

Λ -RINGS AND THE FIELD WITH ONE ELEMENT

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ABSTRACT. The theory of Λ -rings, in the sense of Grothendieck's Riemann–Roch theory, is an enrichment of the theory of commutative rings. In the same way, we can enrich usual algebraic geometry over the ring \mathbf{Z} of integers to produce Λ -algebraic geometry. We show that Λ -algebraic geometry is in a precise sense an algebraic geometry over a deeper base than \mathbf{Z} and that it has many properties predicted for algebraic geometry over the mythical field with one element. Moreover, it does this in a way that is both formally robust and closely related to active areas in arithmetic algebraic geometry.

INTRODUCTION

Many writers have mused about algebraic geometry over deeper bases than the ring \mathbf{Z} of integers. Although there are several, possibly unrelated reasons for this, here I will mention just two. The first is that the combinatorial nature of enumeration formulas in linear algebra over finite fields \mathbf{F}_q as q tends to 1 suggests that, just as one can work over all finite fields simultaneously by using algebraic geometry over \mathbf{Z} , perhaps one could bring in the combinatorics of finite sets by working over an even deeper base, one which somehow allows $q = 1$. It is common, following Tits [60], to call this mythical base \mathbf{F}_1 , the field with one element. (See also Steinberg [58], p. 279.) The second purpose is to prove the Riemann hypothesis. With the analogy between integers and polynomials in mind, we might hope that $\text{Spec } \mathbf{Z}$ would be a kind of curve over $\text{Spec } \mathbf{F}_1$, that $\text{Spec } \mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z}$ would not only make sense but be a surface bearing some kind of intersection theory, and that we could then mimic over \mathbf{Z} Weil's proof [64] of the Riemann hypothesis over function fields.¹ Of course, since \mathbf{Z} is the initial object in the category of rings, any theory of algebraic geometry over a deeper base would have to leave the usual world of rings and schemes.

The most obvious way of doing this is to consider weaker algebraic structures than rings (commutative, as always), such as commutative monoids, and to try using them as the affine building blocks for a more rigid theory of algebraic geometry. This has been pursued in a number of papers, which I will cite below. Another natural approach is motivated by the following question, first articulated by Soulé [57]:

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¹The origins of this idea are unknown to me. Manin [48] mentions it explicitly. According to Smirnov [56], the idea occurred to him in 1985 and he mentioned it explicitly in a talk in Shafarevich's seminar in 1990. It may well be that a number of people have had the idea independently since the appearance of Weil's proof.

Which rings over \mathbf{Z} can be defined over \mathbf{F}_1 ? Less set-theoretically, on a ring over \mathbf{Z} , what should descent data to \mathbf{F}_1 be?

The main goal of this paper is to show that a reasonable answer to this question is a Λ -ring structure, in the sense of Grothendieck's Riemann–Roch theory [31]. More precisely, we show that a Λ -ring structure on a ring can be thought of as descent data to a deeper base in the precise sense that it gives rise to a map from the big étale topos of $\text{Spec } \mathbf{Z}$ to a Λ -equivariant version of the big étale topos of $\text{Spec } \mathbf{Z}$, and that this deeper base has many properties expected of the field with one element. Not only does the resulting algebraic geometry fit into the supple formalism of topos theory, it is also arithmetically rich—unlike the category of sets, say, which is the deepest topos of all. For instance, it is closely related to global class field theory, complex multiplication, and crystalline cohomology.

So let us define an \mathbf{F}_1 -algebra to be a Λ -ring. (The language of \mathbf{F}_1 will quickly feel silly, but for most of this paper, it will be useful as an expository device.) More generally, define an \mathbf{F}_1 -scheme to be a scheme equipped with a Λ -structure. The theory of Λ -structures on schemes was introduced in Borger [6][7]. (Although see Grothendieck [34], p. 506.) Defining a Λ -structure on a general scheme X takes some time, as it does on a general ring. But when X is flat over \mathbf{Z} , there is a simple equivalent definition: it is a commuting family of endomorphisms $\psi_p: X \rightarrow X$, indexed by the set of prime numbers p , such that each ψ_p agrees with the p -th power Frobenius map on the special fiber $X \times_{\text{Spec } \mathbf{Z}} \text{Spec } \mathbf{F}_p$. It is also true that any reduced Λ -scheme is flat over \mathbf{Z} . Keeping these two facts in mind, it is possible to read most of this paper without knowing the definition in general.

If we take this \mathbf{F}_1 -to- Λ dictionary seriously, then the functor that removes the Λ -structure from a Λ -scheme should be thought of as removing the descent data, and hence as the base-change functor from \mathbf{F}_1 to \mathbf{Z} . So for instance, \mathbf{F}_1 itself should be defined to be \mathbf{Z} with its unique Λ -structure, where each ψ_p is the identity map. It is the initial object in the category of Λ -rings.

As indicated above, the definition of a Λ -structure extends not just to the category of schemes but to the entire ambient topos. By this, I mean the big étale topos over \mathbf{Z} , by definition the category of sheaves of sets on the category of affine schemes equipped with the étale topology. It is therefore natural to define the big étale topos over \mathbf{F}_1 to be the category of such sheaves with Λ -structure. The base-change functor v^* that strips off the Λ -structure then induces a well-defined map

$$v: \text{Spec } \mathbf{Z} \longrightarrow \text{Spec } \mathbf{F}_1,$$

not on the level of schemes, which would of course be meaningless, but on the level of big étale toposes.

The functor v^* has not only a right adjoint v_* , as required, but also a left adjoint $v_!$. If we think of v^* as the base-change functor, we should think of v_* as the Weil restriction of scalars functor and $v_!$ as the base-forgetting functor. In terms of definitions rather than interpretations, v_* sends a space to its arithmetic jet space, which is a multi-prime, algebraic version of Buium's p -jet space [9], which is in turn a formal p -adic lift of the Greenberg transform [29][30]. On the other hand, $v_!$ sends a space to its space of big Witt vectors. Both of these are given their natural Λ -structures. These constructions are exotic by the standards of algebraic geometry over fields, but even in the arithmetically local case, where we consider p -adic schemes with only one Frobenius lift ψ_p , they have proven very useful. See, for

example, Buium [9] for applications of the first and Illusie [37] for the second. (In fact, Buium has had similar ideas regarding Frobenius lifts and the field with one element. See [10], the preface of [9], or Buium–Simanca [11]. Manin [47] has also recently interpreted Witt vectors as being related to the field with one element.)

For applications to usual, non- Λ arithmetic algebraic geometry, the most interesting spaces over \mathbf{F}_1 are those obtained from schemes over \mathbf{Z} . Whether we produce them by applying $v_!$ or v_* , the result is almost never a scheme of finite type over \mathbf{F}_1 . So from the perspective of algebraic geometry over \mathbf{Z} , the spaces of principal interest over \mathbf{F}_1 are not those of finite type. In fact, as has been expected, there appear to be very few schemes of finite type over \mathbf{Z} that descend to \mathbf{F}_1 at all. But their infrequency is so extreme that they become interesting on their own terms. For example, it might be possible to describe the category of algebraic spaces of finite type over \mathbf{F}_1 in purely combinatorial terms, without any mention of algebraic geometry or Λ -structures.

This question is probably within reach, and the second purpose of this paper is to take some steps in that direction. For example, we prove the following theorem, stated here under restricted hypotheses:

0.1. Theorem. *Let X be a smooth proper scheme over $\mathbf{Z}[1/M]$, for some integer $M \geq 1$. If X descends to \mathbf{F}_1 (i.e., admits a Λ -structure), then it has the following properties:*

- (a) *The action of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on any p -adic étale cohomology group $H_{\text{ét}}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ factors through its abelianization.*
- (b) *There is an integer N , all of whose prime divisors divide Mp , such that the restriction of the representation $H_{\text{ét}}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ to $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ is isomorphic to a sum of powers of the cyclotomic character.*
- (c) *The (i, j) Hodge number of the mixed Hodge structure on the singular cohomology $H^n(X^{\text{an}}, \mathbf{C})$ is zero when $i \neq j$.*

Another way to express this theorem is that only abelian Artin–Tate motives can be defined over \mathbf{F}_1 . (Compare Soulé [57], 6.4, question 4.) While this means there are no motivically interesting \mathbf{F}_1 -schemes of finite type, the argument that shows this is rather interesting. In the zero-dimensional case, it is an elementary consequence of the Kronecker–Weber theorem and Chebotarev’s density theorem. (See Borger–de Smit [4].) In the higher-dimensional case, the argument also uses the Lefschetz theorem and the proper base change theorem for étale cohomology, on the one hand, and on the other, p -adic Hodge theory, including the potentially semi-stable theorem for arbitrary varieties proved by Kisin [41], following the work of many others.

The theorem above attests to the predicted combinatorial nature of schemes of finite type over \mathbf{F}_1 . I emphasize once again that the combinatorial nature is not built into the foundations of the theory—it is a consequence of hard arithmetic results in the presence of finiteness conditions. Indeed, I expect that the cohomological theory of infinite-dimensional spaces over \mathbf{F}_1 contains, via $v_!$ and de Rham–Witt theory, the full theory of motives, and even in a visible way. It is hard to imagine a theory defined in combinatorial terms having this property.

As strong as the cohomological restrictions above are, they are far from being sharp. In a future version of this paper, I hope to show that all examples of finite type come from toric varieties, in a certain precise sense. Here is a partial result in

this direction whose proof does appear in the present version. It is probably even a necessary step in the proof of the full classification theorem.

0.2. Theorem. *Let X be a Λ -scheme which is of finite type over \mathbf{Z} . Then X has a point with coordinates in a cyclotomic field. When X is proper, there is even a Λ -morphism $\text{Spec } \mathbf{Z} \rightarrow X$.*

In other words, every \mathbf{F}_1 -scheme of finite type has a cyclotomic point and, if proper, has an \mathbf{F}_1 -point. I believe that this result had not been predicted. Observe that in the affine case, the theorem is a new result about Λ -rings, which can be stated independently of the theory. It says that every Λ -ring which is finitely generated as a ring admits a ring map to some cyclotomic field. Even in this case, the proof uses all the deep arithmetic results above.

If we apply theorem 0.1 and the Lefschetz fixed-point formula at various primes p , we obtain the following result, which was essentially predicted (implicitly by the followers of Tits [60], and by definition in Kurokawa [44]):

0.3. Corollary. *Let X be a smooth proper scheme over $\mathbf{Z}[1/M]$, for some integer $M \geq 1$. If X descends to \mathbf{F}_1 , then there is an integer $N \geq 1$ divisible only by the primes dividing M such that for all prime powers $q > 1$ with $q \equiv 1 \pmod{N}$, the number of \mathbf{F}_q -valued points of X is $P(q)$, where P is the Hodge polynomial of X : $P(t) = \sum_{i,n} (-1)^n r_{i,n} t^i$, where $r_{i,n}$ is the (i, i) Hodge number of $H_c^n(X^{\text{an}}, \mathbf{C})$.*

In particular, if X is smooth and proper over \mathbf{Z} , the number of such points is given by the polynomial $P(q)$ for all prime powers $q > 1$.

Let us now consider \mathbf{F}_1 -points. It is not difficult to show, when X is separated and of finite type over \mathbf{F}_1 , that $X(\mathbf{F}_1)$ is finite. So it is tempting to hope that the number of \mathbf{F}_1 -valued points would be $P(1)$, the Euler characteristic of X . This is typically false, but if we restrict to complemented points—those whose complement is an open sub- \mathbf{F}_1 -scheme of X —then it is true in many situations. For instance, it is true for toric varieties, when given a certain natural, “toric” Λ -structure. But it is not always true. It fails for the toric varieties \mathbf{A}^1 and \mathbf{P}^1 if they are given the Chebychev Λ -structure (6.7). On the other hand, if all Λ -varieties can indeed be built from toric varieties with the toric Λ -structure, it might be possible to salvage the formula $X(\mathbf{F}_1) = P(1) = \chi(X)$ by cleverly reinterpreting the definitions.

Our final purpose is to consider variations on the definition of Λ -structure, such as function-field analogues. For example, instead of working over $\text{Spec } \mathbf{Z}$, we can work over a smooth curve S over a finite field k . Then our family of commuting Frobenius lifts would be indexed by the closed points of S , and the same procedure as over \mathbf{Z} gives the notion of a Λ_S -structure, a topos of \mathbf{F}_1^S -spaces, and a topos map $v_S: S \rightarrow \text{Spec } \mathbf{F}_1^S$. Having invented an absolute algebraic geometry relative to S , we can ask how it relates to the usual absolute algebraic geometry relative to S , that is, algebraic geometry over k . The answer is that the structure map $s: S \rightarrow \text{Spec } k$ factors naturally as a composition of topos maps:

$$\begin{array}{ccc} S & \xrightarrow{v_S} & \text{Spec } \mathbf{F}_1^S \\ & \searrow s & \swarrow f \\ & & \text{Spec } k. \end{array}$$

The map f is not an equivalence, but it is not far from being one. For example, f^* embeds the category of reduced algebraic spaces over k fully faithfully into the

category of spaces over \mathbf{F}_1^S , but there can be schemes defined over \mathbf{F}_1^S (given by certain rank-one Drinfeld modules) that do not descend to k . As a check to see if this could lead to a proof of the Riemann hypothesis for \mathbf{Z} , one might examine the translation of Weil’s proof of the Riemann hypothesis for S from algebraic geometry over k to that over \mathbf{F}_1^S . But since the current version of our theory says nothing about the archimedean place of \mathbf{Q} , it is hard to imagine this succeeding without further ideas. Even so, it should be done.

