Generalized symmetries in categories of algebras (following Tall–Wraith, Bergman–Hausknecht,. . .)

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Basic example

1. R = ring $\rightsquigarrow$ adjunction diagram:

$$\begin{tikzcd} \mathsf{Mod}_R \arrow[d, "\mathrm{fgt}"'] \\ \mathsf{Ab} \arrow[u, bend left=60, "\text{induction} = R\otimes -"] \arrow[u, bend right=60, "\mathrm{Hom}(R,-) = \text{co-induction}"'] \end{tikzcd}$$

2. A ring is equivalent to "such a" diagram.
3. Enveloping principle: Any gadget (group, Lie algebra,...) acting on an abelian group should act via some associated ring (group algebra, enveloping algebra,...), which is typically the closure of the gadget under pointwise addition and composition.
4. So, a ring is the ultimate gadget that knows what it means to act on an abelian group.
5. This principle holds in other categories, but it almost completely unknown!

Main points

1. Such an adjunction diagram can be generalized from Ab to any category C that is "algebraic" in the sense that it consists of sets with n-ary operations satisfying universal identities: groups, abelian groups, K-modules, K-algebras, semirings, Lie algebras over K, Jordan rings, heaps, loops, ...
2. $\therefore$ we can produce generalized symmetry objects, the ultimate gadgets that knows what it means to act on an object of C. They play the role in C that rings do in Ab.
3. One can produce them from familiar symmetry objects (groups, ...) by taking the closure under the operations of C applied pointwise and composition (if necessary).
4. For some C, all generalized symmetry objects are built out of familiar symmetry objects.
5. For other C, there are genuinely new symmetry objects.
6. If you want to understand a particular C, it is imperative that you know the structure of its generalized symmetry objects!

Composition objects and representable comonads

1. Today's focus: co-induction $\mathrm{Hom}(R, -)$, viewed as a functor $\mathsf{Ab} \to \mathsf{Ab}$
2. $$\begin{aligned} \{\text{abelian groups}\} &\overset{\text{Yoneda}}{=} \{\text{representable functors } \mathsf{Ab} \to \mathsf{Set}\}^{\mathrm{op}} \\ &\overset{\text{exercise}}{=} \{\text{representable functors } \mathsf{Ab} \to \mathsf{Ab}\}^{\mathrm{op}} \\ \otimes &\leftrightsquigarrow \text{composition} \end{aligned}$$
3. $$\{\text{rings}\} := \{\otimes\!-\!\text{monoids}\} = \{\text{representable comonads on } \mathsf{Ab}\}^{\mathrm{op}}$$
4. Def: A <u>composition structure</u> on an object P of an algebraic category C is the structure of a comonad on $\mathrm{Hom}_{\mathsf{C}}(P, -)$.
5. So $\{\text{composition abelian groups}\} = \{\text{rings}\}$

An example in $\mathrm{CAlg}_{\mathbf{Z}}$=Commutative rings

1. $A \in \mathrm{CAlg}_{\mathbf{Z}}$. A function $d\colon A \to A$ is a derivation if

$$d(x+y) = d(x)+d(y), \quad d(xy) = xd(y)+yd(x), \quad d(1) = 0.$$

2. Differential ring := commutative ring with derivation

$$\begin{tikzcd}
\text{Diff-rings} \arrow[d, "\mathrm{fgt}"'] \\
\mathrm{CAlg}_{\mathbf{Z}} \arrow[u, bend left=60, "\text{free } = \text{``}D \odot -\text{''}"] \arrow[u, bend right=60, "\mathrm{Hom}_{\mathrm{CAlg}_{\mathbf{Z}}}(D,-)"']
\end{tikzcd}$$

D = free differential ring on one generator e

$= \mathbf{Z}[e, d, d^{\circ 2}, \ldots] =$ all polynomial differential operators.

