Exercises for week 5.

(1) Show that the Verschiebung map $V_p: W(R) \to W(R)$ descends to a map $W_n(R) \to W_{n+1}(R)$ (also denoted $V_p$) and induces an exact sequence
$$0 \to W_n(R) \xrightarrow{V_p} W_{n+1}(R) \to R \to 0.$$ 
Use this to show that the canonical map $W_n(R)[1/p] \to W_n(R[1/p])$ is an isomorphism.

(2) Show that the $m$-th ghost component map $w_m: W_n(R) \to R$ is integral, which is to say that every element of the codomain $R$ satisfies a monic polynomial with coefficients in the image of $w_m$. Using this, show that the ghost map $W_n(R) \to R^{[0,n]}$ is integral. (Hint: recall that the sum of two integral elements is integral.)

(3) Prove the universal identity $V_p(x)V_p(y) - pV_p(xy)$. Use this to show that the kernel of the map $\mathbb{F}_p \otimes R \to \mathbb{F}_p \otimes R$ is nilpotent. Can you find a similar formula for $V_m(x)V_m(y)$?

(4) Combine the questions above to show that the ghost map induces an integral morphism
$$\coprod_{[0,n]} \Spec R \to \Spec R^{[0,n]} \to \Spec W_n(R)$$
of schemes, which is an isomorphism away from $p$. Show that modulo $p$, this morphism can be identified with the projection
$$\coprod_{[0,n]} \Spec \mathbb{F}_p \otimes R \to \Spec \mathbb{F}_p \otimes R,$$at least if we work modulo nilpotent elements. Conclude that the ghost map
$$\coprod_{[0,n]} \Spec R \to \Spec W_n(R)$$
is surjective. Thus one might say that, as a topological space, $\Spec W_n(R)$ looks like $n + 1$ copies of $\Spec R$ glued together modulo $p$. Draw a picture of $\Spec W_1(\mathbb{Z}[x])$.

(5) For any $a \in R$, show that the canonical map $W_n(R)[1/[a]] \to W_n(R[1/[a]])$ is an isomorphism, where $[a] = (a,0,0,\ldots,0) \in W_n(R)$ denotes the Teichmüller representative of $a$. 