Let us now list the contents of this paper. In section 1, we recall the foundations of the theory of spaces over \mathbf{F}_1 (that is, Λ -spaces). In section 2, we give examples of \mathbf{F}_1 -schemes. In section 3, we discuss sub- \mathbf{F}_1 -spaces and in particular \mathbf{F}_1 -valued points. In section 4, we discuss function spaces over \mathbf{F}_1 and especially GL_n . In section 5, we show that abelian motives are Artin–Tate. This is a result in usual, non- Λ number theory needed in the following section. In section 6, we discuss the p -adic étale cohomology of \mathbf{F}_1 -schemes of finite type and implications for point counting. And in section 7, we consider variations on our approach to \mathbf{F}_1 for function fields and number fields larger than \mathbf{Q} .

OTHER WORK

In the early days of this project, I was greatly inspired by Manin’s exposition [48], Soulé’s paper [57], and Deninger’s program, for example [25]. In a strict mathematical sense, this paper does not owe them much, but their spiritual effect has been deep.

There are, of course, many approaches to absolute algebraic geometry which I did not mention above. Here are some I know about. One is that an \mathbf{F}_1 -algebra should be some kind of algebraic structure that is set-theoretically weaker than a commutative ring, for example a commutative monoid. From this point a view an \mathbf{F}_1 -vector space is often taken to be a set, perhaps with some additional weak structure. One could investigate the K -theory that comes out of this, and even aspire to see the place at infinity by incorporating archimedean information in these structures. Another approach has been to pursue notions of ζ -functions over \mathbf{F}_1 , perhaps independently of any formal definition of \mathbf{F}_1 . A final approach is to find and prove analogues over number fields of basic geometric results over function fields.

For these approaches see the following references: Baez [3], Connes–Consani [13], Connes–Consani–Marcoli [14], Deitmar [19][18][20], Diers [26], Durov [27], Haran [36] [35], Kapranov [38], Kapranov–Smirnov [39], Kurokawa [42][43][44], Kurokawa–Ochiai–Wakayama [45], Smirnov [55], Tate–Voloch [59], and Toën–Vaquié [61]. Several of these writers have other papers on the subject, but I believe these are representative of their approaches.

ACKNOWLEDGMENTS

My focused work on Λ -algebraic geometry began on October 22, 2003, when I read an email from Ivan Fesenko asking if there were relations between my paper [5] with Ben Wieland and the field with one element. That question instantly gave direction to some scattered thoughts about Λ -algebraic geometry and the p -adic absolute point (7.10). My interests at the time were arithmetically local, and as obvious as the global connection is in retrospect, it had not occurred to me before that day. So I thank him greatly.

I would also like to thank Mark Kisin for many conversations about this project over several years. Most of his influence on this project is on forthcoming work, but even with this paper, if he had not insisted that I begin writing up what I knew in the case of Λ -varieties of finite type over \mathbf{Z} , I never would have uncovered much of what is here. And since the foundational papers [6][7] owe their existence to the present paper, perhaps they also owe some of it to him.

I did some of this work in 2004–2005 at the Institut des Hautes Études Scientifiques and the Max-Planck-Institut für Mathematik. It is a pleasure to thank them for their generous support and nearly ideal working environments. I did more work on this topic in January 2006, as a visitor at the University of Chicago. I thank Alexander Beilinson and Vladimir Drinfeld for making that possible.

I have given a number of lectures on this material, and this paper has benefited from questions many people have asked, especially Alexandru Buium, Jordan Ellenberg, and Kirsten Wickelgren. I would particularly like to thank Wickelgren for some questions that directly inspired the material in section 4.

CONTENTS

| | |
|--|----|
| Introduction | 1 |
| Other work | 5 |
| Acknowledgments | 5 |
| 1. Λ -spaces | 6 |
| 2. Examples | 8 |
| 3. Sub- Λ -spaces | 11 |
| 4. Function spaces | 13 |
| 5. Aside: abelian motives are potentially cyclotomic | 16 |
| 6. p -adic étale cohomology | 19 |
| 7. Variants | 24 |
| References | 29 |

1. Λ -SPACES

1.1. General Λ -spaces. Let us quickly review [7]. Let $W^* = W_\infty^*$ denote the infinite-length big Witt vector functor from the category of spaces to itself. (The category of spaces is, by definition, the category of sheaves of sets on the category of affine schemes under the étale topology.) Then W^* carries a monad structure, and a Λ -structure on a space X is by definition an action of W^* on X . Note that if X is an algebraic space, then the space $W^*(X)$ is ind-algebraic but typically not algebraic. This is just the familiar fact that the ring of infinite-length Witt vectors is naturally a projective limit of rings.

Here are some important examples. If A is a ring, then we have $W^*(\mathrm{Spec} A) = \mathrm{colim}_n \mathrm{Spec} W_n(A)$, where W_n denotes the functor of (big) Witt vectors of length n . Therefore a Λ -structure on the space $\mathrm{Spec} A$ is the same as a Λ -ring structure on A in the usual sense. If X is a flat algebraic space over \mathbf{Z} , then Λ -structure on X is the same as a commuting family of endomorphisms $\psi_p: X \rightarrow X$, one for each prime p , such that ψ_p agrees with the p -th power Frobenius map on $X \times_{\mathrm{Spec} \mathbf{Z}} \mathrm{Spec} \mathbf{F}_p$. If a reduced algebraic space admits a Λ -structure, then it must be flat over \mathbf{Z} . Finally, if X is a Λ -algebraic space, then X_{red} is a closed Λ -algebraic subspace. (Because

a Λ -structure is given by a monad action, a subspace of a Λ -space can have at most one compatible Λ -structure.) Non-reduced Λ -algebraic spaces appear only sporadically in this paper.

The functor W^* has a right adjoint $W_* = W_{\infty*}$, the arithmetic jet space functor, which is a generalized version of Buium’s p -jet space functor [9]. If A is a ring, then $W_*(\text{Spec } A) = \text{Spec } \Lambda \odot A$, where $\Lambda \odot A$ denotes the Λ -ring freely generated by A . For example, $W_*(\mathbf{A}_{\mathbf{Z}}^1) = \text{Spec } \Lambda$, where Λ is the free Λ -ring on one generator, the ring of symmetric functions in infinitely many variables.

Let $S = \text{Spec } \mathbf{Z}$, and let $\mathbf{Sp}_S = \mathbf{Sp}_{\mathbf{Z}}$ denote the category of spaces. Let $\mathbf{Sp}_{S/\Lambda}$ denote the category of Λ -spaces (with Λ -equivariant, or rather W^* -equivariant, morphisms). This can also be described using W_* . By adjunction, W_* inherits a comonad structure from the monad structure on W^* . The category of Λ -spaces is the same as the category of spaces equipped with an action of the comonad W_* . It is therefore a topos, like \mathbf{Sp}_S .

1.2. Categorical structure. Let $v^*: \mathbf{Sp}_{S/\Lambda} \rightarrow \mathbf{Sp}_S$ denote the functor that simply strips off the Λ -structure. The point of this paper is that v^* can also be thought of as the functor that strips off the descent data from \mathbf{Z} to \mathbf{F}_1 , and hence that it can be thought of as the base-change functor from \mathbf{F}_1 to \mathbf{Z} . Therefore, we have the following equation:

$$\begin{array}{ccc}
 \begin{array}{c} \mathbf{Sp}_S \\ \uparrow v^* \\ \mathbf{Sp}_{S/\Lambda} \\ \downarrow v_* \\ \mathbf{Sp}_S \end{array} & = & \begin{array}{c} \mathbf{Sp}_{\mathbf{Z}} \\ \uparrow \text{forget base} \\ S \times_{\mathbf{F}_1} - \\ \downarrow \text{Weil restrict} \\ \mathbf{Sp}_{\mathbf{F}_1} \end{array}
 \end{array}$$

This means that the structure on the right is defined to be that on the left, or that we think of the precisely defined left-hand side using the geometric language of the right-hand side. Each functor is the left adjoint of the one to its right. The left adjoint $v_!$ of v^* sends X to its Witt space $W^*(X)$ with the natural Λ -structure, and the right adjoint v_* sends X to its arithmetic jet space $W_*(X)$, again with the natural Λ -structure. As always the left adjoint of a base-change functor is called base-forgetting, and the right adjoint is called Weil restriction of scalars. In particular, if one accepts the premise of this paper, the space $\text{Spec } \mathbf{Z} \times_{\text{Spec } \mathbf{F}_1} \text{Spec } \mathbf{Z}$ must be defined to be the Witt space $W^*(\text{Spec } \mathbf{Z})$.

These three adjoint functors form, by definition, an essential topos map $v: \mathbf{Sp}_{\mathbf{Z}} \rightarrow \mathbf{Sp}_{\mathbf{F}_1}$. This is what one would hope to have with any algebraic geometry over a deeper base than $\text{Spec } \mathbf{Z}$. Similarly, the base-forgetting functor $v_!$ is faithful but not full. (See [6], 12.2.)

Yet another way of expressing the point of this paper, in the playful tradition of the field with one element, is the nonsense formula “ $\Lambda = \mathbf{Z}[\text{Gal}(\mathbf{Z}/\mathbf{F}_1)]$ ”. The meaning of this is that if descent from \mathbf{Z} to \mathbf{F}_1 were controlled by a finite group, one would call it $\text{Gal}(\mathbf{Z}/\mathbf{F}_1)$. In that case, descent for rings would alternatively be controlled by the plethory $\mathbf{Z}[\text{Gal}(\mathbf{Z}/\mathbf{F}_1)]$ whose underlying ring would be the polynomial ring freely generated by the set $\text{Gal}(\mathbf{Z}/\mathbf{F}_1)$. (See Borger–Wieland [5].) But the plethory that actually controls this is Λ . So while the group $\text{Gal}(\mathbf{Z}/\mathbf{F}_1)$ does not exist, the polynomial algebra it would generate if it did exist does.

Note that everything above extends to the case where S is the spectrum of the ring of integers of any number field or any smooth curve over a finite field. Thus for any such S , there is a topos $\mathbf{Sp}_{\mathbf{F}_1^S} = \mathbf{Sp}_{S/\Lambda_S}$, the topos of spaces over the S -variant of the field with one element. These toposes are all related as S varies, and as one would expect, $S = \mathrm{Spec} \mathbf{Z}$, as above, gives the deepest one. We will return to this in section 7.

1.3. Λ -modules. An important topic we will not discuss in this paper is the Λ -analogue of a quasi-coherent sheaf. The non-linear nature of Λ -structures makes module theory slightly subtler than it is in equivariant algebraic geometry under actions of monoids or Lie algebras, or more generally in the context of Toën–Vaquié [61]. The reason for this, in the language of Borger–Wieland [5], is that the additive bialgebra of the plethory Λ does not agree with the cotangent algebra of Λ . Therefore one cannot properly speak about modules without specifying certain extra information. In the case of Λ , these are essentially the slopes of the Frobenius operators. I mention this here only because \mathbf{F}_1 -modules are a frequent concern in papers on the field with one element and I will not address them.

2. EXAMPLES

2.1. *The point.* The ring \mathbf{Z} has a unique Λ -structure—each ψ_p is the identity. Under this structure, it is the initial object in the category of Λ -rings. It is therefore reasonable to denote it \mathbf{F}_1 and call $\mathrm{Spec} \mathbf{Z}$, viewed as a Λ -space, the absolute point.

2.2. *Monoid algebras.* If M is any commutative monoid, the monoid algebra $\mathbf{Z}[M]$ has a natural Λ -action induced by $\psi_p: m \mapsto m^p$ for any $m \in M$, and so $\mathrm{Spec} \mathbf{Z}[M]$ descends naturally to \mathbf{F}_1 . I will call this Λ -action the *toric* Λ -action. For example, given a choice of coordinates, $\mathbf{A}^r \times \mathbf{G}_m^s$ equals $\mathrm{Spec} \mathbf{Z}[\mathbf{N}^r \times \mathbf{Z}^s]$, which descends naturally to \mathbf{F}_1 .

In fact, we have even more: the monoid scheme structure on $\mathrm{Spec} \mathbf{Z}[M]$ also descends to \mathbf{F}_1 , as does the group structure if M is a group. Indeed, the coalgebra structure on $\mathbf{Z}[M]$ given by $m \mapsto m \otimes m$ for all $m \in M$ is a map of Λ -rings. The same is true of the counit and, if M is a group, the antipode $m \mapsto m^{-1}$. For example, all split tori can be thought of as group schemes over \mathbf{F}_1 . The group scheme $\mu_n = \mathrm{Spec} \mathbf{Z}[x]/(x^n - 1) = \mathrm{Spec} \mathbf{Z}[\mathbf{Z}/n\mathbf{Z}]$ also descends to \mathbf{F}_1 . Following Kapranov–Smirnov [39], Soulé calls this the base change to \mathbf{Z} of \mathbf{F}_{1^n} , the field with 1^n elements [57].

2.3. *Limits and colimits.* The category of Λ -rings has products and coproducts, and their underlying rings agree with the same constructions taken in the category of rings. In fact, this is true for all limits and colimits—in particular pull-backs, push-outs, direct limits, and inverse limits. This is because the forgetful functor $\mathrm{Ring}_\Lambda \rightarrow \mathrm{Ring}_{\mathbf{Z}}$ has a left and a right adjoint.