3. Then $\mathrm{Hom}_{\mathrm{CAlg}_{\mathbf{Z}}}(D, A) \xrightarrow{\sim} \left\{ \sum_{n \geq 0} a_n \frac{t^n}{n!} \right\}$, $a_n = \varphi(d^{\circ n})$,

$d = d/dt$

4. $\operatorname{Hom}(D, -)$ should be a representable comonad on $\mathrm{CAlg}_{\mathbf{Z}}$.
   Q: What does the ring structure on $\operatorname{Hom}(D, A)$ correspond to on D?
   A: D is a $\underline{\text{co-}\mathrm{CAlg}_{\mathbf{Z}} \text{ object in } \mathrm{CAlg}_{\mathbf{Z}}}$:
   Coaddition:

$$\Delta^+ \colon D \to D \amalg D = D \otimes D$$
$$d^{\circ n} \mapsto d^{\circ n} \otimes 1 + 1 \otimes d^{\circ n}$$
$$d^{\circ n}(xy) = d^{\circ n}(x)1(y) + 1(x)d^{\circ n}(y) \qquad \leftarrow \text{meaning}$$

   Comultiplication:

$$\Delta^\times \colon D \to D \amalg D = D \otimes D$$
$$d^{\circ n} \mapsto \sum_i \binom{n}{i} d^{\circ i} \otimes d^{\circ n-i}$$
$$d^{\circ n}(xy) = \sum_i \binom{n}{i} d^{\circ i}(x) d^{\circ n-i}(y) \qquad \leftarrow \text{meaning}$$

   Comultiplication codistributes over coaddition, etc.
5. This is why we need C to be an algebraic category, one defined by sets with n-ary operations satisfying identities.

1. The comonad structure on $\mathrm{Hom}(D, -)$

   $\leftrightsquigarrow$ $\mathrm{Hom}(D, A)$ is naturally a <u>differential</u> ring

   $\leftrightsquigarrow$ D has a composition operation $\circ$

   (the analogue of $\circ$ for $\mathsf{C} = \mathsf{Ab}$ was the multiplication on R)
2. $f \circ g$ on $D = \mathbf{Z}[e, d, d^{\circ 2}, \dots]$ is just composition of differential operators
3. Interpretation: A composition object P of C is a set of abstract operators which
   3.1 is closed under composition (and has an identity),
   3.2 is closed under the operations of C pointwise, and
   3.3 has generalized "Leibniz rules" for all operations m of C, i.e. universal expressions saying how to calculate $f(m(x, y, \dots))$ in terms of $x, y, \dots$ and operators of P applied to them
4. A notable feature: elements of P have personalized Leibniz rules: $d \in D$ is a derivation, but e is a ring homomorphism and $2(d^{\circ 2})^3 d^{\circ 7} - 3e$ is neither but has its own Leibniz rules

Representations of composition objects

1. P = composition object of C

   X = object of C
2. An action of P on X := co-action of the comonad $\mathrm{Hom}_{\mathsf{C}}(P, -)$ on X

   $\Leftrightarrow$ an action of the monoid $(P, \circ)$ on X such that $f \circ x$ is a C-morphism in f and satisfies the Leibniz rules of f in x
3. C = Ab, $P = R$: same as R-module structure on X

   C $= \mathrm{CAlg}$, $P = D$: same as a differential ring structure on X
4. Enveloping principle: All any gadget that knows what it means to act on objects of C does it via a composition object.
5. A composition object in C is the ultimate gadget that knows what it means to act on objects of C.

Familiar examples

1. $\mathsf{C} = \mathsf{Set}$: composition sets = monoids
2. $\mathsf{C} = \mathsf{Ab}$: composition abelian groups = rings
3. $\mathsf{C} = \mathsf{Mod}_K$: composition K-modules = rings R plus a ring map $K \to R$
4. $\mathsf{C}=$ arbitrary, G=monoid: an action of G on $X \in \mathsf{C}$ is the same as an action of the composition object $\mathrm{Free}_{\mathsf{C}}(G)$
   4.1 The co-operations (i.e., Leibniz rules) of $\mathrm{Free}_{\mathsf{C}}(G)$ are determined by the requirement that each $g \in G$ acts homomorphically, and composition is determined by $g \circ h = gh$.
   4.2 For $\mathsf{C} = \mathsf{Ab}$, this is just the usual monoid algebra of G.

