(1) Consider the composition ring \( \Lambda'_p = \colim_{n, \psi} \Lambda_{p,n} \) as defined in the lecture. There we considered two elements \( a, b \in \Lambda'_p \), where \( a \) comes from \( e \in \Lambda_{p,1} \) and \( b \) comes from \( -\delta \in \Lambda_{p,1} \). So formally we could write \( a = \psi^{\delta - 1}, \ b = -\psi^{\delta - 1} \circ \delta \). Then \( a \) and \( b \) satisfy the relation \( e + pb = a^p \), and \( a \) is \( \mathbb{Z} \)-algebra-like (in the sense of ex. 2, q. 4), and \( b \) has the Leibniz rules

\[
\begin{align*}
    b(x + y) &= b(x) + b(y) + \sum_{i=1}^{p-1} \frac{1}{i} \binom{p}{i} a(x)^i a(y)^{p-i} \\
    b(xy) &= a(x)^p b(y) + b(x)a(y)^p + pb(x)b(y) \\
    b(0) &= 0 \\
    b(1) &= 0.
\end{align*}
\]

The converse is also true—any two such operators \( a \) and \( b \) on a ring come from a unique \( \Lambda'_p \)-structure. Give a completely detailed proof of this.

(2) Let \( S \) denote the subring of the product ring \( \mathbb{Z}^N \) consisting of all \( \langle a_0, a_1, \ldots \rangle \) such that \( a_{n+1} = a_n \mod p^{n+1} \) for all \( n \geq 0 \). In lecture it was stated that the ghost map \( W(\mathbb{Z}) \to \mathbb{Z}^N \) (which is injective) has image \( S \), and hence that the ghost map induces an isomorphism \( W(\mathbb{Z}) \to S \). Give a direct proof of this as follows:

First, show that \( S \) is the largest sub-\( \psi \)-ring of \( \mathbb{Z}^N \) on which \( \psi \) is a Frobenius lift; in other words, \( S \) is the largest sub-\( \psi \)-ring which is also a \( \delta \)-ring. Then show that \( S \) satisfies the universal property of \( W(\mathbb{Z}) \), namely that for any \( \delta \)-ring \( R \), any ring map \( R \to \mathbb{Z} \) lifts uniquely to a \( \delta \)-morphism \( R \to S \).

Your proof should use as little of the material developed in the class as possible.

(3) Let \( \Lambda'_p \) be as in question 1, and let \( W' \) denote the associated Witt vector functor. Determine \( W'(\mathbb{F}_p) \) and \( W'(\mathbb{Z}) \). (They are both isomorphic to familiar concrete rings. Hint: Use the concrete descriptions of \( W_n(\mathbb{F}_p) \) and \( W_n(\mathbb{Z}) \).) Are there any nonzero \( \mathbb{F}_p \)-algebras that admit a \( \Lambda'_p \)-ring structure? (Hint: Use the concrete description of \( W'(\mathbb{F}_p) \).)

(4) The functor \( W \) preserves surjections, simply because \( W(R) = R^N \), as a set-valued functor. Does \( W' \) preserve surjections?

(5) In lecture it was shown that the map \( W(\mathbb{Z}_p) \to \mathbb{Z}_p \) sending a Witt vector with ghost components \( \langle a_0, a_1, \ldots \rangle \) to its \( p \)-adic limit \( \lim_n a_n \) induces a ring map \( W(\mathbb{F}_p) \to \mathbb{Z}_p \) and that this map is an isomorphism.

Prove the following more general version of this. Let \( F \) be an extension of \( \mathbb{F}_p \) of degree \( d \). The theory of local fields show there is a unique (up to unique isomorphism) complete discrete valuation ring \( R \) together with an isomorphism \( R/pR \to F \), and that \( R \) has a unique endomorphism \( \sigma \) lifting the Frobenius map. (For example, if \( p = 1 \), then \( F = \mathbb{F}_p \) and \( R = \mathbb{Z}_p \). If \( d = 2 \) and \( p = 3 \mod 4 \), then \( F = \mathbb{F}_p[\iota] \) and \( R = \mathbb{Z}_p[\iota] \).) Find, as above, a morphism \( W(R) \to R \) factoring through \( W(R) \to W(R/pR) \to W(F) \) to an isomorphism \( W(F) \to R \).

Can you generalize this to where \( F \) is allowed to be an arbitrary perfect field of characteristic \( p \)?