Since it is a topos, $\mathbf{Sp}_{S/\Lambda}$ also has all limits and colimits. And since v^* has a left and a right adjoint, the space underlying any limit, or colimit, of Λ -spaces agrees with the limit, or colimit, or the underlying spaces. Further, since algebraization is a left adjoint, it commutes with W^* . Therefore the algebraization of a Λ -space is an algebraic Λ -space. In particular, given a system of algebraic spaces with Λ -actions, the colimit taken in the category of algebraic spaces has a unique compatible Λ -action. (The analogous fact for limits is true simply because the subcategory of \mathbf{Sp}_S consisting of algebraic spaces is closed under limits.)

For the same reason, the affinization of an algebraic Λ -space is an affine Λ -space.

2.4. Toric varieties. A toric variety is a colimit of spectra of monoid rings, where the maps in the system are open immersions induced by maps of the monoids. Therefore the colimit in \mathbf{Sp}_S is an algebraic space (and even a scheme). By 2.2 and 2.3, they carry natural Λ -actions. Thus toric varieties descend to \mathbf{F}_1 . (Compare Soulé [57].) In particular, projective spaces \mathbf{P}^n do, once we choose homogeneous coordinates. Then the Frobenius lifts are given by

$$\psi_p: [x_0, \dots, x_n] \mapsto [x_0^p, \dots, x_n^p].$$

2.5. The Chebychev line. Clauwens [12] has used Ritt’s work [51][52] to argue that, up to isomorphism, the affine line $\text{Spec } \mathbf{Z}[x]$ has exactly one Λ -structure besides the toric one of 2.2. It can be described as follows. The ring $\mathbf{Z}[t^{\pm 1}]$, endowed with the toric Λ -action, has a Λ -involution $t \mapsto t^{-1}$. By 2.3, the fixed subring is naturally a Λ -ring. It is freely generated as a ring by $x = t + t^{-1}$, and this gives the other Λ -action on $\mathbf{Z}[x]$. It also has a simple K -theoretic interpretation as the representation ring of the algebraic group SL_2 , but in this paper we are regarding the connection between Λ -rings and K -theory as a curiosity. The polynomials $\psi_p(x)$ are Chebychev polynomials:

$$\psi_2(x) = x^2 - 2, \quad \psi_3(x) = x^3 - 3x, \quad \psi_5(x) = x^5 - 5x^3 + 5x, \quad \dots$$

More generally, any subgroup of $\text{GL}_n(\mathbf{Z})$ acts on the toric Λ -ring $\mathbf{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and the invariant subrings give more examples of Λ -rings. For example, if we take the permutation representation $S_n \rightarrow \text{GL}_n(\mathbf{Z})$ of the n -th symmetric group, the invariant subring is isomorphic to $\mathbf{Z}[\lambda_1, \dots, \lambda_{n-1}, \lambda_n^{\pm 1}]$ and thus gives a non-toric Λ -action on $\mathbf{A}^{n-1} \times \mathbf{G}_m$.

It would be interesting to generalize Clauwens’ results. For example, are there only finitely many isomorphism classes of Λ -structures on \mathbf{A}^2 ?

2.6. Singular lines. We can divide out, not just by group actions, as in 2.5, but also by any Λ -equivalence relation, whether we take the quotient in the category of affine Λ -spaces, algebraic Λ -spaces, or all Λ -spaces. For instance, on \mathbf{G}_m with the toric Λ -action, we can identify 1 and -1 to make a nodal line

$$A = \{f(z) \in \mathbf{Z}[t^{\pm 1}] \mid f(-1) = f(1)\},$$

or we can identify 1 with itself to order two to make a cuspidal line

$$A' = \{f(z) \in \mathbf{Z}[t^{\pm 1}] \mid f'(1) = 0\}.$$

It is easy to check that these subrings of $\mathbf{Z}[t^{\pm 1}]$ are sub- Λ -rings.

Observe that we can identify the points $1, q \in \mathbf{G}_m$ only when $q = \pm 1$ —otherwise, the ψ_p operators would fail to descend to the quotient. But if we use Λ -algebraic parameter spaces, there are non-discrete moduli. For instance, in the family

$$\mathbf{G}_m \times \mathbf{G}_m \xrightarrow{\text{pr}_1} \mathbf{G}_m,$$

we can identify the diagonal section Δ and the identity section $\text{id}_{\mathbf{G}_m} \times 1$, to get the family of nodal lines $\mathbf{G}_m/(1 \sim q)$ parameterized by $q \in \mathbf{G}_m$. The result is a perfectly legitimate Λ -algebraic family of nodal Λ -quotients of \mathbf{G}_m . The reason why before q could only lie in a finite set is that there we insisted that the endomorphisms ψ_p act trivially on the parameter space. We will see below that an algebraic Λ -space of finite type over \mathbf{Z} has only finitely many points (with coordinates in \mathbf{C} , say) fixed by the ψ_p operators (3.2). Therefore in any Λ -algebraic family, only finitely many

of the fibers will be stable under the ψ_p . In particular this will be true for any universal family, assuming the Λ -moduli space is of finite type. So the finiteness phenomenon above is rather general.

We can also contract any Λ -invariant modulus on \mathbf{G}_m . If $f(x)$ is a product of polynomials the form $x^n - 1$, then the two maps

$$\mathbf{Z}[x^{\pm 1}] \rightrightarrows \mathbf{Z}[x^{\pm 1}]/(f(x)),$$

one given by $x \mapsto x$ and the other by $x \mapsto 1$, are Λ -ring maps. Therefore their equalizer is a Λ -ring. Its spectrum is \mathbf{G}_m with the zero locus of $f(x)$ contracted to a point.

Last, these constructions can be used to make Λ -schemes that cannot be covered by open affine Λ -schemes. In particular, it is inaccurate to say that Λ -schemes are formed by gluing Λ -rings together in the Zariski topology, though it is generally the right idea. The following example is due to Ben Wieland. Consider \mathbf{P}^1 with the toric Λ -structure, and let X be the quotient by the Λ -equivalence relation $0 \sim \infty$. Then there is no open immersion $j: U \rightarrow X$ which has the following properties: U is an affine Λ -scheme, j is a Λ -map, and the nodal point $0 = \infty$ is in the image. Indeed, if there were such a neighborhood U , then since it would be affine, the set $U(\mathbf{C})$ would be the complement in $X(\mathbf{C})$ of a finite nonempty subset T of $\mathbf{G}_m(\mathbf{C})$. But $U \cap \mathbf{G}_m$ would also be a sub- Λ -space of \mathbf{G}_m . Since ψ_n on \mathbf{G}_m is the n -th power map, T would have to be closed under the extraction of n -th roots, which is not the case for any finite nonempty subset of \mathbf{C}^* .

2.7. Zero-dimensional varieties. As shown in Borger–de Smit [4], it follows from class field theory that every Λ -ring which is both finite over \mathbf{Z} and reduced is contained in a product of cyclotomic fields. It is in fact isomorphic to a sub- Λ -ring of a product of toric Λ -rings of the form $\mathbf{Z}[x]/(x^n - 1)$. This can be viewed as an integral version of the Kronecker–Weber theorem. Thus even zero-dimensional Λ -algebraic geometry is somewhat interesting. In fact, the proofs of several theorems about higher-dimensional Λ -varieties use the zero-dimensional theory in key ways.

2.8. Non-example: flag varieties. It follows from a theorem of Paranjape and Srinivas [50] that no flag varieties besides projective spaces \mathbf{P}^n admit even one Frobenius lift ψ_p . Therefore, besides \mathbf{P}^n , flag varieties are not defined over \mathbf{F}_1 .

In 6.2 below, we show that there are strong motivic conditions on a variety for it to descend to \mathbf{F}_1 . But in the case of flag varieties, the obstruction is not in the motive. Indeed, flag varieties are paved by affine spaces and are therefore indistinguishable from them from the point of view of motives. It would be interesting to know whether flag varieties admit a weakened version of a Λ -structure but which is still stronger than being paved by Λ -varieties.

2.9. Non-example: curves of genus $g \geq 1$. Let $C_{\mathbf{Q}}$ be a connected smooth proper curve over \mathbf{Q} of genus $g \geq 1$. Choose in integer $M \geq 1$ such that $C_{\mathbf{Q}}$ has a connected smooth proper model C over $\mathbf{Z}[1/M]$. Then C has no Λ -structure. One can see this as follows.

If $g \geq 2$, let p be a prime number such that $p \nmid M$. Then since $g \geq 2$, the map ψ_p cannot be a constant map on the fiber over \mathbf{Q} because it is not on the fiber over \mathbf{F}_p . Therefore it must be an automorphism on the fiber over \mathbf{Q} . Further, there must exist an integer $n \geq 1$ such that $\psi_p^{o_n}$ is the identity on the fiber over \mathbf{Q} . But

as $C_{\mathbf{Q}}$ is dense in C , we see that $\psi_p^{\circ n}$ is the identity. This contradicts the fact that ψ_p is the Frobenius map on $C_{\mathbf{F}_p}$, which is a nonempty curve.

If $g \geq 1$, then for any prime $p \nmid M$, there is a finite extension K of \mathbf{Q}_p and point $e \in C(K) = C(\mathcal{O}_K)$, where \mathcal{O}_K denote the integral closure of \mathbf{Z}_p in K . Let us now take the group law on $C_{\mathcal{O}_K}$ to be the one for which e is the identity, and let E denote the endomorphism ring $\text{End}(C_{\mathcal{O}_K})$. By enlarging K , we may assume $E = \text{End}(C_K)$. Note that E is an integral domain of rank 1 or 2 over \mathbf{Z} .

Then there is an element $\varphi \in E$ such that $\psi_p(x) = \psi_p(e) + \varphi(x)$, for all points x . Therefore on each fiber of $C_{\mathcal{O}_K}$ over \mathcal{O}_K , the degree of ψ_p is $\varphi\bar{\varphi} \in \mathbf{Z}$. But on the fiber over the residue field of \mathcal{O}_K , the map ψ_p agrees with the base-change of the p -th power Frobenius map, which has degree p . Therefore, we have $\varphi\bar{\varphi} = p$. This rules out $E = \mathbf{Z}$. To rule out the other case, observe that the same equation implies p is not inert in E . But since p was allowed to be any sufficiently large prime, this is impossible.

3. SUB-Λ-SPACES

3.1. Periodic primes. Let X be a Λ -space. Let us say that a prime number p is *periodic* if there exists an integer $m \geq 1$ such that the endomorphism $\psi_p^{\circ m}$ of X is the identity. We also say that m a *period* of p . (Also see Davydov [16].)

3.2. Proposition. *Let X be a separated algebraic Λ -space of finite type over \mathbf{Z} with infinitely many periodic primes. Then X is affine and quasi-finite over \mathbf{Z} .*

Proof. Let us consider quasi-finiteness first. It suffices to assume X is reduced and, hence, flat over \mathbf{Z} . (See [7].) Therefore it is enough to show that $X \times_{\text{Spec } \mathbf{Z}} \text{Spec } \mathbf{Q}$ is finite over $\text{Spec } \mathbf{Q}$.

Let p be one of the given primes, and let X_p denote the fiber of X over p . Then ψ_p is periodic, and therefore the p -th power Frobenius map on X_p is periodic. By 3.3 below, the fiber X_p is finite over \mathbf{F}_p . Because there are infinitely many such p , X is finite over \mathbf{Z} at a dense set of scheme-theoretic points. And because X is of finite type, its fiber over \mathbf{Q} is finite. (See EGA IV (9.2.6.2) [33].)

Affineness follows from quasi-finiteness. Since X is separated, of finite type, and quasi-finite over \mathbf{Z} , Zariski's Main Theorem [32], III 4.4.3, implies it is an open subscheme of a scheme which is finite over \mathbf{Z} , and any such scheme is affine. \square

3.3. Lemma. *Let X be an algebraic space of finite type over \mathbf{F}_p , and let Fr_X denote the p -th power Frobenius map on X . If $\text{Fr}_X^{\circ n}$ is the identity map, then X is a finite disjoint union of spaces of the form $\text{Spec } F$, where F is a field of degree at most n over \mathbf{F}_p .*

Proof. Let $U = \text{Spec } B$ be an affine étale cover of X . Then $\text{Fr}_U^{\circ n}$ is the identity map on U . In particular, B is reduced and is hence a subring of a finite product of fields. Because the p -th power map on B has period n , the image in each field is a field of degree at most n over \mathbf{F}_p . Since U is a finite disjoint union of spectra of finite fields, and since U covers X , X is also such a space. \square

3.4. Proposition. *Let X and Y be separated algebraic Λ -spaces of finite type over \mathbf{Z} . Assume that X is reduced and has infinitely many periodic primes. Then $\text{Hom}_{\Lambda}(X, Y)$ is finite.*

Proof. Since X is reduced, we can assume Y is reduced. We can also assume that Y satisfies the same periodicity conditions as X . Indeed, let p be a periodic prime of X , and let m_p denote its minimal period. Then the equalizer Y_p of $\psi_p^{m_p}$ and the identity map is a closed sub- Λ -space of Y . Therefore, so is the reduced subspace of $Y' = \bigcap_p Y_p$, where p runs over all periodic primes of X . But any map $X \rightarrow Y$ factors through Y' , which as a closed subspace of Y is of finite type over \mathbf{Z} . Therefore it is enough to assume $Y = Y'$, which is to say that Y satisfies the same periodicity conditions as X .