Example 1: C=Groups

1. Theorem: (Kan) Every representable endofunctor on Groups is of the form $X \mapsto X^{S-\{*\}}$ for some pointed set S.
2. Composition corresponds to smash product of S's.
3. Cor: Every representable comonad on Groups is of the form $W(X) = X^G$ for some monoid G.
4. I.e., every composition group P is of the form $\mathrm{Free}_{\mathrm{group}}(G)$ for a (unique) monoid G. An action of P on a group is the same as an action of G.
5. Interpretation: Generalized symmetries for Groups all come from monoids. So they are the same as usual symmetries.

Example 2: C=Monoids

1. New symmetries: anti-homomorphisms!

   On a group, giving an anti-endomorphism φ is equivalent to giving an endomorphism ψ: take $\psi(x) = \varphi(x)^{-1}$. But this is not true for monoids.
2. Let $G \to \{\pm 1\}$ be a monoid map.
3. A signed action of G on a monoid X is an action of G on the set underlying X such that elements of G_+ act homomorphically and those of G_- act anti-homomorphically.
4. Then a signed action of G is the same as an action of the composition algebra $P = \mathrm{Free}_{\mathrm{monoid}}(G)$, where composition is defined using the multiplication on G and the Leibniz rules are such that elements of G_+ are homomorphisms and those of G_- are anti-homomorphisms.
5. OK, anti-homomorphisms are only slightly new...

Genuinely new symmetries on Monoids

1. Consider the category of monoids X equipped with
   1.1 an endomorphism f and
   1.2 an anti-endomorphism g such that
   1.3 $f(x)g(x) = 1$ for all $x \in X$
2. Exercise: This is the category of representations of a composition monoid P. It is not $\mathrm{Free}_{\mathrm{monoid}}(G)$ for any signed monoid $G \to \{\pm 1\}$.
3. Theorem: (Bergman) Every monoid representing an endofunctor is generated by homomorphic elements and anti-homomorphic elements. All relations are of the form $fg = 1$, describable by certain bipartite graphs.
4. Interpretation: There are genuinely new generalized symmetry <u>structures</u> for monoids, but we do not need genuinely new <u>operators</u> to generate them.

   But it's still not clear what form the relations can take. There are probably strong restrictions, so the the full nature of representation theory on monoids is still unresolved.

Example 3: $\mathsf{C} = \mathrm{CAlg}_{\mathbf{Q}}$=Commutative $\mathbf{Q}$-algebras

1. Old: monoids acting by homomorphisms give rise to composition objects in $\mathrm{CAlg}_{\mathbf{Q}}$
2. Slightly new: So do Lie algebras acting by derivations, $P = \mathrm{Sym}_{\mathbf{Q}}(U(\mathfrak{g}))$, all polynomials in $U(\mathfrak{g})$, which consists of linear operators. Then P-actions = $\mathfrak{g}$-actions.
3. Common generalization: Polynomials in operators of any cocommutative bialgebra H: $P = \mathrm{Sym}_{\mathbf{Q}}(H)$
4. Conjecture: Every composition object in $\mathrm{CAlg}_{\mathbf{Q}}$ arises in this way.
5. Interpretation: There should be no genuinely new composition objects in $\mathrm{CAlg}_{\mathbf{Q}}$.

   In fact, by Milnor–Moore, we probably only need monoids and Lie algebras.
6. This should be provable! Bergman–Hausknecht can prove the non-commutative analogue holds!

