mathbb{R} \]

\[ \downarrow \]

\[ \mathbb{R} \rightarrow \mathbb{R}^{d_{0}} \]

Compare

\[ A' \rightarrow \mathbb{P} \leftarrow A' \]

\[ \downarrow \]

\[ A' \leftarrow \mathbb{P} \]

Would like analogy of \( \mathbb{P} \) in our situation.
Neckless components:
If $R$ is $p$-torsion free, then
\[ W(R) \cong R^{\mathbb{N}}. \]

What is the image? Answer: It is the largest sub-$\mathbb{Z}$-ring on which $\Pi$ is a Frob. lift. This is not explicit!

Thus if $R$ is $p$-torsion free and has a $S$-structure, then
\[ \langle x_0, x_1, \ldots \rangle \in R^{\mathbb{N}} \]
is in the image of the ghost map if and only if
\[ x_{n+1} \equiv \Pi(x_n) \mod p^{n+1}R \]
for all $n \in \mathbb{N}$. \\
Ex $W(\mathbb{Z}) = \left\{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}^{\mathbb{N}} \mid x_{n+1} \equiv x_n \mod p^{n+1} \mathbb{Z} \right\}$

so consists of certain $\mathbb{Z}$-adic Cauchy sequences.
Correction: The square

\[ \Lambda_p \rightarrow \Lambda_p \langle y^{q-1} \rangle \]

\[ \uparrow \quad \uparrow \]

\[ \mathbb{Z} \langle e \rangle \rightarrow \Lambda_p \]

\[ \text{is not cocartesian, and} \]

\[ W(R) \leftarrow \lim_{n \to \infty} W(R) \]

\[ \downarrow \]

\[ R \leftarrow \lim_{n \to \infty} W_n(R) \]

\[ \text{is not cartesian. For example,} \]

\[ W(F_p) \leftarrow \lim_{n \to \infty} W(F_p) \]

\[ \downarrow \]

\[ F_p \leftarrow \lim_{n \to \infty} W_n(F_p) \]

\[ \text{is obviously not cartesian.} \]

Necklace components:

\[ \text{Then if } R \text{ is a } p\text{-tors. free } \mathcal{O}\text{-ring, then} \]

\[ W(R) \rightarrow R \]

\[ \text{has image } \langle x_0, x_1, \ldots \rangle \mid x_n \equiv \gamma(x_{n-1}) \text{ mod } p^nR \]
Reformulate statement:

\[ W_n(R) = \text{Hom}_{\text{ring}}(\Lambda_p, R) \oplus a_i \]

\[ = \text{Hom}_{\text{ring}}(\Lambda_p \oplus \Lambda_p, R) \oplus a'_i \]

The ghost coordinates are

\[ x_n = a(\gamma^n) = a'(e \circ \gamma^n) \]

so

\[ \gamma(x_{n-1}) = \gamma(a(\gamma^{n-1})) \]

\[ = a'(e \circ \gamma^{n-1}) \]

Hence,

\[ x_n = \gamma(x_{n-1}) + \rho^n \]

or

\[ a'(e \circ \gamma^n - \gamma \circ \gamma^{n-1}) = \rho^n \]

Natural to consider \( c_0 = e \circ e \) and

\[ c_n = \frac{1}{\rho^n} (e \circ \gamma^n - \gamma \circ \gamma^{n-1}) \]

\[ \in \Lambda_p \oplus \Lambda_p, \quad n \geq 1 \]
Then (2) will be a \( \alpha'(cn) \). So theorem will follow from:

Thus the elements

\[ c_0, c_1, \ldots, c_n \in \Delta p \circ \Delta p, m \{ \theta \} \]

lie in \( \Delta p \circ \Delta p, m \) and generate it freely as a \( \Delta p \)-ring.

\textbf{Pf: Calculate:}

\[
e \circ \Psi^m = e \circ \sum_{\text{os i} \in n} p^i \theta_i^{p^{n-1-i}}
\]

\[
\Psi \circ \Psi^{m-1} = \Psi \circ \sum_{\text{os i} \in n} p^i \theta_i^{p^{n-1-i}}
\]

\[
= \sum_{\text{os i} \in n} p^i (\Psi \circ \theta_i)^{p^{n-1-i}}
\]

\[
= \sum_{\text{os i} \in n} p^i (e^{p \circ \theta_i} + p \circ \theta_i)^{p^{n-1-i}}
\]

\[
= p^{-1} (e^{p \circ \theta_{n-1}} + p \circ \theta_{n-1})
\]

\[
+ \sum_{\text{os i} \in n-1} p^i (e^{p \circ \theta_i})^{p^{n-1-i}} + p^n \theta
\]

\(( \theta \in \Delta p, m \circ A[\theta_0, \ldots, \theta_{n-2}] )\)

\[
= p^m \circ \theta_{n-1} + \sum_{\text{os i} \in n} p^i (e^{p \circ \theta_i})^{p^{n-1-i}} + p^n \theta
\]

so we find
\[ p^n(c \circ 
abla^n - \nabla \circ \nabla^{n-1}) \]
\[ = c \circ \Theta_n - \delta \circ \Theta_{n-1} - f, \]

and hence,

\[ c_n = c \circ \Theta_n - \delta \circ \Theta_{n-1} \]

modulo \( \Lambda p,1 \circ \mathbb{Z}[\Theta_0, \ldots, \Theta_{n-2}] \). Now, by induction on \( n \geq 0 \), the case \( n=0 \) being trivial, we find

\[ \Lambda p \circ \Lambda p, n = \Lambda p \circ \mathbb{Z}[\Theta_0, \ldots, \Theta_n] \]
\[ = \Lambda p \circ (\Lambda p, n-1 \circ \mathbb{Z}[\Theta_n]) \]
\[ = (\Lambda p \circ \Lambda p, n-1) \circ (\Lambda p \circ \mathbb{Z}[\Theta_n]) \]
\[ = (\Lambda p \circ \mathbb{Z}[c_0, \ldots, c_{n-1}) \circ (\Lambda p \circ \mathbb{Z}[\Theta_n]) \]

which, by the formula above becomes

\[ = (\Lambda p \circ \mathbb{Z}[c_0, \ldots, c_{n-1}) \circ (\Lambda p \circ \mathbb{Z}[c_n]) \]
\[ = \Lambda p \circ \mathbb{Z}[c_0, \ldots, c_n] \].

This proves the induction step. \( \Box \)
Rule (1) The composition map

$$\Delta p \circ \Delta p \rightarrow \Delta p$$

maps $c_0$ to $e$ and $c_n$ with $n > 1$ to zero. Accordingly,

$$S(\Theta n) = \Theta n+1 + \frac{1}{n}$$

with $\frac{1}{n} \in \Delta p_{11} : \mathbb{Z}(\Theta_0, \ldots, \Theta_n)$. Note that we use the $\Theta_n$ to prove this; there should be some simplification, not making use of the $\Theta_n$.