By 3.2, there are generically finite rings A and B such that $X = \text{Spec } A$ and $Y = \text{Spec } B$. Because A and B are reduced Λ -rings, they are torsion free, and thus

$$\text{Hom}_\Lambda(X, Y) = \text{Hom}(B, A) \subseteq \text{Hom}(\mathbf{Q} \otimes_{\mathbf{Z}} B, \mathbf{Q} \otimes_{\mathbf{Z}} A).$$

By Galois theory, there are only finitely many ring maps between two finite étale algebras over a field; so $\text{Hom}_\Lambda(X, Y)$ is finite. \square

3.5. Corollary. *Let X be a separated algebraic space of finite type over \mathbf{F}_1 . Then there are only finitely many Λ -maps $\mu_n \rightarrow X$, where μ_n is defined in (2.2). In particular, X has only finitely many \mathbf{F}_1 -valued points.*

3.6. Primitive Λ -spaces and complemented sub- Λ -spaces. Let $f : Y \rightarrow X$ be a map of Λ -algebraic spaces which is a closed (resp. open) immersion. Then Y (or better, f) is said to be *complemented* if the complementary open (resp. reduced closed) algebraic subspace T admits a Λ -structure such that the map $T \rightarrow X$ is a Λ -map. (Compare SGA 4 IV 9.1.13c [1].) Note that because $T \rightarrow X$ is a monomorphism, the Λ -structure on T , when it exists, must be unique.

We also say that a Λ -space X is *primitive* if it is nonempty and its only nonempty complemented closed sub- Λ -space is itself.

Observe that if $X' \rightarrow X$ is a Λ -morphism and Y is a complemented closed (resp. open) algebraic subspace of X , then its preimage $X' \times_X Y$ is a complemented closed (resp. open) algebraic subspace of X' . Also it is clear that finite intersections of complemented closed (resp. open) Λ -algebraic subspaces are again complemented. Therefore the same is true for finite unions.

For example, consider \mathbf{A}^1 with the toric Λ -structure. The \mathbf{Z} -valued Λ -points of \mathbf{A}^1 are in bijection with $\text{Hom}_\Lambda(\mathbf{Z}[x], \mathbf{Z})$, which agrees with

$$\{x \in \mathbf{Z} \mid x^p = x \text{ for all primes } p\} = \{0, 1\}.$$

So 0 and 1 are the only \mathbf{F}_1 -valued points of the toric \mathbf{A}^1 . The point 0 is complemented, because its complement is the toric \mathbf{G}_m . But the point 1 is not complemented, because the preimage of 1 under ψ_2 , say, is μ_2 , which is not contained in $\{1\}$. More generally, the set of \mathbf{F}_1 -valued points of \mathbf{A}^d is $\{0, 1\}^d$, but the only complemented point is the origin.

3.7. Proposition. *The Λ -space \mathbf{G}_m^d is primitive, for any integer $d \geq 0$.*

Proof. Let Z be a nonempty complemented closed sub- Λ -space of \mathbf{G}_m^d . Since Z_{red} is flat over \mathbf{Z} ([7]) and nonempty, the space Z has a \mathbf{C} -valued point z . Write $z = (e^{w_1}, \dots, e^{w_d})$ for some numbers $w_1, \dots, w_d \in \mathbf{C}$. For each integer $s \geq 1$, the point

$$z_s = (e^{w_1/s}, \dots, e^{w_d/s})$$

satisfies $\psi_s(z_s) = z \in Z$. Because Z is complemented, we then have $z_s \in Z$. But in the analytic topology we have

$$\lim_{s \rightarrow \infty} z_s = (1, \dots, 1).$$

Since Z is closed (in the Zariski and hence analytic topology), it contains $(1, \dots, 1)$, and because it is complemented, it must contain $\psi_n^{-1}(1, \dots, 1) = \mu_n^d$ for any integer $n \geq 1$. On the other hand, $\cup_n \mu_n$ is Zariski dense in \mathbf{G}_m . Therefore $\cup_n \mu_n^d$ is Zariski dense in \mathbf{G}_m^d , and so we have $Z = \mathbf{G}_m^d$. \square

3.8. Proposition. *Let X be a toric variety. Then a closed sub- Λ -space is complemented if and only if it is a union of closures of torus orbits.*

For background on toric varieties, see Fulton’s notes [28], especially sections 2.1 and 3.2.

Proof. Let Z be a closed sub- Λ -space of X . Suppose Z is a union of closures Z_i of torus orbits. Then each Z_i is the toric subvariety corresponding to a fan, and is therefore complemented. Since there are only finitely many torus orbits, the union Z of the Z_i is complemented.

Now assume instead that Z is complemented. Then its intersection with any torus orbit Y is either Y or \emptyset , by 3.7. Therefore Z is a union of torus orbits. Since Z is closed, it is also a union of their closures. \square

3.9. Corollary. *The complemented \mathbf{F}_1 -points of a toric variety X are the fixed points of the torus action. In particular, the number of complemented \mathbf{F}_1 -points is the Euler characteristic.*

For example, the only complemented \mathbf{F}_1 -valued point of the toric \mathbf{A}^n is the origin. The toric \mathbf{P}^n has exactly $n+1$ complemented points with values in \mathbf{F}_1 . They are $[1, 0, \dots, 0]$, $[0, 1, 0, \dots, 0]$, and $[0, \dots, 0, 1]$. These facts have been predicted in earlier speculations on the field with one element [60], p. 285.

4. FUNCTION SPACES

In this section, we discuss GL_n over \mathbf{F}_1 . The group scheme GL_n does not descend to \mathbf{F}_1 . (See Buium [8].) But we can realize GL_n as the automorphism group of something that does descend to \mathbf{F}_1 . This might be surprising, because formation of function spaces commutes with base change. Indeed, that is one important way in which the topos map $v: \mathrm{Sp}_{\mathbf{Z}} \rightarrow \mathrm{Sp}_{\mathbf{F}_1}$ is not like a true map of spaces.

For discussion of GL_n in other approaches to \mathbf{F}_1 , see for example Connes–Consani [13] and Toën–Vaquié [61].

This section was directly inspired by some questions Kirsten Wickelgren asked me.

4.1. Let X and Y be objects of $\mathrm{Sp}_{\mathbf{F}_1}$. Write $\mathrm{Hom}_{\mathbf{F}_1}(X, Y)$ for the set of maps from X to Y , and write $\mathrm{Hom}_{\mathbf{Z}}(X, Y)$ for the set of maps $v^*X \rightarrow v^*Y$ between the underlying objects of $\mathrm{Sp}_{\mathbf{Z}}$. Let $\underline{\mathrm{Hom}}_{\mathbf{F}_1}(X, Y)$ denote the usual $\mathrm{Sp}_{\mathbf{F}_1}$ -object of maps from X to Y . It is defined by

$$\underline{\mathrm{Hom}}_{\mathbf{F}_1}(X, Y): T \mapsto \mathrm{Hom}_T(X \times T, Y \times T),$$

for any space $T \in \mathrm{Sp}_{\mathbf{F}_1}$. We define $\underline{\mathrm{Hom}}_{\mathbf{Z}}(X, Y)$ similarly.

Then we have a map

$$(4.1.1) \quad \underline{\mathrm{Hom}}_{\mathbf{F}_1}(X, Y) \longrightarrow v_*(\underline{\mathrm{Hom}}_{\mathbf{Z}}(X, Y)),$$

which sends an \mathbf{F}_1 -map $a: X \times T \rightarrow Y \times T$ to its underlying \mathbf{Z} -map $v^*(a)$. This map is clearly injective.

Adjunction then gives another map

$$(4.1.2) \quad v^*: v^*(\underline{\mathrm{Hom}}_{\mathbf{F}_1}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{Z}}(X, Y).$$

Note that this map is generally neither a monomorphism nor an epimorphism. This is one way in which the topos map $v: \mathrm{Sp}_{\mathbf{Z}} \rightarrow \mathrm{Sp}_{\mathbf{F}_1}$ is different from one induced by a true map of spaces. For example, if $X = W(\mathrm{Spec} \mathbf{Q})$, then (4.1.2) is identified with the map $Y^{\mathbf{Z}} \rightarrow Y^{\mathbf{N}}$, which is rarely a monomorphism. On the other hand, if $X = Y = \mathbf{A}_{\mathbf{F}_1}^1$, then the map is not an epimorphism.

Recall that an endomorphism $\mathbf{A}^n \rightarrow \mathbf{A}^n$ is a linear transformation if and only if it is equivariant under the action of \mathbf{G}_m on \mathbf{A}^n given by scalar multiplication:

$$\mathbf{Z}[x_1, \dots, x_n] \xrightarrow{x_i \mapsto x_i \otimes z} \mathbf{Z}[x_1, \dots, x_n] \otimes_{\mathbf{Z}} \mathbf{Z}[z, z^{-1}].$$

Now observe that if we give \mathbf{G}_m and \mathbf{A}^n their toric Λ -structures, then this action morphism is Λ -equivariant. Therefore it is reasonable to define

$$(4.1.3) \quad M_{n/\mathbf{F}_1} = \underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^n),$$

where for general \mathbf{F}_1 -spaces X, Y with \mathbf{G}_m -actions, we define $\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(X, Y)$ to be the sub- \mathbf{F}_1 -space of $\underline{\mathrm{Hom}}_{\mathbf{F}_1}(X, Y)$ consisting of the \mathbf{G}_m -equivariant maps. Let us also set

$$(4.1.4) \quad \mathrm{GL}_{n/\mathbf{F}_1} = \underline{\mathrm{Aut}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n),$$

to be locus of M_{n/\mathbf{F}_1} consisting of invertible maps. Of course, M_{n/\mathbf{F}_1} is a monoid object in $\mathrm{Sp}_{\mathbf{F}_1}$ and $\mathrm{GL}_{n/\mathbf{F}_1}$ is a group object. In 4.3 below, we will describe them concretely.

Note that because v^* is faithful, we can view $v^*(\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(X, Y))$ as a subspace of $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(v^*X, v^*Y)$. Consider this when $X = \mathbf{A}^n$ and $Y = \mathbf{A}^1$, both with the toric Λ -structures. Then for any ring B , a B -valued point of $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1)$ is just a B -module homomorphism $B^n \rightarrow B$, or simply a $1 \times n$ matrix (b_1, \dots, b_n) with entries in B . This furnishes an identification

$$(4.1.5) \quad \underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1) = \mathrm{Spec} \mathbf{Z}[b_1, \dots, b_n] = \mathbf{A}^n$$

4.2. Proposition. *Under the identification (4.1.5), the subspace of \mathbf{A}^n corresponding to the subspace $\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1)$ of $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1)$ is the union of the axes*

$$Z = \mathrm{Spec} \mathbf{Z}[b_1, \dots, b_n]/(b_i b_j : i \neq j).$$

Further the operators on Z induced by ψ_p take each coordinate b_i to b_i^p .

Proof. We just calculate the B -valued points of $v^* \underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1)$ for any ring B . We have

$$(v^* \underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1))(\mathrm{Spec} B) = (\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1))(v_!(\mathrm{Spec} B)).$$

But since $\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1)$ sits inside $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1)$, which is affine, any map $v_!(\mathrm{Spec} B) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1)$ factors through the affinization $\mathrm{Spec} W(B)$ of $v_!(\mathrm{Spec} B)$. Therefore we have

$$(v^* \underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1))(\mathrm{Spec} B) = (\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1))(\mathrm{Spec} W(B)).$$

If we think of points of \mathbf{A}^n as n -dimensional column vectors, then it is natural to think of the points of $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1)$ as $1 \times n$ matrices. Given such a matrix

$$(a_1, \dots, a_n) \in W(B)^n,$$

the corresponding map in $\underline{\mathrm{Hom}}_{\mathbf{G}_m}(\mathbf{A}^n, \mathbf{A}^1)(\mathrm{Spec} W(B))$ is an \mathbf{F}_1 -map if and only if the ring map

$$(4.2.1) \quad \varphi: \mathbf{Z}[t] \longrightarrow \mathbf{Z}[x_1, \dots, x_n] \otimes W(B)$$

determined by $t \mapsto \sum_j x_j \otimes a_j$ is a Λ -ring map, where $\mathbf{Z}[x_1, \dots, x_n]$ has the toric Λ -structure.

Thus to finish the proof, it suffices to show that necessary and sufficient conditions for φ to be a Λ -ring map are, first, that each element a_j is the Teichmüller lift of some element $b_j \in B$ and, second, that $b_i b_j = 0$ for all $i \neq j$.

Let us first show the necessity. So assume φ is a Λ -ring map. Consider, for each i , the map

$$\pi_i: \mathbf{Z}[x_1, \dots, x_n] \otimes W(B) \longrightarrow W(B)$$

given by $x_j \mapsto \delta_{ij}$ and by the identity on $W(B)$. Then $\pi_i \circ \varphi$ is a Λ -ring map under which the image of t is a_i .

It follows that a_i must be a Teichmüller lift. Indeed, giving a Λ -map $\mathbf{Z}[t] \rightarrow W(B)$ is by adjunction the same as giving a ring map $\mathbf{Z}[t] \rightarrow B$. This in turn is the same as an element of B . Tracing through these identifications shows that $a_i = [b_i]$, and thus the necessity of the first condition above.

To see the necessity of the second condition, use the operator $\lambda_2 \in \Lambda$:

$$0 = \varphi(0) = \varphi(\lambda_2(t)) = \lambda_2(\varphi(t)) = \lambda_2\left(\sum_j x_j \otimes [b_j]\right) = \sum_{i < j} x_i x_j \otimes [b_i][b_j].$$

Thus for $i \neq j$, we have $[b_i b_j] = [b_i][b_j] = 0$ and hence $b_i b_j = 0$.