Example 4: $\mathrm{C} = \mathrm{CAlg}_{\mathbf{Z}}$=Commutative $\mathbf{Z}$-algebras

1. $A \in \mathrm{CAlg}_{\mathbf{Z}}$, prime p. Buium–Joyal p-derivation $\delta\colon A \to A$

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$
$$\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$$
$$\delta(1) = 0.$$

   Meaning: δ is a p-derivation $\Rightarrow \psi(x) = x^p + p\delta(x)$ is a ring map.

2. $\Lambda_p = \mathbf{Z}[e, \delta, \delta^{\circ 2}, \dots]$. Define Δ^+ and $\Delta^\times$ and $\circ$ in the evident way from the Leibniz rules above.
3. Fact: The linear operators in Λ_p are the linear combinations of the iterates of $\psi = e^p + p\delta$. They generate Λ_p over $\mathbf{Z}[1/p]$, but not over $\mathbf{Z}$.
4. $\therefore$ integrality restrictions force non-linear operators on us!

Commutative $\mathbf{Z}$-algebras, continued

5. The corresponding comonad $\mathrm{Hom}(\Lambda_p, -)$ is the *p*-typical Witt vector functor. The non-linear nature of Λ_p is why it seems so exotic and is hard to understand.
6. Conjecture: All composition objects in $\mathrm{CAlg}_{\mathbf{Z}}$ can be built from linear ones and "similar congruence constructions".
7. This should be provable! (once made precise...)

Example 5: $C = \mathrm{CAlg}_{\mathbf{N}}$=commutative semirings

1. Q: Are there nonlinear composition objects in $\mathrm{CAlg}_{\mathbf{N}}$?
   A: Yes, in fact there are nonlinear ones over $\mathbf{R}_{\geq 0}$!
2. So, both positivity and integrality restrictions force us to admit nonlinear operators!
3. Positivity is an analogue of integrality at the place ∞ of $\mathbf{Q}$
4. Λ=symmetric functions in $x_1, x_2, \dots$.
   I.e., $\Lambda = \mathbf{Z}[e_1, e_2, \dots]$, where we imagine

$$e_1 = x_1 + x_2 + \cdots, \quad e_2 = x_1x_2 + x_1x_3 + \cdots, \quad \dots$$

5. $\Lambda_{\mathbf{N}}$=symmetric functions whose coeffs are ≥ 0.

$$\Delta^+ : f \mapsto f(\dots, x_i \otimes 1, 1 \otimes x_i, \dots)$$
$$\Delta^\times : f \mapsto f(\dots, x_i \otimes x_j, \dots)$$
$$f \circ \sum_i m_i = f(\dots, m_i, \dots),$$

where the m_i are monomials.

Commutative semirings, continued

6. Then $\Lambda_{\mathbf{N}}$ is a composition object in $\mathrm{CAlg}_{\mathbf{N}}$.
7. $\therefore \mathbf{R}_{\geq 0} \otimes_{\mathbf{N}} \Lambda_{\mathbf{N}}$ is a composition object in $\mathrm{CAlg}_{\mathbf{R}_{\geq 0}}$.
8. Fact: Every generating set of $\mathbf{R}_{\geq 0} \otimes_{\mathbf{N}} \Lambda_{\mathbf{N}}$ must contain scalar multiples of $e_2, e_3, \ldots$, and hence many nonlinear operators.
9. The comonad on $\mathrm{CAlg}_{\mathbf{Z}}$ represented by $\mathbf{Z} \otimes_{\mathbf{N}} \Lambda_{\mathbf{N}}$ is the big Witt vector functor, which combines all the p-typical ones.
10. So the comonad on $\mathrm{CAlg}_{\mathbf{N}}$ represented by $\Lambda_{\mathbf{N}}$ could be viewed as incorporating further nonlinear structure, some "∞-typical" Witt vector information.

References

1. Tall–Wraith
2. Bergman–Hausknecht (especially)
3. Borger–Wieland
4. Stacey–Whitehouse