The $c_i$ give a new bijection

$$W_n(\mathbb{R}) \rightarrow \mathbb{R}^{[0, n]}$$

$$a \mapsto [a'(c_0), \ldots, a'(c_n)]$$

call the $a'(c_i)$ the necklace components of $a$. To understand $+$ and $\times$ in necklace components, recall that

$$c \circ \gamma^i = \gamma \circ \gamma^{i-1} + p \gamma^i c_i,$$

and hence,

$$a'(c \circ \gamma^i) = \gamma(a'(c \circ \gamma^{i-1})) + p \gamma^i a'(c_i).$$
So writing \( \langle x_0, \ldots, x_n \rangle \) for the ghost comp. of \( a \) and \([b_0, \ldots, b_n]\) for the necklace comp. we have

\[
x_i = \gamma(x_{i-1}) + p^{i}b_i,
\]

from which we find

\[
[b_0, b_1, \ldots] + [b'_0, b'_1, \ldots]
\]

\[
= [b_0 + b'_0, b_1 + b'_1, \ldots]
\]

\[
[b_0, b_1, \ldots] \times [b'_0, b'_1, \ldots]
\]

\[
= [b_0 b'_0, \gamma(b_0) b'_1 + b_1, \gamma(b'_0) + p b_1 b'_1, \ldots]
\]

\[
[b_0, b_1, \ldots] \gamma^{-1}[\gamma(b_0) + p b_1, p b_2, \ldots]
\]

To work these formulas out, use formula for ghost coordinates in terms of necklace coordinates:

\[
x_0 = b_0
\]

\[
x_1 = \gamma(b_0) + p b_1
\]

\[
x_2 = \gamma^2(b_0) + p \gamma(b_1) + p^2 b_2
\]

\[
\vdots
\]
Recall that

\[ W_n(R) = \text{Hom}_{\text{ring}}(\Lambda_p^n, R) \]

and that we have the different components

\[ \mathbb{Z}[c, s, \ldots, s^m] \xrightarrow{\sim} \mathbb{Z}[c, y, \ldots, y^m] \xrightarrow{\sim} \Lambda_p^n \xrightarrow{\sim} R \]

\[ \mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n] \xrightarrow{\sim} \]

inducing, respectively,

\[ \mathbb{R}^{[0,n]} \xrightarrow{\sim} \text{s-components} \]

\[ W_n(R) \xrightarrow{\sim} \mathbb{R}^{[0,n]} \xrightarrow{\sim} \text{ghost components} \]

\[ \mathbb{R}^{[0,n]} \xrightarrow{\sim} \text{with components} \]

If \((R, S)\) is a \(\Lambda_p\)-ring, then also

\[ W_n(R) = \text{Hom}_{\Lambda_p\text{-ring}}(\Lambda_p \circ \Lambda_p^n, R) \]

and

\[ \Lambda_p \circ \mathbb{Z}[c_0, \ldots, c_n] \xrightarrow{\sim} \Lambda_p \circ \Lambda_p^n \xrightarrow{\sim} R, \]
\[ c_i = \begin{cases} 
  e \circ e & (i=0) \\
  p^{-i} (e \circ \gamma^0_{i-1} \circ \gamma^0_0) & (0 < i \leq n),
\end{cases} \]

gives the necklace components

\[ W_n(R) \xrightarrow{\gamma} R^{[0,n]} \]

\text{Proof: Here } R \text{ may have } p \text{-torsion.}

Let \( \langle x_0, \ldots, x_n \rangle \) and \( [b_0, \ldots, b_n] \) be the ghost comp. and necklace comp. of \( a \in W_n(R) \), respectively. Recursively,

\[ x_0 = b_0 \]

\[ x_i = \gamma(x_{i-1}) + p^i b_i \quad (0 < i \leq n). \]

Can work out with vector arithmetic in necklace components, e.g.,

\[ [b_0, b_1, \ldots] \cdot [b'_0, b'_1, \ldots] \]

\[ = [b_0 b'_0, \ldots, \sum_{0 \leq s+t \leq i} p^{s+t} \gamma^{s}(b_s) \gamma^{t}(b'_t), \ldots] \]
Cor (Dwork's lemma) If $(R, S)$ is a $\Lambda$-ring, then a vector $\langle x_0, \ldots, x_n \rangle$ is in the image of the ghost map

$$\psi : W_n(R) \to R^\times [0, n]$$

if and only if

$$x_i \equiv \psi(x_{i-1}) \mod p^i R,$$

for all $1 \leq i \leq n$.  \\

If, in addition, $R$ is $p$-torsion free, then the ghost map is injective. Hence, in this case Dwork's lemma identifies $W_n(R)$ as a subring of $R^\times [0, n]$.

Ex. Spec $W_n(\mathbb{Z})$:
Similarly, 

\[ W(\mathbb{Z}_p) = \{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}_p^\infty \mid x_i \equiv x_{i-1} \pmod{p} \} \]

so have ring homomorphism:

\[ W(\mathbb{Z}_p) \xrightarrow{\text{w_0}} \mathbb{Z}_p \]

\[ \langle x_0, x_1, \ldots \rangle \mapsto x_0 := \lim_{i \to \infty} x_i \]

It is clearly surjective, and its kernel is

\[ I = \{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}_p^\infty \mid x_i \in p \mathbb{Z}_p \} \]

We claim that I is also the kernel of the surjective ring homomorphism

\[ W(\mathbb{Z}_p) \twoheadrightarrow W(\mathbb{F}_p) \]

induced by the unique ring homomorphism from \( \mathbb{Z}_p \) to \( \mathbb{F}_p \). First show that the latter map takes I to zero. Let \( (a_0, a_1, \ldots) \) be the Witt comp. corresponding to \( \langle x_0, x_1, \ldots \rangle \) in I. We must show that \( a_i \in p \mathbb{Z}_p \), for all \( i > 0 \). This is clear for \( i = 0 \), since \( a_0 = x_0 \in p \mathbb{Z}_p \), and
for $i \geq 1$, it follows inductively from
\[ x^i = qa_0 + pa_1 + \cdots + pa_i. \]

Running this argument backwards, we find that if all Witt comp. $a_0, a_1, \ldots$ are divisible by $p$, then the $i$th ghost comp. $x^i$ is divisible by $p^i$. So $w_p$ annihilates the kernel of $W(Z) \rightarrow W((F_p))$. We therefore conclude, finally, that
\[ Z \rightarrow W((F_p)). \]

Remark that $\Delta_p \circ \Delta_{p,n}$ represents

\[ \Delta_p \text{-rings} \xrightarrow{w_n} \text{Rings} = \mathbb{Z}[c] \text{-rings}; \]

so $\Delta_p \circ \Delta_{p,n}$ is a $\Delta_p \text{-Z}[c]$-biring.

Also, while $\Delta_{p,n}$ is not generated by additive elements, $\Delta_p \circ \Delta_{p,n}$ is generated by the necklace elements, which are additive. I.e., $\Delta_{p,n}$ is not (weakly) linear, but becomes linear rel. to $\Delta_p$ after applying the "base-change" $\Delta_p \circ -$.

More generally, \ldots
Question: Is it true that for every plethory $P$, the $P$-ring $P \circ P$ is generated by additive elements?

If $X$ is a scheme, then its $n$th jet space $J^nX$ is defined by

$$(J^nX)(R) = X(R[t]/(t^{n+1})),$$

it is again a scheme. Similarly, define the $n$th arithmetic jet space

$$(J^n_pX)(R) = X(W_n(R)),$$

it, too, is a scheme. We write

$$J^n_p \text{Spec}(A) = \text{Spec}(A \otimes \mathbb{P} \otimes A)$$

and also

$$\text{Sch} \xleftarrow{W_n} \text{Sch} \xrightarrow{J^n_p} \text{Sch}.$$ 

Since $J^n_p$ has a left adjoint, it preserves limits; in particular, it takes group schemes to group schemes.
Verschiebung and Teichmüller:

Define Teichmüller representative of \( a \in \mathbb{R} \) to be the Witt vector

\[ [a] = (a, 0, 0, \ldots) \in W(\mathbb{R}). \]

Universal example:

\( a \in \mathbb{Z}[a] \)

with \( \Delta_p \)-structure \( \xi(a) = a^p \), i.e.

\[ \xi^m(a) = \begin{cases} a & \text{if } m = 0 \\ 0 & \text{if } m > 0. \end{cases} \]

The resulting ring homomorphism:

\[ \mathbb{Z}[a] \longrightarrow W(\mathbb{Z}[a]) \]

takes \( a \) to \([a]_p\); more generally,

\[ \xi(a) = (a, \xi(a), \xi^2(a), \ldots). \]

This ring homomorphism defines a map of sets:

\[ \mathbb{R} \longrightarrow W(\mathbb{R}). \]
In ghost components,
\[ [a] = \langle a, a^p, a^{p^2}, \ldots \rangle, \]
which shows that
\[ [ab] = [a] [b]. \]
Indeed, this formula is satisfied in the universal case \( R = \mathbb{Z}[\alpha, \beta], \)
in which case the ghost map is injective; and hence, it is satisfied in general.