Sufficiency is similar. Suppose φ is defined by

$$\varphi(t) = \sum_j x_j \otimes [b_j]$$

with $b_i b_j = 0$ for all $i \neq j$. To show φ is a Λ -ring map, it is enough to show it commutes with λ_n for $n \geq 2$. But we have that $\lambda_n(x_j \otimes [b_j]) = 0$ for all $n \geq 2$. Therefore $\lambda_n(\sum_j x_j \otimes [b_j])$ is the n -th elementary symmetric function in the $x_j \otimes [b_j]$. Because $b_i b_j = 0$ for $i \neq j$, each of the elementary monomials is zero when $n \geq 2$. Therefore we have

$$\lambda_n(\varphi(t)) = \lambda_n\left(\sum_j x_j \otimes [b_j]\right) = \lambda_n(0) = \varphi(0) = \varphi(\lambda_n(t)),$$

for $n \geq 2$. Thus φ is a map of Λ -rings. □

4.3. Corollary. *We have the following equalities of subspaces of M_n :*

$$(4.3.1) \quad M_{n/\mathbf{F}_1} = Z^n \quad \text{and} \quad \mathrm{GL}_{n/\mathbf{F}_1} = S_n \times \mathbf{G}_m^n.$$

Proof. The first equality follows from the universal property of products and 4.2:

$$\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^n) = (\underline{\mathrm{Hom}}_{\mathbf{G}_m/\mathbf{F}_1}(\mathbf{A}^n, \mathbf{A}^1))^n = Z^n.$$

For the second equality, recall that a morphism of Λ -spaces is an isomorphism if and only if it becomes one after applying v^* . Therefore we have

$$\mathrm{GL}_{n/\mathbf{F}_1} = \mathrm{GL}_n \cap M_{n/\mathbf{F}_1} = S_n \times \mathbf{G}_m^n.$$

□

4.4. Corollary. (a) $M_{n/\mathbf{F}_1}(\mathbf{F}_1)$ is the set of $n \times n$ matrices with the property that every entry is either 0 or 1 and every row has at most one 1.

(b) $\mathrm{GL}_{n/\mathbf{F}_1}(\mathbf{F}_1) = S_n$.

4.5. Remarks. Note that none of the \mathbf{F}_1 -valued points of $\mathrm{GL}_{n/\mathbf{F}_1}$ are complemented. Also note that the determinant map $\mathrm{GL}_{n/\mathbf{F}_1} \rightarrow \mathbf{G}_m$ is not a Λ -map, because it fails to commute with ψ_2 . It does commute with the other ψ_p though.

5. ASIDE: ABELIAN MOTIVES ARE POTENTIALLY CYCLOTOMIC

Let p be a prime number. The purpose of this section is to establish the result (5.3) in usual, non- Λ algebraic geometry that abelian p -adic Galois representations of geometric origin are Artin–Tate. The proof is an result in p -adic Hodge theory. It is well within the scope of established techniques, but to my knowledge, it is not actually in the literature. For results of a similar flavor, see Wang [63], Kisin–Lehrer [40], and van dan Bogaart–Edixhoven [62].²

5.1. Proposition. *Let X be a separated scheme of finite type over $\mathbf{Z}[1/p]$. Let $f: X \rightarrow \mathrm{Spec} \mathbf{Z}[1/p]$ denote the structure map. Then $R^n f_!(\mathbf{Q}_p)$ is lisse away from a finite set of primes. If X is smooth and proper over $\mathbf{Z}[1/M]$, then the sheaf is lisse at all primes not dividing M .*

Recall that we can define cohomology with compact support because, by Nagata’s theorem, any separated morphism of finite type between noetherian schemes is compactifiable. (See Conrad [15], theorem 4.1. Actually, according to a forthcoming paper of Conrad–Lieblich–Olsson, the scheme-theoretic hypotheses can be removed.)

Proof. The sheaf $R^n f_!(\mathbf{Z}/p\mathbf{Z})$ is constructible (Deligne SGA 4 1/2 [21] [Arcata] IV (6.2)) and hence lisse away from a finite set T of primes. For $m \geq 0$, if $R^n f_!(\mathbf{Z}/p^m\mathbf{Z})$ is lisse, then all subsheaves and quotient sheaves of $R^n f_!(\mathbf{Z}/p^m\mathbf{Z})$ are lisse away from T . On the other hand, by the long exact sequence of cohomology, $R^n f_!(\mathbf{Z}/p^{m+1}\mathbf{Z})$ is an extension of a subsheaf of $R^n f_!(\mathbf{Z}/p\mathbf{Z})$ by a quotient sheaf of $R^n f_!(\mathbf{Z}/p^m\mathbf{Z})$ and is therefore also lisse away from T . By induction $R^n f_!(\mathbf{Z}/p^m\mathbf{Z})$ is lisse away from T for all m . Therefore $R^n f_!(\mathbf{Z}_p)$ and hence $R^n f_!(\mathbf{Q}_p)$ are lisse away from T .

The final statement follows from Deligne, SGA 4 1/2 [21] [Arcata] V (3.1). □

5.2. Cyclotomic representations. Let E be a finite extension of \mathbf{Q}_p . Let K be a finite extension of \mathbf{Q} or \mathbf{Q}_p , and let V be a d -dimensional continuous E -linear representation of $\mathrm{Gal}(\bar{K}/K)$. Let us say that V is *cyclotomic* if there exist integers j_1, \dots, j_d such that V is isomorphic to $E(j_1) \oplus \dots \oplus E(j_d)$. Let us say that V is *potentially cyclotomic* if there exists a finite extension L of K such that V is cyclotomic as a representation of $\mathrm{Gal}(\bar{K}/L)$.

Observe that these concepts are independent of E in the sense that for any finite extension E' of E , we have

$$(5.2.1) \quad E' \otimes_E V \text{ is (potentially) cyclotomic if and only if } V \text{ is.}$$

²The style of this section is somewhat clumsy. I hope to improve it in a future version.

5.3. Theorem. *Let X be a separated scheme of finite type over \mathbf{Q} . Suppose that $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ through its abelianization. Then there is an integer $N \geq 1$ such that the restriction of $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ to $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ is cyclotomic.*

If X has a smooth and proper model over $\mathbf{Z}[1/M]$, then N can be taken such that all its prime divisors are divisors of Mp .

Proof. By the main theorem of p -adic Hodge theory, $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ is a potentially semi-stable representation of $\text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$. (This is the work of many people. See [41] for the final form of the theorem.) By 5.1, the set of ramified primes is finite, and if X has a smooth proper model over $\mathbf{Z}[1/M]$, it contains only primes dividing Mp . The theorem is then an immediate consequence of 5.4. \square

5.4. Theorem. *Let E be a finite extension of \mathbf{Q}_p . Let T be a finite set of prime numbers containing p , and let V be a finite-dimensional representation of $G_{\mathbf{Q}}$ satisfying the following properties:*

- (a) $G_{\mathbf{Q}}$ acts on V through its abelianization,
- (b) V is unramified away from T ,
- (c) V is potentially semi-stable at p .

Then there is an integer N divisible only by the primes in T such that the restriction of V to $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ is cyclotomic.

Proof. By the Kronecker–Weber theorem, there is an integer $n \geq 1$, divisible only by primes in T , such that the action of $G_{\mathbf{Q},S}$ on V factors through the cyclotomic character $G_{\mathbf{Q},S} \rightarrow (\mathbf{Z}/n\mathbf{Z})^* \times \mathbf{Z}_p^*$. Let H be the torsion subgroup of $(\mathbf{Z}/n\mathbf{Z})^* \times \mathbf{Z}_p^*$. Let us show that it is enough to consider the case where H acts trivially on V .

Let G denote the quotient $(\mathbf{Z}/n\mathbf{Z})^* \times \mathbf{Z}_p^*/H$. Since $G \cong \mathbf{Z}_p$, we have

$$(\mathbf{Z}/n\mathbf{Z})^* \times \mathbf{Z}_p^* \cong G \times H.$$

By (5.2.1), we can assume every irreducible E -linear representation of H is one-dimensional. For each character $\rho : H \rightarrow E^*$, let V_ρ denote the summand of V on which H acts via ρ . Since $G \times H$ is abelian, V_ρ is stable under $G \times H$. Therefore we have a decomposition of Galois representations

$$V = \bigoplus_{\rho} V_{\rho}.$$

It is enough to show each V_ρ is potentially cyclotomic. On the other hand, because each V_ρ satisfies (a)–(c), it is enough to assume $V = V_\rho$ for some ρ . Now observe that ρ , viewed as a Galois representation by the composition

$$G \times H \longrightarrow H \xrightarrow{\rho} E^*,$$

is potentially cyclotomic, since it is potentially trivial. Therefore it is enough to show $V \otimes \rho^{-1}$ is potentially cyclotomic. Similarly, ρ^{-1} satisfies (a)–(c) above, and hence so does $V \otimes \rho^{-1}$. But H acts trivially on $V \otimes \rho^{-1}$. Therefore it is indeed enough to consider the case where H acts trivially on V .

Let g be a pro-generator of G/H . By (5.2.1), it is sufficient to assume that E contains all the eigenvalues of g acting on V . Then V has a g -stable filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d = V$$

by sub- E -vector spaces such that $\dim_E(V_i/V_{i-1}) = 1$, for $i = 1, \dots, d$. Because g is a pro-generator of G/H , the filtration is G/H -stable.

Because V is potentially semi-stable, so is each subquotient V_i/V_{i-1} . For each i , the lemma 5.5 implies there is finite extension K_i of \mathbf{Q}_p in $\mathbf{Q}_p(\zeta_{p^\infty})$ such that the action of $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/K_i)$ on V_i/V_{i-1} is an integral power of the cyclotomic character. Take an integer $r \geq 0$ such that $\mathbf{Q}_p(\zeta_{p^r})$ contains K_1, \dots, K_d . Then the action of $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p(\zeta_{p^r}))$ on V_i/V_{i-1} is an integral power of the cyclotomic character. In other words, V is an iterated extension of $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p(\zeta_{p^r}))$ -representations of the form $E(j)$. But by 5.6, it must then be a direct sum of such representations. Thus it is cyclotomic as a representation of $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p(\zeta_{p^r}))$. But because the natural map

$$\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p(\zeta_{p^r})) \longrightarrow \text{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_{p^r}))$$

is an isomorphism, V is cyclotomic as a representation of $G_{\mathbf{Q}(\zeta_{p^r})}$. \square

5.5. Lemma. *Let E be a finite extension of \mathbf{Q}_p , and let V be a one-dimensional E -vector space with a continuous E -linear action of $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$. Assume further that V is potentially semi-stable (as a \mathbf{Q}_p -representation of $G_{\mathbf{Q}_p}$). Then there is a finite extension K of \mathbf{Q}_p such that for some integer i , we have $V \cong E(i)$ as representations of $\text{Gal}(K(\zeta_{p^\infty})/K)$.*

Proof. Let $d = [E : \mathbf{Q}_p]$. Let M be the weakly admissible module associated to V . Then M inherits from V an action of E . That is, every $e \in E$ acts as a morphism of weakly admissible modules. Therefore the Hodge filtration on M is a filtration by sub- E -modules and the Frobenius and monodromy operators are E -linear. Because $\dim_E(M) = \dim_{\mathbf{Q}_p}(M)/d = 1$, there is an integer j such that $F^j M = M$ and $F^{j+1} M = 0$, and there is an element $\alpha \in E$ such that the endomorphism φ of M is multiplication by α . Therefore the the Hodge number of $\det_{\mathbf{Q}_p}(M)$ is dj and the slope is $dv_{\mathbf{Q}_p}(\alpha)$. Because M is weakly admissible, these two must be equal, and so we have $v_{\mathbf{Q}_p}(\alpha) = j$. Replacing M by $M(-j)$, it suffices to assume $j = 0$. We now want to show that M becomes trivial after a finite extension.

Let k denote the residue field of K . Let $M' = M \otimes_{W(k)} W(\bar{k})$. Then M' is a one-dimensional potentially semi-stable weakly admissible E -module over $W(\bar{k})$ with $j = 0$. Let us show that M' is isomorphic to the trivial weakly admissible module.

Let us now show that we can change basis of M' so that $\alpha = 1$. (This is a consequence of Manin's theorem when $E = \mathbf{Q}_p$.) The ring $E \otimes W(\bar{k})$ has an endomorphism $\text{id} \otimes \sigma$. The ratio of this with the identity map gives a group endomorphism f of $(E \otimes W(\bar{k}))^*$ sending $a \otimes x \mapsto a \otimes \sigma(x)/x$. We need to show this is a surjection. Define a filtration A^i of $(E \otimes W(\bar{k}))^*$ by setting A^i

$$(A \otimes W(\bar{k}))^* \longrightarrow (E/\mathfrak{m}^i \otimes W(\bar{k}))^*,$$

where \mathfrak{m} is the maximal ideal of E . Because the ring $E \otimes W(\bar{k})$ is \mathfrak{m} -adically complete, the group $(E \otimes W(\bar{k}))^*$ is complete with respect to this filtration. Therefore it is enough to show f is surjective on the associated graded abelian group. We have $\text{gr}^0 A = (E/\mathfrak{m} \otimes \bar{k})^*$ and for $i \geq 1$, we have $\text{gr}^i A \cong (\mathfrak{m}^i/\mathfrak{m}^{i+1} \otimes \bar{k})$ by the map $x \mapsto x - 1$. The map f becomes $x \mapsto x^{p-1}$ on $\text{gr}^0 A$, and it becomes $a \otimes x \mapsto a \otimes (x^p - x)$ on $\text{gr}^i A$ for $i \geq 1$. Because \bar{k} is algebraically closed, both these maps are surjective.

Let us now show that the monodromy operator N is zero. But this holds because φ and N are two E -linear endomorphisms of a one-dimensional vector space

satisfying $\varphi N = pN\varphi$ and $\varphi \neq 0$. And so the weakly admissible module associated to the $G_{\mathbf{Q} \otimes W(\bar{k})}$ -representation V is trivial.

Therefore there is a finite extension L of $\mathbf{Q} \otimes W(\bar{k})$ such that G_L acts trivially on V . Therefore there is a finite extension K of \mathbf{Q}_p such that the inertia group I_K of G_K acts trivially on V . Since the action of $G_{\mathbf{Q}_p}$ on V factors through $\text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$, the action of G_K . \square

5.6. Lemma. *Let E and K be finite extensions of \mathbf{Q}_p , and let V be a finite-dimensional continuous E -representation of $\text{Gal}(K(\zeta_{p^\infty})/K(\zeta_{p^b}))$. If V is Hodge–Tate and has a semi-simplification which is isomorphic to a sum of cyclotomic representations, then V itself is isomorphic to a sum of cyclotomic representations.*

Proof. Let G denote $\text{Gal}(K(\zeta_{p^\infty})/K(\zeta_{p^b}))$. Giving a representation of G is the same as giving a representation of its Lie algebra, which is one-dimensional; and so this is the same as giving a matrix θ . We have assumed that θ is upper-triangular with only integers on the diagonal. One can change basis to make θ a block-diagonal matrix, where each block is upper-triangular and has a single integer on the diagonal. Therefore it is enough to assume θ has is upper triangular and has a single integer on the diagonal. Twisting by a cyclotomic character, we can assume θ is nilpotent.

Thus it suffices to assume that V has a trivial semi-simplification. By induction on $\dim_E(V)$, we need only prove that any Hodge–Tate extension W of the trivial representation by itself is split. Therefore the representation is given by a group map $G \rightarrow E$ into the upper-right corner of the matrix. Since the Hodge–Tate weights of W are both 0, it must in fact be \mathbf{C}_p -admissible. Therefore by Sen’s theorem, it factors through finite quotient of G . But the only group map $G \rightarrow E$ with this property is the trivial map. Therefore V is a split extension. \square

6. p -ADIC ÉTALE COHOMOLOGY

Let X be a separated Λ -scheme of finite type over \mathbf{Z} . Let $r_{m,n}$ denote the Hodge number $h^{m,m}$ of $H_c^n(X^{\text{an}}, \mathbf{C})$, where X^{an} is the complex-analytic space underlying X . (See Deligne [23], (2.3.7), for the definition of the Hodge numbers of a mixed Hodge structure and Deligne [22] for the fact that $H_c^n(X^{\text{an}}, \mathbf{C})$ carries a natural mixed Hodge structure.) Let us write

$$P(t) = \sum_{m,n} (-1)^n r_{m,n} t^m \in \mathbf{Z}[t].$$

Finally, let us fix an algebraic closure $\bar{\mathbf{Q}}$ of \mathbf{Q} .

6.1. Theorem. *For any integer $s > 0$, the action of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Z}/s\mathbf{Z})$ factors through $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})^{\text{ab}}$.*

Proof. By Deligne [21], [Arcata] IV (6.2), the sheaf $R^n f_!(\mathbf{Z}/s\mathbf{Z})$ is constructible. Therefore, there exists a finite Galois extension K/\mathbf{Q} and an integer $N > 0$ such that the restriction of $R^n f_!(\mathbf{Z}/s\mathbf{Z})$ to $\text{Spec } \mathcal{O}_K[1/N]$ is the constant sheaf associated to an abelian group V . Here, \mathcal{O}_K denotes the ring of integers of K . By functoriality V has an action of $G = \text{Gal}(K/\mathbf{Q})$. Let us also assume that N is a multiple of the discriminant of K . By the base-change theorem, we have an isomorphism

$$H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Z}/s\mathbf{Z}) \cong V$$

of representation of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ (which acts on V via the map to G). Therefore it is enough to show the action of G on V is through its abelianization.

Let p be a prime not dividing N . Let \mathfrak{p} be a prime of K over p , let $k_{\mathfrak{p}}$ denote $\mathcal{O}_K/\mathfrak{p}$, and let D denote the decomposition subgroup of G corresponding to \mathfrak{p} . Because $p \nmid N$, the map $D \rightarrow \text{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$ is an isomorphism. By the proper base-change theorem, we have an isomorphism

$$V \cong H_{\text{ét},c}^n(X_{\bar{\mathbf{F}}_p}, \mathbf{Z}/s\mathbf{Z})$$

of representations of $\text{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$. Further, the endomorphism ψ_p^* of V (induced by the endomorphism ψ_p of X) corresponds to the endomorphism Fr_p^* of the right-hand side. By generalities about the Frobenius map, Fr_p^* acts on $H_{\text{ét},c}^n(X_{\bar{\mathbf{F}}_p}, \mathbf{Z}/s\mathbf{Z})$ as the inverse of the residual arithmetic Frobenius element $\text{Frob}_p \in D/I$, the automorphism defined by $\text{Frob}_p(x) = x^p$. Therefore ψ_p^* acts on V in the same way as Frob_D^{-1} where Frob_D is the element of D mapping to Frob_p .

On the other hand, the endomorphisms ψ_p of X commute with each other as p varies. Therefore the endomorphisms ψ_p^* of $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Z}/s\mathbf{Z})$ commute with each other. By the above, the action of any Frobenius elements Frob_D for $p \nmid N$, commute with each other. By Chebotarev's theorem (see Neukirch [49], V (6.4)), every element of G is such a Frobenius element. Therefore G acts on V through its abelianization. \square

6.2. Corollary. *For any prime number p , there is an integer $N \geq 1$ such that there is an isomorphism of representations of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$:*

$$(6.2.1) \quad H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p) \cong \bigoplus_m \mathbf{Q}_p(-m)^{r_{m,n}}.$$

If X is smooth and proper over $\mathbf{Z}[1/M]$, for some integer $M > 0$, then N can be taken such that all its prime divisors are divisors of Mp .

Let us call an integer $N > 0$ satisfying the conclusion of 6.2 a *conductor* of X .

Proof. By 6.1 and 5.3, there exists an integer $N \geq 1$ such that $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ is a cyclotomic representation of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$. The fact that the multiplicities are given by Hodge numbers as shown is because $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ is potentially semi-stable. See Kisin [41] (3.2).

Last, when X is smooth and proper over $\mathbf{Z}[1/M]$, the Galois representation $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ is unramified at primes not dividing Mp , by 5.1. Then 5.3 implies that N can be taken as asserted. \square

6.3. Remark. Note that the cohomology of the nodal curve of 2.6 when $q = -1$ is non-pure mixed Tate, but the extension class is $q = -1 \in \mathbf{Q}^*$, which being torsion vanishes when coefficients are taken in \mathbf{Q}_p . Therefore the \mathbf{Q}_p -cohomology is pure mixed Tate, and there is no contradiction. On the other hand, the previous theorem would be false with cohomology with coefficients in \mathbf{Z}_2 . It would be interesting to see which other mixed Tate motives with torsion classes can be realized in Λ -algebraic geometry.

6.4. Corollary. *Let N be a conductor for X . Then there is a finite set T of prime numbers such that for any finite field k whose cardinality q is relatively prime to every element of T and satisfies $q \equiv 1 \pmod{N}$, the number of k -valued points of X is $P(q)$.*

More precisely, the set T can be taken such that it contains only prime numbers p with the property that for every prime $\ell \neq p$ the sheaf $R^n f_!(\mathbf{Q}_\ell)$ is not lisse at p , where f denotes the map

$$X \times \text{Spec } \mathbf{Z}[1/\ell] \rightarrow \text{Spec } \mathbf{Z}.$$

Proof. Fix a prime number $\ell \neq p$. Let T denote the set of prime numbers at which $R^n f_!(\mathbf{Q}_\ell)$ is not lisse. By 5.1, T is finite.

Because of the restrictions on q , we have a factorization

$$\text{Spec } k \xrightarrow{a} \text{Spec } \mathbf{Z}[\zeta_N, T^{-1}] \xrightarrow{b} \text{Spec } \mathbf{Z}.$$

Let D_a denote a decomposition group in $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ at the point a , and let \bar{k} denote the corresponding algebraic closure of k . Then since $b^* R^n f_!(\mathbf{Q}_\ell)$ is lisse, the proper base change theorem implies

$$H^n(X_{\bar{k}}, \mathbf{Q}_\ell) \cong H^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$$

as representations of D_a , where D_a acts on the left side through $\text{Gal}(\bar{k}/k)$. In particular, the trace of the geometric Frobenius element $F \in \text{Gal}(\bar{k}/k)$ on $H^n(X_{\bar{k}}, \mathbf{Q}_\ell)$ agrees with the trace of the inverse of an arithmetic Frobenius element Frob_a of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\zeta_N))$ on $H^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$.

Therefore by the Lefschetz fixed-point formula (Houzel, SGA 5, exp. XV [2]) and 6.2, the number of \mathbf{F}_q -valued points of X is

$$\begin{aligned} \sum_n (-1)^n \text{tr}(F | H_{\text{ét},c}^n(X_{\bar{\mathbf{F}}_q}, \mathbf{Q}_\ell)) &= \sum_n (-1)^n \text{tr}(\text{Frob}_a^{-1} | H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)) \\ &= \sum_n (-1)^n \text{tr}(\text{Frob}_a^{-1} | \bigoplus_m \mathbf{Q}_\ell(-m)^{r_{m,n}}) \\ &= \sum_n (-1)^n \sum_m r_{m,n} q^m = P(q). \end{aligned}$$

□

6.5. Corollary. *Let X be a smooth proper Λ -scheme over $\mathbf{Z}[1/M]$. Then there is an integer N divisible only by the primes dividing M such that for all prime powers $q > 1$ with $q \equiv 1 \pmod{N}$, the number of \mathbf{F}_q -valued points of X is $P(q)$.*

Proof. The representation $H_{\text{ét},c}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ is unramified away from M , and so there is an integer N satisfying the property of 6.2 and such that it has the same prime divisors as M .

Now let $q > 1$ be a prime power with $q \equiv 1 \pmod{N}$. Let p be the prime number dividing q , let $\ell \neq p$ be another prime number. Observe that $p \nmid M\ell$.

The map

$$f: X \times \text{Spec } \mathbf{Z}[1/\ell] \rightarrow \text{Spec } \mathbf{Z}[1/M\ell]$$

is smooth and proper, and so by Deligne (SGA 4 1/2, [Arcata] V (3.1) [21]), the sheaf $R^n f_*(\mathbf{Q}_\ell)$ is lisse. In particular, it is lisse at p . Therefore by 6.4, there is a set of primes T not containing p such that the conclusion of 6.4 holds. Since $q \equiv 1 \pmod{N}$, the number of \mathbf{F}_q -valued points of X is $P(q)$. □

6.6. Corollary. *Let X be a smooth proper scheme over \mathbf{F}_1 . Then for all prime powers $q > 1$, the number of \mathbf{F}_q -valued points of X is $P(q)$.*

Proof. This is 6.5 in the case $M = 1$. □

6.7. Remark. The number of complemented \mathbf{F}_1 -valued points of X equals $P(1)$ when $X = \mathbf{A}^n$ or $X = \mathbf{P}^n$, with their toric Λ -structures. On the other hand, if X is the Chebychev line (2.5), it has no complemented \mathbf{F}_1 -valued points, but $P(1) = 1$.

6.8. Remark. Some conditions on q of the kind in 6.4 are necessary. For example, let $X = \text{Spec } \mathbf{Z}[\zeta_N, 1/NM]$. For primes p dividing NM , let $\psi_p = \text{id}$; and for all other p , let ψ_p be the unique Frobenius lift.

Then $r_{0,0} = \phi(N)$ and all other $r_{m,n}$ are zero. Therefore $P(t) = \phi(N)$. On the other hand, there are $\phi(N)$ points in $X(\mathbf{F}_q)$ if and only if both $NM \in \mathbf{F}_q^*$ and $q \equiv 1 \pmod{N}$. Furthermore the sheaf $R^n f_{1!}(\mathbf{Q}_\ell)$ is lisse exactly at the prime numbers that do not divide NM .

6.9. Proposition. *Let X be a nonempty separated Λ -scheme of finite type over \mathbf{Z} . Let $P(t)$ and N be as in 6.1. Then for every prime $p \gg 0$ and every integer $r \geq 1$ with $p^r \equiv 1 \pmod{N}$, there is an \mathbf{F}_q -valued point of X .*

Proof. By 6.4, for every prime p not in set T supplied and for every integer r with $p^r \equiv 1 \pmod{N}$, the number of \mathbf{F}_{p^r} -valued points of X is $P(p^r)$.

On the other hand, it follows from general facts about Hodge numbers of varieties proved by Deligne [24], (8.2.4), that $P(t) \rightarrow \infty$ as $t \rightarrow \infty$. Indeed, since $X_{\mathbf{C}}$ is nonempty, its dimension d is non-negative, the degree of $P(t)$ is $2d$, and the coefficient of t^{2d} is the number of connected components of $X_{\mathbf{C}}$, which is positive.