Define Verschiebung operator
\[ \mathcal{W}(R) \xrightarrow{V_p} \mathcal{W}(R) \]
to be the set map given in Witt components by the shift
\[ (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots). \]
In ghost components, this becomes
\[ \langle x_0, x_1, \ldots \rangle \mapsto \langle 0, px_0, px_1, \ldots \rangle, \]
which shows that \( V_p \) is additive, but not multiplicative.
Rank The definition of $V_p$ implicitly uses the choice of generator $p$ of the maximal ideal $p \mathbb{Z} \subset \mathbb{Z}$. We could as well have used the generator $-p$ to get $V_{-p}$.

Similarly, by checking in ghost components, we find that

$$\gamma(V_p(a)) = p - a,$$

$$a = \sum_{i > 0} V_p[V_i a_i],$$

where $a_i$ is the $i$th Witt comp. of $a$. 
Frobenius lifts in general:

So far: Lifting of $p$'th power Frob. on $R/pR$.

Here $p$ plays two roles:

- exponent
- generator of ideal $pR$

Wish to consider generalization to $p \in \mathbb{Q}$ — $R, [K : \mathbb{Q}] < \infty$. So wish to generalize both occurrences of $p$ above.

Context:

$A$: ring ("base")

$\mathfrak{p} \subset A$: ideal, invertible as $A$-mod.

$K = A/\mathfrak{p}$

$F \in k[\ell]$ $k$-algebra like:

$\Delta^+(F) = Fe_1 + 1eF; \quad \Delta^x = (Fe_1)(1eF), ...$

so corepresents $k$-alg. homom.
Def. A pre-
\[ \Lambda \text{A}, p, F \text{-ring} \] is a pair \((R, \mathfrak{p})\) of an \(\Lambda\)-algebra \(R\) and an \(\Lambda\)-alg. homom. \(\mathfrak{p} : R \to R\) s.t.

\[
\begin{array}{ccc}
R & \xrightarrow{\mathfrak{p}} & R \\
\downarrow & & \downarrow \\
R/\mathfrak{p}R & \xrightarrow{\mathfrak{p}} & R/\mathfrak{p}R
\end{array}
\]

commutes.

Ex. \( [k : \mathbb{Q}] < \infty, A = \mathbb{O}_K, \mathfrak{p} \subset A \) maximal ideal, \( \mathfrak{p} = A/\mathfrak{p} \), \( F = e^q \) where \( q = (\# k)^r \) for some \( r > 0 \).

Ex. As in previous example, but with \( K \) a local field or global field (\( A = \mathbb{F}_p[[t]], \mathbb{Z}_p, \mathbb{F}_p[[t]]^{\infty}, \ldots) \).

Ex. Can consider \( A, \mathfrak{p} \) any ring and ideal and \( F = e^q \).

Ex. \( A = \mathbb{Z}, \mathfrak{p} = \mathfrak{p}_1 \mathfrak{p}_2 \mathbb{Z}, \mathfrak{p}_i \) prime, \( \mathfrak{p} = \mathbb{F}_{p_1} \times \mathbb{F}_{p_2}, F \in \mathbb{F}_{p_1}[e] \times \mathbb{F}_{p_2}[e], 
\]

\[
F = (e^{p_1^{r_1}}, e^{p_2^{r_2}}).
\]

Remark Other variants possible: Could ask for endom. \( \mathfrak{p} \) s.t. \( \mathfrak{p} \) on lifts \( F \).
Want: $A$-plethory $\Lambda A_{1, \beta, F}$ s.t. for $\beta$-torsion $A$-algebras,

$\Lambda A_{1, \beta, F}$ - structure

$\Lambda A_{1, \beta, F}$ - structure

**Note** We'll abbreviate

$\Lambda = \Lambda A_{1, \beta, F}$.

**Ex.** For $[K : \mathbb{Q}] < \infty$, suppose $\beta \in \mathcal{O}_K$ is principal. Choosing a uniformizer $\pi \in \mathfrak{p}$, the Frobenius lift cond. becomes

$$\gamma(x) = x^q + \pi \delta(x),$$

where, as before, $\delta$ satisfies

$$\delta(x + y) = \delta(x) + \delta(y) - \sum_{0 < i < q} \frac{1}{\pi} \left( \frac{q}{i} \right) x^i y^{q-i},$$

$$\delta(xy) = x^q \delta(y) + \delta(x) y^q + \pi \delta(x) \delta(y),$$

$$\delta(1) = 0.$$

Note that $\frac{1}{\pi} \left( \frac{q}{i} \right) \in \mathcal{O}_K$. Define

$\Lambda = \Lambda A_{1, \beta, F} = \mathcal{O}_K [\varepsilon, \delta, \delta^{a_2}, \ldots]$.
and

\[ W(\mathbb{R}) = W_{A_{11}^{\beta,\gamma}}(\mathbb{R}) = \mathbb{R}^N \]

with ring structure defined as earlier. Two issues:

(1) How does this depend on \( \alpha \)?

(2) What if \( \beta \) is not principal?

One solution is to ask for operators

\[ S_\gamma(x) = \gamma \cdot (\gamma(x) - x^2) \]

for every \( \gamma \in \beta^{-1} \). However, this will necessitate to understand rel.

\[ S_{\gamma_1 + \gamma_2} = \ldots \]

\[ S_{\gamma_1} \circ S_{\gamma_2} = \ldots \]

which is obviously a huge mess.

It is clearly enough to consider \( S_\gamma \) for \( \gamma \) ranging over a family of generators of \( \beta^{-1} \). In the
case \( A = \mathcal{O}_E \), \([E:Q]< \infty\), always have \( \rho^{-1} = (1, y) \), so only have two operators \( \delta_1 \) and \( \delta_2 \). Still need to show that the resulting theory is independent on choice of \( y \). So not the way to go.

**Better approach:**

\[
\begin{align*}
(\text{\( p \)-tors. free} & \quad \text{pre-\( \Lambda \)-rings} \\
| & \quad \text{\( \Lambda \)-rings} \\
\uparrow \text{w thai} & \quad \text{\( \Lambda \)-rings} \\
\text{\( p \)-tors. free} & \quad \text{\( \Lambda \)-rings} \\
\text{\( A \)-algebras} & \quad \text{\( \Lambda \)-rings}
\end{align*}
\]

It is better to produce the comonad \( \mathcal{J} \circ \mathcal{W} \) than to produce the monad \( \mathcal{J} \circ \Lambda_{\text{o-}} \). Reason: the latter does not preserve \( p \)-tors. freeness, so first produce

\[
\mathcal{W} \Lambda = \mathcal{W}_{\text{\( p \)-tors-free}}
\]

and use Kan extension to get \( \mathcal{W} \Lambda \).
Def An $A$-module $M$ is $\mathfrak{p}$-torsion-free, if the $A$-module
\[ \text{Tor}_1^A(M, A/\mathfrak{p}) \]
is trivial. 
Equivalently, an $A$-module $M$ is $\mathfrak{p}$-torsion-free if the map
\[ \mathfrak{p} \otimes_A M \rightarrow M \]
induced by the canonical incl. is injective.

Def Fix $(A, \mathfrak{p}, F)$ as above and for $A$-algebras $R$, define a sequence of sub-$A$-algebras
\[ W^{(0)}(R) \supset \cdots \supset W^{(m)}(R) \supset \cdots, \]
recursively, by
\[ W^{(0)}(R) = R \]
\[ W^{(m+1)}(R) = \{ a \in W^{(m)}(R) \mid \forall (a) \equiv F(a) \mod \mathfrak{p} W^{(m)}(R) \}. \]
Prop (1) $W^{(m)}(R) \subset R^m$ is a sub-$\gamma$-ring.

(2) $W^{(m)}(R)$ is the Frobenius lift property.

(W) $R$ is the cofree pre-$\Lambda$-ring on $R$.

Proof (1) Proof by induction on $m \geq 0$, the case $m = 0$ being trivial. By definition,

\[ W^{(m+1)}(R) \to W^{(m)}(R) \to W^{(m)}(R)/\beta W^{(m)}(R) \]

is an equalizer, so $W^{(m+1)}(R)$ is a sub-$\Lambda$-alg. of $W^{(m)}(R)$. Must prove that it is stable under $\gamma$. So let $a \in W^{(m+1)}(R)$, i.e.

\[ \gamma(a) \equiv F(a) \mod \beta W^{(m)}(R). \]

So

\[ \gamma^2(a) \equiv \gamma(F(a)) \mod \gamma(\beta W^{(m)}(R)) \]

\[ \equiv F(\gamma(a)) \]
and

\[ \varphi (\bar{\psi} W^{(m)} (R)) = \bar{\psi} \varphi (W^{(m)} (R)) \]

\[ \leq \bar{\psi} W^{(m)} (R) \]

by inductive hypothesis. So

\[ \varphi (\varphi (a)) = F (\varphi (a)) \mod \bar{\psi} W^{(m)} (R) \]

which shows that \( \varphi (a) \in W^{(m+1)} (R) \)

as desired.

(2) Since \( P \) is finitely generated and projective (if this does not follow from \( P \) being invertible, then we will assume it),

\[ \lim \bar{\psi} W^{(m)} (R) \]

\[ = \lim \text{ im } (\bar{\psi} \otimes W^{(m)} (R) \to R^{m}) \]

\[ = \text{ im } (\lim \bar{\psi} \otimes W^{(m)} (R) \to R^{m}) \]

\[ = \text{ im } (\bar{\psi} \otimes \lim W^{(m)} (R) \to R^{m}) \]

which is what we need. We'll justify \( \bullet \bullet \) next time and prove (3).
Continue with proof from last time.
Rewrite equalizer diagram defining $W^{(m)}(R)$ as

\[ W^{(m)}(R) \overset{\gamma}{\longrightarrow} W^{(m)}(R) \overset{\phi}{\longrightarrow} k \otimes_A W^{(m)}(R) \]

Taking limits over $m$, we get the equalizer diagram

\[ W^{(\infty)}(R) \overset{\gamma}{\longrightarrow} W^{(\infty)}(R) \overset{\phi}{\longrightarrow} \varprojlim_{m} k \otimes_A W^{(m)}(R) \]

What we wish to prove is that $W^{(\infty)}(R)$ is an equalizer, so we need to show that the canonical map

\[ k \otimes \varprojlim_{m} W^{(m)}(R) \longrightarrow \varprojlim_{m} k \otimes_A W^{(m)}(R) \]

is a monomorphism. The sequence

\[ 0 \rightarrow \beta \rightarrow A \rightarrow k \rightarrow 0 \]

is exact. We will now assume that $R$ is torsion free. Then $\beta$ is $W^{(m)}(R)$, so we get a diagram with exact rows.
\( p \otimes \lim W^{(m)} \to \lim W^{(m)} \to \ker \lim W^{(m)} \to 0 \)
\[
\begin{array}{c}
\downarrow p \otimes \downarrow 1 \\
0 \to \lim p \otimes W^{(m)} \to \lim W^{(m)} \to \lim \ker p \otimes W^{(m)} \to 0
\end{array}
\]

where we abbreviate \( W^{(m)} = W^{(m)}(R) \).

So we may instead show that the map \( p \) is surjective. Claim:

(i) \( p \) is finitely generated as an \( A \)-module.

(ii) \( p \) is projective as an \( A \)-module.

Granting this, \( p \) is dualizable as an \( A \)-module, and hence, the functor \( p \otimes - \) preserves limits. This shows that \( p \) is an isomorphism.

So actually \( A \cong B \) an isomorphism.

It remains to justify the claims (i)–(ii). First, by EGA I.5.4.1, an \( A \)-module \( M \) is invertible \( p \) and only if \( M \) is locally free of rank 1. Now, by EGA IV.2.5.3 and Bourbaki, Comm. Alg., Chap. I, §3, no. 6, Prop. 11, the properties of being finitely generated and finitely presented are local properties for the Fpqc
topology. Since we know that, locally for the Zariski topology, \( P \) is free of \( \text{rk. } 1 \), we conclude that \( P \) is finitely presented. So (i) holds. Similarly, being flat is a local property, so if \( P \) is flat. But flat and finitely presented is projective by Bourbaki, ibid., §2, p. 64. So (iii) holds.

Finally, we also used that \( W^{(m)}(R) \) is \( \beta \)-torsion free if \( R \) is. Again, we may prove this locally on \( A \), so we can assume that \( P \) is principal. But

\[
W^{(m)}(R) \twoheadrightarrow \mathbb{R}^n \\
\downarrow \pi \\
W^{(m)}(R) \twoheadrightarrow \mathbb{R}^n
\]

shows that \( W^{(m)}(R) \) is \( \beta \)-torsion free. This completes the proof of (2); but we had to assume \( R \) to be \( \beta \)-torsion free.

The remaining statement (3) was that \( W^{(m)}(R) \) is the cofree pre-\( A \)-ring on \( R \), if \( R \) is
\( p \)-torsion free. So let \( S \) be a pre-\( A \)-ring and let \( \gamma : S \to R \) be an \( A \)-algebra map. We wish to show that \( \gamma \) admits a unique lifting to a map of pre-\( A \)-rings, or equivalently, a map of \( A \)-rings.

\[
\exists ! \tilde{\gamma} : W^{(m)}(R) \to W^{(m)}(R)
\]

Uniqueness is clear, since

\[
\tilde{\gamma}(S) = \langle \gamma(S), \gamma(\gamma(S)), \gamma(\gamma^2(S)), \ldots \rangle
\]

and to prove existence, it suffices to show that for all \( m \geq 0 \),

\[
\tilde{\gamma}(S) \subseteq W^{(m)}(R) \subseteq R
\]

We proceed by induction, the case \( m = 0 \) being trivial. So we assume \( \tilde{\gamma}(S) \subseteq W^{(m-1)}(R) \) and show that \( \tilde{\gamma}(S) \subseteq W^{(m)}(R) \). We calculate
\[ \gamma(\xi(s)) - F(\xi(s)) \]
\[ = \gamma(\xi(s)) - \gamma(F(s)) \]
\[ = \gamma(\xi(s) - F(s)) \]
\[ \leq \gamma(\rho S) \]
\[ \leq \rho \gamma(S) \]
\[ \leq \rho W^{(m-1)}(R) \quad \text{(by induction)} \]

which proves the induction step, by the definition of \( W^{(m)}(R) \) (as an equalizer).
We proceed to show that the functor $W(\omega)$ from $\mathfrak{p}$-torsion free $A$-algebras to pre-$A$-rings is representable. We use the representing object to extend $W(\omega)$ to all $A$-algebras.

We first define $A[\mathfrak{p}^{-1}]$. If $\mathfrak{p} = (\alpha)$, then we have $A[\alpha^{-1}]$, which we wish to generalize. The $A$-module

$$\mathfrak{p}^{-1} = \text{Hom}_A(\mathfrak{p}, A)$$

is invertible, and the canonical inclusion $\mathfrak{p} \subset A$ induces a map of $A$-modules $A \rightarrow \mathfrak{p}^{-1}$. This map again is injective, since $\mathfrak{p}$ is locally principal. We define $A[\mathfrak{p}^{-1}]$ to be the pushout

$$\begin{array}{ccc}
\text{Sym}_A(A) & \rightarrow & A \\
\downarrow & & \downarrow \\
\text{Sym}_A(\mathfrak{p}^{-1}) & \rightarrow & A[\mathfrak{p}^{-1}]
\end{array}$$

The morphism $A \rightarrow A[\mathfrak{p}^{-1}]$ is injective and flat, since being so are local properties and these are true locally.
We first show that each $W^{(m)}$ is representable.

Def. The family of $A$-algebras

$(\Lambda^{(m)} \mid m \in \mathbb{N})$

are defined recursively by:

$\Lambda^{(0)} = \Phi = A[e, t, t^2, \ldots]$

$\Lambda^{(m)} = \text{sub-} \Lambda^{(m-1)}$-algebra of $A[\beta^{-1}] \otimes_A \Phi$ generated by

$(f \circ S_y \mid f \in \Lambda^{(m-1)}, y \in \beta^{-1})$, where $S_y = y(1 - F)$.