Thus for sufficiently large p and all r as above, we see that the number of \mathbf{F}_{p^r} -valued points is $P(p^r)$ and that $P(p^r)$ is positive. \square

6.10. Theorem. *Let X be a nonempty separated Λ -scheme of finite type over \mathbf{Z} . Then there is a nonempty closed Λ -subscheme Z of X which is étale over \mathbf{Z} .*

Proof. Let us first reduce to the case where X is reduced and quasi-finite over \mathbf{Z} . Define a sequence $X_0 \supseteq X_1 \supseteq \dots$ of nonempty closed sub- Λ -schemes of X recursively as follows: Let $X_0 = X$. For $n \geq 0$, assume X_n has already been defined. Then let $p = p_{n+1}$ be a prime number distinct from p_1, \dots, p_n such that X_n has an \mathbf{F}_p -valued point x . This exists by 6.9. (For definiteness, we can take p to be the smallest such prime, say.) Let X_{n+1} be the fixed locus of X_n under ψ_p , which is to say the equalizer in the category of Λ -spaces of $\psi_p: X_n \rightarrow X_n$ and the identity map. Because X_n is separated, X_{n+1} is a closed sub- Λ -scheme of X_n . Because x is an \mathbf{F}_p -valued point of X_n , it is fixed by the Frobenius map on the special fiber of X_n over p . Therefore it is fixed by ψ_p , and so x is also a point of X_{n+1} . In particular, X_{n+1} is nonempty.

Let $Z = \bigcap_{n \geq 0} X_n$. Because X is of finite type over \mathbf{Z} , it is noetherian. Therefore there is an integer $n \geq 0$ such that $Z = X_n$. It follows that Z is nonempty and, by 3.2, it is also affine and quasi-finite over \mathbf{Z} . Therefore we can assume $X = Z = \text{Spec } B$, where B is quasi-finite over \mathbf{Z} . Since the reduced subscheme of any nonzero Λ -ring of finite type over \mathbf{Z} is the same, we can also assume B is reduced.

Now let us show that B has a quotient Λ -ring which is étale over \mathbf{Z} . Suppose B is not étale over \mathbf{Z} at some prime p . For each integer $m \geq 1$, let I_m denote the kernel of $\psi_m: B \rightarrow B$. The Λ -ideals I_m are ordered by divisibility on m , and the ordering is cofinal. Let I denote the Λ -ideal $\bigcup_m I_m$, and let C denote the quotient of B/I by the ideal of torsion elements. Then C is a Λ -ring quotient of C and, hence, of B . Note that $1 \notin I$, so B/I is a nonzero Λ -ring. But 1 is not a torsion

element in any nonzero Λ -ring. Therefore C is nonzero and is flat, quasi-finite, and of finite type over \mathbf{Z} . Let us finally show that C is actually étale over \mathbf{Z} .

For each integer $m \geq 1$, the endomorphism ψ_m of C is injective. Indeed, if b is a lift to B of any element of the kernel, then $n\psi_m(b) = 0$ for some integer $n \geq 1$. Therefore $\psi_m(nb) = 0$ and hence $nb = 0$ and hence b is torsion. Therefore the image of b in C is 0.

Now let p be a prime. Since ψ_p is an injective endomorphism of C , it induces an injective endomorphism of $\mathbf{Q} \otimes_{\mathbf{Z}} C$. Since $\mathbf{Q} \otimes_{\mathbf{Z}} C$ is finite over \mathbf{Q} , this endomorphism is in fact an automorphism of finite order. Since C is flat over \mathbf{Z} , we have $C \subseteq \mathbf{Q} \otimes_{\mathbf{Z}} C$, and so ψ_p is an automorphism of C . Therefore the Frobenius endomorphism of C/pC is an automorphism. Thus C/pC is reduced, and so C is étale at p . \square

6.11. Corollary. *Let X be a nonempty proper scheme over \mathbf{F}_1 . Then $X(\mathbf{F}_1) \neq \emptyset$.*

Proof. The scheme Z supplied by 6.10 is proper and étale over \mathbf{Z} . Therefore each of its finitely many connected components must be $\text{Spec } \mathbf{Z}$, by Minkowski's theorem. But the only Λ -structure on such a space is the disjoint-union Λ -structure. Indeed, it is flat over \mathbf{Z} , so it suffices to show that every Frobenius lift ψ_p is the identity. But the Frobenius map on each special fiber is the identity. Since Z is a disjoint union of copies of $\text{Spec } \mathbf{Z}$, each ψ_p must be the identity.

Therefore Z , as a nonempty disjoint union of copies of $\text{Spec } \mathbf{F}_1$, has an \mathbf{F}_1 -valued point, and hence so does X . \square

6.12. Corollary. *Let U a nonempty open Λ -subscheme of a Λ -scheme X which is proper over \mathbf{Z} . Then U has a \mathbf{Q} -valued Λ -point.*

Proof. By the theorem above, there is a nonempty closed Λ -subscheme Z of U which is étale over \mathbf{Z} . Let Y denote the closure of Z in X with the reduced subscheme structure. Then Y is a closed Λ -subscheme of X . (Basic property of Λ -ideals.) Because Y is reduced, it is flat. Because it is the closure of Z it is generically finite over \mathbf{Z} . A closed subscheme of X , it is therefore finite over \mathbf{Z} . By 6.11, it has an \mathbf{Z} -valued Λ -point, and hence a \mathbf{Q} -valued Λ -point. Since we have

$$\text{Spec } \mathbf{Q} \times_{\text{Spec } \mathbf{Z}} Y = \text{Spec } \mathbf{Q} \times_{\text{Spec } \mathbf{Z}} Z \subseteq \text{Spec } \mathbf{Q} \times_{\text{Spec } \mathbf{Z}} U,$$

we see that U has a \mathbf{Q} -valued Λ -point. \square

6.13. Remark. The condition that such a compactification X exists cannot be dropped. For example, for any integer $n > 0$, we can make $\mathbf{Z}[\zeta_n, 1/n]$ a Λ -ring by taking ψ_p to be anything for $p \mid n$, and to be the unique choice $\zeta_n \mapsto \zeta_n^p$ for $p \nmid n$. But this ring has no maps to \mathbf{Q} .

6.14. For any \mathbf{F}_1 -valued point x of X , let Z_x denote the closure of the pre-images of x under the maps ψ_p , viewed as a reduced closed subscheme. Then Z_x is a complemented closed Λ -subspace of X . Therefore the assignment $x \mapsto Z_x$ defines a function

$$\{\mathbf{F}_1\text{-points of } X\} \longrightarrow \{\text{complemented reduced closed } \Lambda\text{-subspaces of } X\}.$$

6.15. Corollary. *Let X be a proper Λ -scheme over \mathbf{Z} . Assume that X is irreducible as a scheme. Then there is a \mathbf{Z} -valued Λ -point x such that $Z_x = X$.*

Proof. Since the set $X(\mathbf{F}_1)$ is finite and complementary closed Λ -subschemes are stable under finite union, the union

$$Z = \bigcup_{x \in X(\mathbf{F}_1)} Z_x,$$

with its reduced scheme structure is a complementary closed Λ -subscheme. By construction, there are no \mathbf{Q} -valued Λ -points of $X - Z$. Therefore by 6.12, this is only possible if $X = Z$. Finally since Z is irreducible, we must have $Z_x = X$ for some $x \in X(\mathbf{F}_1)$. \square

7. VARIANTS

One of the motivations behind work on the field with one element has been to imitate over number fields the theory of function fields over a finite field k , where we can work over the absolute point $\text{Spec } k$. But Λ -algebraic geometry works perfectly well over function fields, too. So we can compare k -algebras to the function-field analogues of \mathbf{F}_1 -algebras.

7.1. $\Lambda_{S,E}$ -spaces. Let S be a scheme of finite type over \mathbf{Z} , and let E be a set of regular closed points of codimension 1. For each point $s \in E$, let q_s denote the cardinality of s . Let X be a flat algebraic space over S . We can then define a $\Lambda_{S,E}$ -action on X just as we define Λ -actions, but now we use commuting endomorphisms $\psi_s: X \rightarrow X$, one for each point $s \in E$, such that ψ_s agrees with the q_s -power Frobenius operator on the fiber X_s . (See [7].) If E is the set of all regular closed points of codimension 1, then we write $\Lambda_S = \Lambda_{S,E}$. For example, if $S = \text{Spec } \mathbf{Z}$, then a Λ_S -space is just a Λ -space.

The general theory works just as well when S is arbitrary. Thus we get a topos $\mathbf{Sp}_{\mathbf{F}_1^{S,E}}$, whose objects we call spaces over the generalized field with one element $\mathbf{F}_1^{S,E}$. If S' and E' are another instance of this data, and $a: S \rightarrow S'$ is a map such that $a(E) \subseteq E'$, we have a diagram of toposes

$$(7.1.1) \quad \begin{array}{ccc} \mathbf{Sp}_S & \xrightarrow{v} & \mathbf{Sp}_{\mathbf{F}_1^{S,E}} \\ \downarrow a & & \downarrow b \\ \mathbf{Sp}_{S'} & \xrightarrow{v'} & \mathbf{Sp}_{\mathbf{F}_1^{S',E'}} \end{array}$$

rendered commutative by a certain invertible 2-morphism. Here b is as in [7].

In particular, $\mathbf{Sp}_{\mathbf{F}_1}$ is the deepest.

7.2. *Function fields.* Let q denote the cardinality of k , and let S be a smooth geometrically connected curve over k . We will now construct a factorization of topos maps

$$(7.2.1) \quad \begin{array}{ccc} \mathbf{Sp}_S & \xrightarrow{v} & \mathbf{Sp}_{\mathbf{F}_1^S} \\ & \searrow s & \swarrow f \\ & \mathbf{Sp}_{\text{Spec } k} & \end{array}$$

Let us first define $f^*(T)$ for affine (or algebraic) T . As a space, set

$$f^*(T) = S \times_k T.$$

Since $S \times_k T$ is flat over S , giving a Λ_S -action on $S \times_k T$, is the same as giving a commuting family of Frobenius lifts. For any maximal ideal \mathfrak{m} of \mathcal{O}_S , define $\psi_{\mathfrak{m}}: S \times_k T \rightarrow S \times_k T$ by $\psi_{\mathfrak{m}} = \text{id}_S \times \text{Fr}_T^{\mathfrak{m}}$, where $\text{Fr}_T^{\mathfrak{m}}$ denotes the endomorphism of T which on \mathcal{O}_T acts as $x \mapsto x^{q_{\mathfrak{m}}}$, where $q_{\mathfrak{m}}$ denotes the cardinality of the residue field $\mathcal{O}_S/\mathfrak{m}$. It is clear the $\psi_{\mathfrak{m}}$ commute with each other and lift the appropriate Frobenius maps.

The functor $\text{Aff}_S \rightarrow \mathbf{Sp}_{\mathbf{F}_1^S}$ just defined preserves covering families. Indeed, a map $U \rightarrow V$ in $\mathbf{Sp}_{\mathbf{F}_1^S}$ is an epimorphism if and only if the induced map $v^*U \rightarrow v^*V$ is. But v^*f^* preserves covering families. Therefore f^* does.

For a similar reason, f^* sends products to products. Therefore it extends uniquely to a topos map $f: \mathbf{Sp}_{\mathbf{F}_1^S} \rightarrow \mathbf{Sp}_{\text{Spec } k}$, which yields the commutative diagram (7.2.1).

In fact, f is essential, meaning that f^* has a left adjoint, denoted $f_!$. We let $f_!(U)$ be the colimit of the coequalizers of the diagrams

$$U \begin{array}{c} \xrightarrow{\text{Fr}_U^{\mathfrak{m}}} \\ \xrightarrow{\psi_{\mathfrak{m}}} \end{array} U$$

over all \mathfrak{m} . In other words, $f_!(U)$ is the largest quotient U' of U on which each $\psi_{\mathfrak{m}}$ acts as $\text{Fr}_{U'}^{\mathfrak{m}}$. This is clearly the left adjoint of f^* .

7.3. *f is not an isomorphism of toposes.* For example, let T be an affine space over S , the unit of the adjunction $f_! \dashv f^*$ at $v_!(T)$ is

$$(7.3.1) \quad v_!(T) \longrightarrow f^* f_! v_!(T) = f^* s_!(T)$$

which, by definition, is a map

$$(7.3.2) \quad W_S(T) \longrightarrow S \times_k T,$$

where $W_S := W_{S,E}$ is the E -typical Witt vector functor over S . (See [7].) The composition

$$\coprod_{\mathbf{N}^{(E)}} T \xrightarrow{\gamma} W_S(T) \longrightarrow S \times_k T.$$

with the ghost map γ is the map that, on the component $n \in \mathbf{N}^{(E)}$, is simply

$$T \xrightarrow{(\text{pr}, \text{Fr}_T^n)} S \times_k T$$

where pr denotes the structure map $T \rightarrow S$, and Fr_T^n is the Frobenius map defined on functions by $x \mapsto x^{q^{\deg(n)}}$, where $\deg(n)$ denotes $\sum_{\mathfrak{m}} n_{\mathfrak{m}}[\mathcal{O}_S/\mathfrak{m} : k]$, the degree of the effective divisor corresponding to n .