Prop. The canonical inclusion

$\Lambda^{(m)} \hookrightarrow \Phi$

becomes an isomorphism after base-change along $A \leftarrow A[\beta^{-1}]$.

Proof by induction on $m \in \mathbb{N}$.
the case \( m=0 \) being trivial. To prove the induction step, consider

\[
\begin{align*}
A[p^{-1}] & \otimes A^{(m)} \rightarrow A[p^{-1}] \otimes A \\
\downarrow & \\
A[p^{-1}] & \otimes A^{(m-1)}
\end{align*}
\]

The left-hand slanted map is an isomorphism by the definition of \( A^{(m)} \), since

\[
A[p^{-1}] \otimes A \rightarrow A[p^{-1}] \otimes A[p^{-1}]
\]

is an isomorphism.

Prop. The \( A \)-algebra \( A^{(m)} \) is \( p \)-torsion free and represents the endofunctor \( W^{(m)} \) on the category of \( p \)-torsion free \( A \)-algebras.

Prf. Wish to show

\[
\begin{align*}
\text{Hom}_{A\text{-alg}} (A^{(0)}, R) & \rightarrow \mathbb{R}^n \\
\uparrow & \\
\text{Hom}_{A\text{-alg}} (A^{(m)}, R) & \rightarrow \mathbb{R}
\end{align*}
\]

and proceed by induction on \( m \).
the case \( m=0 \) being trivial. To prove induction step, given
\[
\Lambda^{(m-1)} \xrightarrow{\alpha} R,
\]
consider
\[
\Lambda^{(m-1)} \xrightarrow{\alpha} R \quad \text{injective since } R \text{ is potent, free}
\]
and calculate:
\[
\{ \Lambda^{(m)} \xrightarrow{\alpha} R \} = \{ \Lambda^{(m-1)} \xrightarrow{\alpha} R \mid \forall y \in \Sigma, \forall \psi \in \Lambda^{(m-1)} \}
\]
inductive hypothesis
\[
= \{ \alpha \in W^{(m-1)}(R) \mid \forall y, \delta_y(a') \Lambda^{(m-1)} \subseteq R \}
\]
\[
= \{ a \in W^{(m-1)}(R) \mid \forall y, \delta_y(a') \in W^{(m-1)}(R) \}
\]
\[
= \{ a \in W^{(m-1)}(R) \mid \forall y, y \mid \psi(a') - F(a') \in W^{(m-1)}(R) \}
\]
\[ \{ a \in W^{(m-1)}(R) \mid \varphi(a') - F(a') \in \beta W^{(m-1)}(R) \} \]
\[ = W^{(m)}(R), \]
\[ \text{Cor } \Lambda^{(\infty)} = U \Lambda^{(m)} \text{ represents } W^{(\infty)} \text{ on } \beta\text{-torsion free } A\text{-algebras.} \]

Remark: The canonical map \( A \rightarrow A[p^{-1}] \) identifies \( \text{Spec}(A[p^{-1}]) \) with the open subscheme of \( \text{Spec}(A) \) on which \( j: \beta \rightarrow A \) is invertible.

Recall:

\[ \begin{array}{ccc}
\beta\text{-tors. free} & \hookrightarrow & \Lambda\text{-rings} \\
\text{pre-} \Lambda\text{-ring} & & \\
\downarrow \phi^* & & \downarrow \phi^* \\
W^{(\infty)} & \hookrightarrow & W^{\Lambda} \\
\beta\text{-tors. free} & \hookrightarrow & A\text{-algebras}
\end{array} \]

Here \( W^{(\infty)}(R) \) is the maximal sub-\( \varphi \)-ring of \( R^{\infty} \) on which \( \varphi \) has the Frobenius lift property. We have now proved that the composite functor \( \phi^* \circ W^{(\infty)} \) is represented by \( \Lambda^{(\infty)} \). Addition and multiplication of \( A\)-algebras are represented by
\[ \Lambda^{(\infty)} \stackrel{\Delta^+, \Delta^x}{\longrightarrow} (\Lambda^{(\infty)} \otimes \Lambda^{(\infty)}) / \beta \text{-torsion} \]

coprod. in \( \beta \)-tors. free \( \Lambda \)-alg.

If \((\beta \text{-tors. free}) \otimes (\beta \text{-tors. free})\) again is \( \beta \text{-tors. free} \), then we obtain that \( \Lambda^{(\infty)} \) is an \( \Lambda \)-\( \Lambda \)-biring and hence, represents the functor \( W \Lambda \) that we wish to define. But this is not true in the generality that we work in here. The following result gets us around this difficulty.

Prop If \( \beta = 1 \pi \), then

\[ \Lambda^{(m)} = \Lambda[e, s, \ldots, s^m, y_0, y_0^2, y_0^3, \ldots] \]

where \( s = \frac{1}{\pi}(\tau - F) \).

Pf Induction, \( m = 0, 1, K \),

\[ \Lambda^{(m)} = \text{sub-} \Lambda^{(m-1)} \text{-alg. gen. by} \]

\[ \{ \pi \delta y \mid y \in \Lambda^{(m-1)}, y \in \pi^{-1} \} \]

\[ = \text{sub-} \Lambda^{(m-1)} \text{-alg. gen. by} \]

\[ \{ \pi \delta y \mid y \in \Lambda^{(m-1)} \} \].
Use Leibniz' rules to prove equality. So enough to let \( \delta \) range over a family of generators of the \( \Lambda \)-alg. Hence,

\[
\Lambda^{(m)} = \text{sub-} \Lambda^{(m-1)} \text{-alg. gen. by }
\]

\[
(\cos, \sin, \ldots, \delta^{(m-1)} \delta, \gamma \delta^{(m-1)} \delta, \ldots)
\]

where we use the inductive hypothesis. Using it again, we get

\[
\Lambda^{(m)} = \text{sub-} A \text{-algebra gen. by the family consisting of }
\]

\[
e, \delta, \ldots, \delta^{(m-1)} \delta, \gamma \delta^{(m-1)} \delta, \gamma^2 \delta^{(m-1)} \delta, \ldots
\]

and

\[
\delta, \ldots, \delta^{(m-1)} \delta, \delta \delta, \gamma \delta \delta, \gamma^2 \delta \delta, \ldots
\]

We claim that the subfamily of

\[
e, \delta, \ldots, \delta \delta, \gamma \delta \delta, \gamma^2 \delta \delta, \ldots
\]

generates \( \Lambda^{(m)} \) as an \( A \)-algebra. This follows from the calcula-
Indeed, \( F \in \mathcal{A}(\mathcal{E}) \) and \( \pi \in \mathcal{A} \), so claim follows by induction on \( \mathcal{E} \).
Finally, the generators \( \delta^0 \circ i \) and \( \gamma^0 \circ i \) are differential operators of order \( i \) and \( j + m \) respectively, and therefore, are algebraically independent.

Cor. \( \Lambda^{(\infty)} \) is locally free as an \( A \)-algebra, and hence, as an \( A \)-module.

In particular, for all \( m, n \), the \( A \)-module

\[
(\Lambda^{(m)})^{\otimes n}
\]

is locally free. Hence, the \( A \)-mod.

\[
(\Lambda^{(\infty)})^{\otimes n} = \text{colim} \ (\Lambda^{(m)})^{\otimes n}
\]

is flat, and hence, \( \pi \)-torsion free.
It follows that $\Lambda^{(m)}$ and $\Lambda^{(\infty)}$ are $A - A$-bimodules; the co-ring axioms hold because $(\Lambda^{(m)})\otimes^3$ and $(\Lambda^{(\infty)})\otimes^3$. 
Recall the context we are considering:

\[ A : \text{ring}, \]

\[ \beta \subset A : \text{ideal, invertible as an } A\text{-module}, \]

\[ F \in A/\beta [c] : A/\beta \text{-algebra-like} , \]

Have constructed left-hand vertical functor in the diagram

\[
\begin{array}{ccc}
\{ \beta \text{-tors., free} \} & \xrightarrow{\iota} & \text{Alg}_A \\
\downarrow \text{fgt} & \downarrow \text{fgt} & \downarrow \text{fgt} \text{to } W \\
\{ \beta \text{- tors., free} \} & \xrightarrow{\iota} & \text{Alg}_A
\end{array}
\]

and we may define the right-hand functor by left Kan extension,

\[
(\text{fgt} \text{to } W)(R) = \text{colim } (\text{fgt} \text{to } W^{(\omega)})(\tilde{R}).
\]

It is not clear that this extension exists in the universe of discourse; we know that it does,
because it is represented by the (p-torsion-free) \( A \)-algebra representing \( \text{fgto} \ W^{(0)} \). We note that, since \( i \) is fully faithful, we have

\[
\text{fgto} \ W^{(0)} \circ i = i \circ \text{fgto} \ W^{(0)}.
\]

Question: Why is \( \Lambda := i_{\ast} (\Lambda^{(0)}) \) a plethory?

In other words, why is \( \text{fgto} \ W \) a comonad?

Answer: Because \( \text{fgto} \ W^{(0)} \) is.

In detail, by Yoneda,

\[
\begin{align*}
\{ \text{fgto} \ W^{(0)} & \rightarrow \text{fgto} \ W^{(0)} \circ \text{fgto} \ W^{(0)} \} \\
& = (\text{fgto} \ W^{(0)} \circ \text{fgto} \ W^{(0)}) \ (\Lambda^{(0)}) \\
& = (\text{fgto} \ W \circ \text{fgto} \ W) \ (\Lambda) \\
& = \{ \text{fgto} \ W \rightarrow \text{fgto} \ W \circ \text{fgto} \ W \}
\end{align*}
\]

which gives the comonadial (co-)product on \( \text{fgto} \ W \), and the counit is defined similarly.
It satisfies the comonad axioms because \( \text{fgt} \circ \mathbb{W}^{(n)} \) does. This defines a canonical comonad structure, or equivalently, a canonical plethory structure on

\[ \Lambda = \Lambda_{A, \beta, F} \]

Def: A \( \Lambda_{A, \beta, F} \)-structure on an \( A \)-algebra \( R \) is an action of the plethory \( \Lambda_{A, \beta, F} \).

We may define the category of \( \Lambda_{A, \beta, F} \)-rings, equivalently, to be the category of coalgebras for \( \text{fgt} \circ \mathbb{W} \) with the comonad structure defined above. If \( R \) is an \( A \)-algebra, then the \( A \)-algebra \( \text{fgt} \circ \mathbb{W}^{(n)}(R) \) has a canonical \( \Lambda_{A, \beta, F} \)-structure; this gives the functors

\[
\begin{align*}
\{ \beta-\text{tors.}, \beta-\text{free} \} & \xrightarrow{\mathbb{W}} \Lambda_{A, \beta, F}\text{-rings} \\
\{ \text{pre-} \Lambda \text{-rings} \} & \xrightarrow{\mathbb{W}} \Lambda_{A, \beta, F}\text{-rings} \\
\{ \beta-\text{tors.}, \beta-\text{free} \} & \xrightarrow{\text{fgt}} \text{Alg}_A
\end{align*}
\]
Classical structures in general contexts:

1. **Ghost components**: canonical $\simeq \gamma$-operators

2. **S-components**: exist if $\psi = (\pi)$; $\simeq S$-operators depend on $\pi$

3. **Verschibung filtr.**: canonical $\simeq W_h(\mathbb{R})$

4. **Verschibung operator**: exist under Witt components, $\Theta_n$-operators

   Further assumptions, e.g., $A/\mathfrak{f} = \text{fin. field}$
   $F = \mathbb{C}^q$, $q = 1$, $\psi = (\pi)$.

**Explain:**

1. By def.

2. $\Lambda = A[e, s, s^2, \ldots], s = \frac{1}{q}(2-F)$,
   $W(R) \xrightarrow{\cong} \mathbb{R}^n$

   $(\Lambda \xrightarrow{X} \mathbb{R}^n) \cong (x(e), x(s), x(s^2), \ldots)$
NB. The $S$-components depend on the choice of $x$ in a complicated way.

(3) Define \( \Psi_n = A[e, \Psi, \ldots, \Psi^{\otimes n}] \) \( \Lambda_n \longrightarrow \Lambda \)

\( \text{pullback} \)

\( A[\beta^1] \otimes \Psi_n \longrightarrow A[\beta^1] \otimes \Psi \)

Locally, $\beta = 1_{n}$, so \( \Lambda_n = A[e, S, \ldots, S^{\otimes n}] \)

$S$ is a retract of $\Lambda$; therefore, the $A$-$A$-bining structure on $\Lambda$ defines one on $\Lambda_n$:

\( \Lambda_n \longrightarrow \Lambda \)

\( \text{pullback} \)

\( \Lambda_n \otimes \Lambda_n \longrightarrow \Lambda \otimes \Lambda \)

\( \text{by c locally a retract} \)

Consider three tensor factors to verify
\[ W_n(R) := \text{Hom}_{A^{\text{alg}}}(\Lambda^{	ext{alg}}, R) \]
\[ \uparrow \quad \uparrow \]
\[ W(R) := \text{Hom}_{A^{\text{alg}}}(\Lambda, R) \]
\[ \uparrow \quad \uparrow \]
\[ V^{\text{alg}} W(R) = \text{kernel} \]

Since \( \text{colim}_n \Lambda_n \xrightarrow{\sim} \Lambda \),
\[ W(R) \xrightarrow{\sim} \lim_n W_n(R), \]

(4) Assume \( \ell = (\pi) \), \( A/\ell \approx k \) a finite field, and \( F = c^q \), \( q \neq 1 \). Note that we need \( q > 1 \) to get

\[ x \equiv y \mod \ell \Rightarrow \]
\[ x q^n \equiv y q^n \mod \ell^{n+1} . \]

(For \( q=1 \), we have in fact,

\[ \theta_2 = \frac{1}{\kappa} (\theta_1 + \theta_1 \circ \theta_1) \]
\[ = \frac{1}{\kappa} S + S \circ S . \)
We may now define an operator

\[ V_\pi : W(R) \rightarrow W(R) \]

in \( \pi \)-Witt coordinates by the usual formula

\[(a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots).\]

In ghost components,

\[ \langle x_0, x_1, \ldots \rangle \mapsto \langle 0, \pi x_0, \pi x_1, \ldots \rangle, \]

so \( V_\pi \) does depend on the choice of \( \pi \); \( (F \circ V_\pi)(x) = \pi x \).

Can remove dependence on \( \pi \) as follows: The map

\[ \beta^{\pi \circ a} : W(R) \rightarrow W(R) \]

defined locally by

\[ \pi^{\beta a} \circ a \mapsto V_\pi^{\beta a} \]

does not depend on \( \pi \) and exists even if \( \beta \) is not principal. But the assumptions \( F = e^q, q \geq 1 \), etc. are needed.
For it is given in ghost coord by

\[ y \circ (x_0, x_1, \ldots) \]

\[ \langle 0, \ldots, 0, yx_0, yx_1, \ldots \rangle \]
Witt Vectors

Geometry, $\mathbb{F}_1$-picture

Multiple primes: $\mathcal{O}_K$ ring of integers in a number field $K$

Def: $\text{pre} = \Lambda(\ldots)$-ring

i) An $\mathcal{O}_K$-algebra $R$

ii) All primes $p \mid \mathcal{O}_K$, an $\mathcal{O}_K$-alg endo $\psi_p : R \rightarrow R$

iii) $\psi_{p_1} \circ \psi_{p_2} = \psi_{p_1 p_2}$

iv) $\psi_p$ reduces to the $q^\ell$th power Frob mod $pR$ where $q = \# (\mathcal{O}_K/p)$

Ex: $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$ primes $(p)$, $p$ prime

Commuting ring maps $\psi_p : R \rightarrow R$ s.t.