In particular, if $T = S$, then the image of this map is the union in $S \times_k S$ of the graphs of all powers of the Frobenius map on S . These components are not disjoint. For instance, let x and y be two distinct closed points of S with the same residue field. Then the components of $W_S(S)$ of indices x and y are distinct but have the same image in $S \times_k S$. So (7.3.1) is not a monomorphism. Therefore $f_!$ is not faithful, and hence f is not an isomorphism of toposes. One can also show that (7.3.1) is not an epimorphism.

But the image of (7.3.2), or equivalently (7.3.1), does see much of the geometry of $S \times_k T$. Another way of expressing the point of this paper is that it is reasonable to think of it as being almost an isomorphism. For example, we have the following result.

7.4. Proposition. *The map*

$$W_S(T) \longrightarrow S \times_k T$$

of (7.3.2) has Zariski dense image.

Proof. Let \mathfrak{m} be a maximal ideal of \mathcal{O}_S with residue cardinality r . It suffices to show the image of the composition

$$\mathbf{N} \times T \longrightarrow \mathbf{N}^{(E)} \times T \longrightarrow W_S(T) \longrightarrow S \times_k T$$

is dense. The map satisfies $(n, t) \mapsto (h(t), \text{Fr}_{r^n}(t))$.

We may assume S and T are affine. Indeed, it suffices to show density locally. Therefore we can replace S with an affine open subscheme S' and replace T with $T' = S' \times_S T$. We can then replace T' with an affine scheme T'' mapping to T' by an étale map.

Let us then write $S = \text{Spec } R$ and $T = \text{Spec } B$, where R is a Dedekind domain and B is an R -algebra. In terms of rings, the map in question is

$$(7.4.1) \quad R \otimes_k B \longrightarrow B \times B \times \dots$$

and is defined by $a \otimes b \mapsto \langle ab, ab^r, ab^{r^2}, \dots \rangle$. We need to show that any element of its kernel is nilpotent.

So let $\sum_{j=1}^d a_j \otimes b_j$ be an element of its kernel. Assume without loss of generality that the elements a_j are linearly independent over k . Then for every $m \in \mathbf{N}$, we have $\sum_j a_j b_j^{r^m} = 0$. Applying various powers of Fr_r , we have the system of equations

$$\sum_{j=1}^d a_j^{r^i} b_j^{r^d} = 0$$

where $i = 0, \dots, d-1$. This can be expressed as the matrix equation

$$(a_j^{r^i})_{ij} \cdot (b_j^{r^d})_j = 0.$$

Since the family a_1, \dots, a_d is linearly independent, the matrix $(a_j^{r^i})$ has nonzero determinant. (This is the Moore matrix from the theory of function fields. The vanishing of its determinant is equivalent to the linear dependence of the a_j . The proof is by a degree argument, just as for the familiar, analogous result about Vandermonde matrices.) Therefore we have $b_j^{r^d} = 0$ for all j . It follows that the element

$$\left(\sum_{j=1}^d a_j \otimes b_j \right)^{r^d}$$

is zero. So every element of the kernel of (7.4.1) is nilpotent. \square

7.5. Corollary. *Let X and Y be separated reduced algebraic spaces over k . Then any map $f_! f^* X \rightarrow Y$ factors uniquely through the map $f_! f^* X \rightarrow X$.*

Proof. Consider the composite map

$$S \times X \longrightarrow f_! f^* X \longrightarrow Y,$$

and let $\Gamma = (S \times X) \times_Y (S \times X)$ denote the induced equivalence relation on $S \times X$. Recall that $f_! f^* X$ is defined to be the quotient of $S \times X$ by the equivalence relation Γ' generated by the image of $W(S \times X)$ in $S \times S \times X$ under the map (7.3.2) when $T = S \times_k X$. Therefore Γ contains Γ' . On the other hand, by 7.4, Γ' is dense in

$\Gamma'' = S \times S \times X = (S \times X) \times_X (S \times X)$. Therefore Γ agrees with Γ'' , and so the map $S \times X \rightarrow Y$ factors uniquely through the quotient of $S \times X$ by Γ'' , which is just X . \square

7.6. Corollary. *The functor $f^* : \mathrm{Sp}_k \rightarrow \mathrm{Sp}_{\mathbf{F}_1^S}$ is faithful, and it embeds the full subcategory of separated reduced algebraic spaces over k fully faithfully in the category $\mathrm{Sp}_{\mathbf{F}_1^S}$.*

Some restriction on nilpotent elements is necessary. For example, suppose $S = \mathrm{Spec} R$. Let A be a non-reduced k -algebra. Choose a square-zero element $a \in A$ and an element r in R but not in k . Then the element $r \otimes a$ of $R \otimes_k A$ does not lie in the subring A . On the other hand, we have $(r \otimes a)^{q_{\mathfrak{m}}} = 0 = r \otimes a^{q_{\mathfrak{m}}} = \psi_{\mathfrak{m}}(r \otimes a)$ for any maximal ideal \mathfrak{m} of R . Therefore the map $f_! f^* \mathrm{Spec} A \rightarrow \mathrm{Spec} A$ is not an isomorphism, because $r \otimes a$ is a function on the affinization of $f_! f^* \mathrm{Spec} A$ that does not come from A .

7.7. Analogy. It is rare for the pull-back functor for a map of spaces to be fully faithful. So let us consider a similar, but more familiar situation where this happens. Let S be a complex algebraic space and let Γ be an equivalence relation on S which is Zariski dense in $S \times S$. For instance, Γ could be given by the action of a discrete group with a dense orbit or, if S is connected, by the formal neighborhood of the diagonal. Then algebraic spaces (perhaps under some mild conditions) form a full subcategory of the category of Γ -equivariant spaces over S . The condition for a Γ -equivariant algebraic space X over S to descend to the point is then a property on X , rather than a structure. We might then say the Γ -action is uniform, or constant.

Therefore, following the previous corollary, it is natural to interpret a descent datum on a reduced algebraic space X over S to the point $\mathrm{Spec} k$ as being a descent datum to \mathbf{F}_1^S with a similar algebraic uniformity property. So, objects of \mathbf{F}_1^S are generalized—but not weakened—versions of separated reduced algebraic spaces over the point $\mathrm{Spec} k$. Of course this makes essential use of equal characteristic. The corresponding interpretation of Λ -spaces in the usual sense, over \mathbf{Z} , would be that while it is possible to say what it means to descend an algebraic space to \mathbf{F}_1 —that is, to give it a Λ -action—we do not know if there is a uniformity property, which is what we would need to create a true arithmetic analogue of the base point $\mathrm{Spec} k$.

(Buium has come to similar ideas independently. He proposed in conversation that it might be reasonable to consider a Λ -structure on a scheme as being an isotrivialization relative to \mathbf{F}_1 .)

7.8. Drinfeld modules. Using the theory of Drinfeld modules, we can give examples of objects of $\mathrm{Sp}_{\mathbf{F}_1^S}$ that do not descend to Sp_k .

Let C be a connected smooth projective curve over \mathbf{F}_p , let $\infty \in C$ be a closed point, and let $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. Let S be a smooth \mathbf{F}_p -curve over which there is a Drinfeld A -module

$$\varphi : A \rightarrow \mathrm{End}_S(\mathbf{G}_a)$$

of rank 1 and of generic characteristic. (See Laumon [46] section (1.2), say.)

Then for any closed point s of S , the fiber of φ over s gives a Drinfeld R -module φ_s over s of rank 1. Because of the assumption that φ has generic characteristic, the characteristic of φ_s is s . A basic result of Drinfeld's ([46] (2.2.2)(ii)) then implies that there is a unique element $\Pi_s \in A$ such that $\varphi_s(\Pi_s)$ is the q_s -th power Frobenius

endomorphism of \mathbf{G}_a over s , where q_s denotes the residue cardinality of s . If we set $\psi_s = \varphi(\Pi_s)$, then the various ψ_s are commuting endomorphisms of \mathbf{G}_a over S , each agreeing with the q_s -power Frobenius map on the fiber over s . This gives a Λ_S -structure on \mathbf{G}_a (which also respects the group structure).

For example, the Carlitz module is defined when $A = \mathbf{F}_p[t]$ and $S = \text{Spec } A$ by $\rho(t) = t + \tau$, where τ is the Frobenius map of \mathbf{G}_a . Then for each maximal ideal \mathfrak{m} of $k[t]$, the operator $\psi_{\mathfrak{m}}$ is $\rho(f(t))$, where $f(t)$ denotes the monic generator of \mathfrak{m} .

Observe that none of these ‘‘Drinfeld Λ_S -structures’’ on \mathbf{A}_S^1 descends from $\mathbf{Sp}_{\mathbf{F}_1^S}$ to \mathbf{Sp}_k . Indeed, for every object in the image of f^* , the operators ψ_s act as zero on the conormal sheaf of the identity section $S \subset \mathbf{G}_a$. But φ was assumed to have generic characteristic. Therefore every Π_s acts faithfully on the conormal sheaf, and hence so does every ψ_s .

Another important use of the construction $S \times X$ is in the study of shtukas, also due to Drinfeld. Indeed, it is possible to mimic this using $W_S(X)$ instead of $S \times X$. And this concept can be translated to number fields. This paper is not, however, the place to discuss this in any detail.

7.9. Complex multiplication by number fields. Let R be a Dedekind domain whose field of fractions is a number field. Assume that there is an abelian scheme X over R of dimension d having the property that $\mathbf{Q} \otimes \text{End}_R(X)$ contains a field F of degree $2d$ over \mathbf{Q} .

As above, X has a natural Λ_R -structure. For each maximal ideal \mathfrak{m} of R , there is a unique element $\pi_{\mathfrak{m}} \in F \cap \text{End}_R(X)$ such that $\pi_{\mathfrak{m}}$ induces the $q_{\mathfrak{m}}$ -th power Frobenius map on the fiber of X over \mathfrak{m} . (See Serre–Tate [54].) Because each $\pi_{\mathfrak{m}}$ lies in F , they all commute. Therefore putting $\psi_{\mathfrak{m}} = \pi_{\mathfrak{m}}$ is a Λ_R -structure on X .

Observe that we can modify X to make Λ_R -varieties that are not CM varieties in the usual sense. For instance, let G is a finite subgroup of $\text{Aut}_R(X)$. Because every automorphism commutes with the complex multiplications, G acts Λ_R -equivariantly on X . Therefore the quotient X/G is also a Λ_R -space. (Because G is finite, the quotient is an algebraic space, by Artin’s theorem.) For instance, if X is an elliptic curve and $G = \text{Aut}(X)$, then X/G is a projective line.

In particular, it seems likely that explicit class field theory for imaginary quadratic fields could be expressed in terms of Λ_R -structures on \mathbf{P}^1 , just like in the case of \mathbf{Q} and function fields.

Suppose instead that R is a Dedekind domain whose field of fractions is a real quadratic number field. In light of Ritt’s work [51][52], it seems unlikely that there are Λ_R -actions on \mathbf{P}^1 which do not come from $\Lambda_{\mathbf{Z}}$ -actions. It might, however, be possible to find such Λ_R -actions on surfaces or higher-dimensional varieties, and any example would without a doubt lead to another example of an explicit class field theory. Of course, it would only be interesting if it could see more than the maximal cyclotomic extension of R . On the other hand, it would also be interesting to prove that no such examples exist.

Here are some precise questions. Is there an algebraic Λ_R -space X of finite type over R with the property that for any abelian étale R -algebra R' , the Λ_R -space $\text{Spec } R' \times_{\text{Spec } R} X$ is not isomorphic as a Λ_R -space to one of the form $\text{Spec } R' \times_{\text{Spec } \mathbf{Z}} Y$, for any algebraic $\Lambda_{\mathbf{Z}}$ -space Y ? (*Abelian* here means that $\mathbf{Q} \otimes_{\mathbf{Z}} R'$ is a product of abelian extensions of $\mathbf{Q} \otimes_{\mathbf{Z}} R$.) Are there any algebraic Λ_R -spaces X of finite type over R whose generic fiber is geometrically connected and which do not descend to

Λ \mathbf{Z} -spaces? I do not even know if many particular varieties can be ruled out: are there any Λ $_R$ -structures on $X = \mathbf{P}_R^2$ with this property?

7.10. Local number fields. Let $S = \text{Spec } \mathbf{Z}$, let p be a prime number, and let $E = \{p\}$. Let us write $\Lambda_{S,E} = \Lambda_p$. The corresponding Witt functor $W_n^* = W_{S,E,n}^*$ is, up to re-indexing, the p -typical Witt vector functor defined by Witt [65] in 1936. Of course, Λ $_p$ -rings and p -typical Witt vectors are now ubiquitous in work on p -adic cohomology. (See [17] or [37], say.)

Now let A be a complete discrete valuation ring with perfect residue field k of characteristic p . If A is of equal characteristic, then the map $A \rightarrow k$ has a unique section. Now suppose A has mixed characteristic. Of course $A \rightarrow k$ cannot have a section defined over \mathbf{Z} , but remarkably, it does have a unique section defined over $\mathbf{F}_1^{S,E}$, the p -typical field with one element, and hence over \mathbf{F}_1 . By definition, this means that the map $W(A) \rightarrow W(k)$ of Λ $_p$ -rings has a unique section. Indeed, because k is perfect, there is a unique ring map $W(k) \rightarrow A$ compatible with the projections to k . (See [53], say.) By adjointness, this then lifts to a unique map $W(k) \rightarrow W(A)$ of Λ $_p$ -rings, which is easily seen to be a section of the map in question.

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