$\psi_p(x) \equiv x^p \mod pR$

Ex: $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$

$p = (1 + i)$

$p \equiv 3 \pmod{4}$, $p = (p)$

$p \equiv 1 \pmod{4}$, $p = (a + ib)$, $p = (a - (b))$, $a^2 + b^2 = p$

$\psi_{1 + i}(x) \equiv x^2 \mod (1 + i)R$

$\psi_p(x) \equiv x^p \mod pR$

$\psi_{a + bi}(x) \equiv x \cdot x^p \mod (a + bi)R$
Witt Vectors

Also works for function fields
\((\text{Fin ext. of}) \quad \mathbb{F}_p(t) = K\)
\[\mathbb{F}_p[t] = 0_K\]

We pass from pre-\(\bigwedge_K\) rings to \(\bigwedge_K\) rings in the usual way.

\(i)\quad p = (\prod p) \quad \text{all } p\)

Define \(S_p\) operators

\(ii)\quad \text{Usual case } \not\exists \text{ ghost ring } R'N^n\)

Multiple prime case \(R^{id(K)}Z_N^n\)

\(I\) any ideal \(I = p_1 \cdots p_n\)

\(Id(K) = \text{monoid of non-zero ideals under mult}\)

\[\mathbb{N} \oplus \mathbb{N}\]

max ideals \(p\)

\(R^{id(K)}\) is the cofree \(4_K\) ring on \(R\)
Witt Vectors

Ex: $R = O_K[z]$  
$\psi_p : z \mapsto z^{q_p}$

$K = q, \quad R = \mathbb{Z}[z], \quad z \mapsto z^p$

$\wedge_k^\times (\rightarrow \downarrow \leftarrow) W_k$

$\wedge_k - \text{rings}$
$O_k - \text{alg}$

Most important example: $K = q, \quad W_k = \text{big Witt-vectors}$
$W_q \text{ ring} = \lambda - \text{ring in } K - \text{theory}$

Thm: $\wedge_q = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, ... ]$

Like $\Theta_1, \Theta_2, ...$

g-compact

NB: $\text{Spec}(R^N) \neq \bigoplus \text{Spec}(R), \quad R \neq (0)$

Only true for finite sets $

X \text{ scheme } \quad W_{k,n}(X)$  
Want colimit, clever way

Cover by $\text{Spec}(R)$ to be well behaved
Witt Vectors

\[ X = \text{spec}(\mathbb{F}_p) \]

Work with the usual \( p \)-typical \( W \)

\[ W_n(\mathbb{F}_p) = \mathbb{Z}/p^{n+1}\mathbb{Z} \]
\[ W(\mathbb{F}_p) = \mathbb{Z}_p \]

\[ \text{Spec}(\mathbb{Z}_p) \]

\[ \text{spec} \underset{\text{colim}}{\lim} \text{Spec}(\mathbb{Z}/p^{n+1}\mathbb{Z}) =: \text{Spf}(\mathbb{Z}_p) \]

\[ W(\text{Spec}(\mathbb{F}_p)) = \text{Spf} \mathbb{Z}_p \]

\[ R \text{ ring} \]

\[ \text{Hom}(\text{Spec}(R), \text{Spec}(\mathbb{Z}_p)) = \text{Hom}(\mathbb{Z}_p, R) \]
\[ \text{Hom}(\text{Spec}(R), \text{Spf}(\mathbb{Z}_p)) \]
\[ = \text{Hom}(\text{Spec}(R), \text{colim Spec}(\mathbb{Z}_p/p^{n+1}\mathbb{Z})) \]
\[ = \text{colim Hom}(\text{Spec} R, \text{spec} \mathbb{Z}/p^{n+1}\mathbb{Z}) \]
\[ = \text{colim Hom}(\mathbb{Z}/p^{n+1}\mathbb{Z}, R) \]

\[ \text{Spec}(\mathbb{Z}_p) \text{ ignores top. on } \mathbb{Z}_p \]

\[ \text{Spf}(\mathbb{Z}_p) \text{ uses the } p \text{-adic topology} \]
Witt Vectors

\[ Z_p = \lim_{n \to \infty} \mathbb{Z}/p^{n+1}\mathbb{Z} \]

\[ \text{Spf} (Z_p) = \operatorname{colim}_n \text{ Spec } (Z/p^{n+1}Z) \]

Remembering \( Z_p \) is a limit.

Fact: \( W_n (R) = \lim_n (W^{(p_1)} \circ \ldots \circ W^{(p_n)} (R)) \)

\[ p_1, \ldots, p_n \text{ first } n \text{ primes} \]

\[ = \ldots \circ W^{(5)} \circ W^{(3)} \circ W^{(2)} (R) \]

Big Witt Vectors

\[ W_n (F_p) = \bigoplus_{n \geq 1} Z_p \]

\[ \bigoplus_{p \mid n} \]

\[ W_n (\mathbb{Z}) \rightarrow \mathbb{Z} \times \mathbb{Z} \times \ldots \]

\[ \text{ghost map} \]

\[ \mathbb{Z}^{[3]}_{[1, 2, \ldots]} \]

\[ p^{\text{th}} \text{ frob mult. by } p \]

(\( p \)-typ case \( \Rightarrow \) shift \( \frac{(0, 1, 2, \ldots)}{3} \))

\[ (p^2, p^3, p^3, \ldots) \]

\[ W_n (\mathbb{Z}) = \{ \langle a_1, a_2, \ldots \rangle \in \mathbb{Z}^\infty \mid \forall \text{ prime } p \}

\[ a_{np} \equiv a_n \mod p^{1+\operatorname{ord}_p(n)} \]
Witt Vectors

\[ \forall \mathfrak{q} \in \mathbb{R} \quad \frac{1}{\mathfrak{q}} \in R \quad \text{for all } \mathfrak{q} \notin \mathfrak{p} \text{ i.e. } R = \mathbb{Z}_{(\mathfrak{p})} - \text{alg} \]

\[ W_{\mathfrak{q}}(R) = \prod_{n=1}^{\infty} \frac{W^{(p^n)}(R)}{p^n} \]

\[ W_{K} \text{ is a comonad on } \mathcal{O}_K - \text{alg} \]

\[ \Rightarrow W_{K} \text{ will be a monad on } \mathcal{O}_K - \text{schemes} \]

\[ \xymatrix{ \text{"} \Lambda_{K} - \text{schemes"} \ar@{^{(}->}[rr] & & I_{K} \ar@{^{(}->}[rr] \ar[d] & & \mathcal{O}_{K} - \text{schemes} \ar@{^{(}->}[rr] & & \text{anitmetic jet space} \ar@{^{(}->}[rr] & & K - \text{alg} \ar@{^{(}->}[rr] \ar[d] & & L - \text{alg} \ar@{^{(}->}[rr] & & G = \text{Gal}(L/K) } \]
Witt Vectors

\[ \begin{align*}
K \text{-mod} & \rightarrow L \text{-mod} \text{ with } G \text{-action} \\
L \otimes_K & \xrightarrow{fg \cdot t} \text{Hom}_K(L, -) \\
L \text{-mod} & \xrightarrow{fg \cdot t} \Pi \otimes_G L \text{-mod}
\end{align*} \]

\[ K \rightarrow L \text{ subalgebra} \]
\[ G = \text{Gal}(L/K) \]

- G-action is descend data from L to K

\[ K = \mathbb{Q} \]

\[ \begin{align*}
\text{schemes} & \quad \text{Z - schemes} \\
\Lambda_{\mathbb{Q}} \text{-schemes} & \quad \text{F}_1 \text{-schemes}
\end{align*} \]

\[ \begin{align*}
\mathbb{Z} & \quad \text{Weil restrict. of scalars} \\
\mathbb{Z} \otimes_{\mathbb{F}_1} & \quad \text{base change}
\end{align*} \]

\[ \begin{align*}
\mathbb{F}_q[t] & \quad \mathbb{Z} \\
\uparrow & \quad \uparrow \\
\mathbb{F}_q & \quad \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{F}_1
\end{align*} \]

\[ \Lambda_{\mathbb{Q}}\text{-structure is descend data from } \mathbb{Z} \text{ to } \mathbb{F}_1 \]

Like $\$
Witt Vectors

\[ \text{Spec}(\mathbb{Z}) \]

\[ W(\text{Spec}(\mathbb{Z})) = \text{"Spec } \mathbb{Z} \text{ viewed as an } \mathbb{F}_1 \text{-scheme"} \]

as a \( \Lambda_{\mathbb{Q}} \)-scheme

\[ W(\text{Spec}(\mathbb{Z})) \text{ viewed } = \text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} \]

as a scheme \( \text{Spec } \mathbb{F}_1 \)

\[ \Lambda_{\mathbb{Q}} = \mathbb{Z}[\text{Gal}(\mathbb{Z}/\mathbb{F}_1)] \]

\[ = \mathbb{Z}[\mathfrak{g} \mid \mathfrak{g} \in \text{Gal}(\mathbb{Z}/\mathbb{F}_1)] \]

\[ \mathbb{Q} \otimes \Lambda_{\mathbb{Q}} = \mathbb{Z}[\psi_1, \psi_2, \ldots] \]

monoid \( \mathbb{N}^+ \)

Can do the real picture: \( \mathbb{F}_p[\text{EtJ}] \)-schemes

not over \( \mathbb{Z} \),

\[ \mathbb{F}_p \text{-schemes} \]
Witt vectors and semirings:

First recall big Witt vectors and $\lambda$-rings / $\mathbb{Z}$. Context: $A = \mathbb{Z}$; a pre-$\lambda$-str. on $A$-alg. $R$ is a family of commuting Frobenius lifts $\phi^p$, one for each prime number $p$. The general Witt vector construction produces a $\mathbb{Z}$-plethory $\Lambda$ and a Witt vector functor $W$.

In this case,

$\Lambda$-ring $\lambda$-ring (Grothendieck)

but this fact requires proof. The proof uses:

Thm (Wilkerson, Joyal) If $R/\mathbb{Z}$ is flat, then every pre-$\lambda$-structure is a $\lambda$-ring structure. 

To extend the above to semirings, we wish to find a "model" for $\Lambda \mathbb{Z}$ over $\mathbb{Z} N = \{0, 1, 2, \ldots \}$. This means a semiring $\Lambda N$ together with semiring homomorphisms

$$\delta^+ : \Lambda N \rightarrow \Lambda N \otimes \Lambda N$$

$$\delta^- : \Lambda N \rightarrow \Lambda N \otimes \Lambda N$$
\[ \Lambda_N \xrightarrow{\epsilon^*} \Lambda \]

and a ring homomorphism

\[ \Lambda_N \otimes \Lambda_N \rightarrow \Lambda_N \]

that is an isomorphism of \( \mathbb{Z} \)-plethories.

\[ \text{Rank } K_0 \text{ (reasonable cat.) } = \Lambda \]-ring,

in this situation, would expect semiring of \( \Lambda \)-classes to have a canonical \( \Lambda \)-semiring str. To prove this, one should probably look at all Schur functors acting on this semiring."

Preparation on symmetric functions:

Fact: \( \Lambda_\mathbb{Z} = \) the ring of symm. fets.

\[ = \varprojlim \mathbb{Z}[x_1, \ldots, x_n]^{\mathbb{E}_n} \]

where the limit is formed in the category of graded rings and deg \( x_i \) = 1. By Newton,
\[ \Lambda_Z = \mathbb{Z} \langle e_1, e_2, e_3, \ldots \rangle \]

where \( e_n \) is the \( n \)'th elementary symm. function

\[ e_n = x_1x_2 \cdots x_n + \text{all perm.} \]

For example,

\[ e_1 = x_1 + x_2 + x_3 + \cdots \]

\[ e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots \]

We note that

\[ e = e_1 \]

\[ \gamma_p = x_1^p + x_2^p + \cdots \]

\[ S_p = \frac{(x_1^p + x_2^p + \cdots) - (x_1 + x_2 + \cdots)^p}{p} \quad \text{coeff. in } \mathbb{Z} / p \mathbb{Z} \]

Now, as a \( \mathbb{Z} \)-module, \( \Lambda_Z \) has a basis with respect to which all plethory structure constants are in \( \mathbb{Z} \), and hence \( \Lambda_Z \) descends to \( \Lambda_{\mathbb{Z}} \) defined to be the \(\mathbb{Z}\)-span of said basis. (Remark: \( \mathbb{Z} \) is not flat over \( \mathbb{Z} \); the zero ring is the only ring flat over \( \mathbb{Z} \).)
Define a partition $\lambda$ to be a weakly decreasing sequence in $\mathbb{N}$, the terms of which tend to zero, e.g.,

$$\lambda = (5, 5, 4, 2, 1, 1, 0, 0, \ldots)$$

The associated monomial fact. $B$

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots + \text{perm.}$$

For instance,

$$e_n = m((1, \ldots, 1, 0, \ldots)) \quad n \text{ 1's.}$$

$$y_n = m((n, 0, 0, \ldots))$$

Easy fact: $\{m_\lambda\}_\lambda$ part. is a $\mathbb{Z}$-basis of $\Lambda_\mathbb{Z}$.

$$\Lambda_\mathbb{Z} \overset{\Delta^+}{\to} \Lambda_\mathbb{Z} \oplus \Lambda_\mathbb{Z} \oplus \Lambda_\mathbb{Z}$$

$$f(x_1, x_2, \ldots) \mapsto f(x_1 e_1, 1 e x_1, x_2 e_1, 1 e x_2, \ldots)$$

$$\Lambda_\mathbb{Z} \overset{\Delta^+}{\to} \Lambda_\mathbb{Z} \oplus \Lambda_\mathbb{Z} \oplus \Lambda_\mathbb{Z}$$

$$f(x_1, x_2, \ldots) \mapsto f(\ldots, x_i e x_j, \ldots)$$

The definition of $\Delta^+$ (resp. $\Delta^+$)
uses a choice of a bijection \( \mathbb{N} \approx \mathbb{N} \) (resp. \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \)), which one we choose does not matter, since we consider symm. functions.

Ex. Check that \( \Delta^p \) is ring-like:

\[
\Delta^+ (\Delta^p) = \Delta^+ (x_1^p + x_2^p + \cdots )
\]

\[
= (x_1 \otimes 1)^p + (1 \otimes x_1)^p + (x_2 \otimes 1)^p + (1 \otimes x_2)^p + \cdots 
\]

\[
= \Delta^p + \Delta^p
\]

\[
\Delta^x (\Delta^p) = \Delta^x (x_1^p + x_2^p + \cdots )
\]

\[
= \sum_{i,j} (x_i \otimes x_j)^p = (\sum x_i^p) \otimes (\sum x_j^p)
\]

\[
= \Delta^p \otimes \Delta^p.
\]

Finally,

\[
\Lambda _Z \otimes _Z \Lambda _Z \stackrel{\alpha}{\longrightarrow} \Lambda _Z
\]

\[
f \circ g \longrightarrow f (y_1, y_2, \cdots )
\]

\[
\text{If } g = y_1 + y_2 + \cdots \text{ with } y_j \text{ monomials in the } x_i.
\]
All structure maps on $\Lambda Z$ descend to the sub-semiring $\Lambda N \subset \Lambda Z$ defined to be the $\Lambda N$-span of the monomial basis $(m_N)^2$ part. So, for $R$ a semiring, define $W(R) = \text{Hom}_{\Lambda N-\text{alg}}(\Lambda N, R)$.

We note that Newton's theorem fails: as a $\Lambda N$-algebra, $\Lambda N$ is not free. So the underlying set of $W(R)$ is not a product of copies of $R$ unless $R$ is a $\mathbb{Z}$-algebra. The semirings $W(\mathbb{R}_{\geq 0}), W(\Lambda N), W(\mathbb{N})$ are all nice semirings, the latter two of which are countable. In addition, $W(\Lambda N)$ is an integral domain, $W(\Lambda N) \subset \Omega_{\text{hol}}(\text{Re}(s) > 1)$.

What is $W(\mathbb{R}_{\geq 0}^{\text{trig}})$?