Ramanujan and Euler's Constant

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In memory of Ed McMillan
1907 – 1991

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Motivation

Ramanujan gave many beautiful formulas for π and $1/\pi$. See, for example, J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley and Sons, New York, 1987; also (same authors) "Ramanujan and Pi", *Scientific American*, February 1988, 66–73.

Euler's constant

$$\gamma = -\Gamma'(1) \simeq 0.577$$

is more mysterious than π. For example, unlike π, we do not know any quadratically convergent iteration for γ. We do not know if γ is transcendental. We do not even know if γ is irrational, though this seems likely. All we know is that if $\gamma = p/q$ is rational, then q is large. This follows from a computation of the regular continued fraction expansion for γ.

Analogy with $\zeta(3)$

Apéry proved $\zeta(3)$ irrational using the series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k! k!}{(2k)! k^3}$$

and, in Chapter 9 of his Notebooks, Ramanujan gives several similar series, some involving $\zeta(3)$.

Ramanujan rediscovered Euler's formula

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2},$$

where

$$H_k = \sum_{j=1}^{k} \frac{1}{j}$$

is a Harmonic number. Harmonic numbers also occur in formulas involving γ (examples later).

Thus, it is natural to look in the work of Ramanujan for formulas involving γ, in the hope that some of these might be useful for computing accurate approximations to γ, or even for proving that γ is irrational.

Ramanujan's Papers and Notebooks

Ramanujan published one paper specifically on γ: "A series for Euler's constant γ", *Messenger of Mathematics* 46 (1917), 73–80. In this paper he generalizes an interesting series which was first discovered by Glaisher:

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)}.$$

This family of series all involve the Riemann zeta function or related functions, so they are not convenient for computational purposes.

Much of Ramanujan's work was not published during his lifetime, but was summarized in his Notebooks. Edited editions have been published by Berndt [1]. In the following, page numbers refer to Berndt's edition (Part I for Chapters 1–9, Part II for Chapters 10–15).

γ in Ramanujan's Notebooks

Scanning Berndt, we find many occurrences of γ. Some involve the logarithmic derivative $\psi(x)$ of the gamma function, or the sum

$$H_x = \sum_{k=1}^{x} 1/k,$$

which we can interpret as $\psi(x+1)+\gamma$ if x is not necessarily a positive integer (Ch. 8, pg. 181). There are also applications of the result

$$H_n = \ln n + \gamma + O(1/n)$$

as $n \to \infty$.

Other interesting formulas involving γ occur in Chapters 14–15, e.g. Ch. 15, Entry 1, examples (i–ii), pp. 303–304.

Chapter 4, Entry 9

We shall concentrate on Chapter 4, Entry 9, Corollaries 1–2 (pg. 98), because these are potentially useful for computing γ. Corollary 1 is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k!k} \sim \ln x + \gamma \tag{1}$$

as $x \to \infty$. In fact, Euler showed that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k!k} - \ln x - \gamma = \int_x^{\infty} \frac{e^{-t}}{t} dt = O\left(\frac{e^{-x}}{x}\right)$$

and this has been used by Sweeney and others to compute Euler's constant (one has to be careful because of cancellation in the series). In Ch. 12, Entry 44(ii), Ramanujan states Euler's result that the error is between $e^{-x}/(1+x)$ and e^{-x}/x.

A Generalization

Ramanujan's Corollary 2, Entry 9, Chapter 4 (page 98) is that, for positive integer n,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{nk} \left(\frac{x^k}{k!} \right)^n \sim \ln x + \gamma \tag{2}$$

so (1) is just the case $n = 1$.

Berndt shows that (2) is false for $n \geq 3$. In fact, the function defined by the left side of (2) changes sign infinitely often, and grows exponentially large as $x \to \infty$. However, Berndt leaves the case $n = 2$ open.

We shall sketch a proof that (2) is true in the case $n = 2$. In fact, we shall obtain an exact expression for the error in (2) as an integral involving the Bessel function $J_0(x)$, and deduce an asymptotic expansion.

The exact expression for $n = 2$ is a special case of a formula given on page 48 of Y. L. Luke, *Integrals of Bessel Functions*, 1962. However, Luke does not comment on the connection with Ramanujan.

Avoiding Cancellation

In Chapter 3, Entry 2, Cor. 2, page 46, Ramanujan states that the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k!k}$$

occurring in (1) can be written as

$$e^{-x} \sum_{k=0}^{\infty} H_k \frac{x^k}{k!}.$$

This is easy to prove (Berndt, page 47). Thus (1) gives

$$\sum_{k=0}^{\infty} H_k \frac{x^k}{k!} \Big/ \sum_{k=0}^{\infty} \frac{x^k}{k!} \sim \ln x + \gamma. \qquad (3)$$

This is more convenient than (1) for computation, because there is no cancellation in the series when $x > 0$. Later we indicate how Ramanujan might have generalized (3) in much the same way that he attempted to generalize (1).

Ramanujan's Corollary for $n = 2$

The following result from [3] shows that (2) is valid for $n = 2$. Recall that

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!k!}$$

is a Bessel function of the first kind and order zero.

Theorem

Let

$$e(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k} \left(\frac{x^k}{k!}\right)^2 - \ln x - \gamma.$$

Then, for real positive x,

$$e(x) = \int_{2x}^{\infty} \frac{J_0(t)}{t} dt.$$

Sketch of Proof

Proceed as on pg. 99 of Berndt, and use the fact that

$$\int_0^\infty \left(\frac{e^{-t} - J_0(2t)}{t} \right) dt = 0. \tag{4}$$

A slightly more general result than (4) is given in equation 6.622.1 of Gradshteyn and Ryzhik, and is attributed to Nielsen. An independent proof is given in [3].

Corollary

Let $e(x)$ be as in Theorem 1. Then, for large positive x, $e(x)$ has an asymptotic expansion

$$e(x) = \frac{1}{2\pi^{1/2}x^{3/2}} \left(\cos\left(2x + \frac{\pi}{4}\right) + \frac{13\sin\left(2x + \frac{\pi}{4}\right)}{16x} + O\left(\frac{1}{x^2}\right) \right).$$

Comparison of $n = 1$ and $n = 2$

We see that, for computational purposes, it is much better to take $n = 1$ than $n = 2$ in (2), because the error for $n = 1$ is $O(e^{-x}/x)$, but for $n = 2$ it is $\Omega_{\pm}(x^{-3/2})$.

A Different Generalization

We obtained (2) from (1) by replacing $x^k/k!$ by $(x^k/k!)^n/n$. A similar generalization of (3) is

$$\sum_{k=0}^{\infty} H_k \left(\frac{x^k}{k!}\right)^n \bigg/ \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)^n \sim \ln x + \gamma \qquad (5)$$

as $x \to \infty$. (3) is just the case $n = 1$.

It is easy to show that (5) is valid for all positive integer n. An essential difference between (2) and (5) is that there is a large amount of cancellation between terms on the left side of (2), but there is no cancellation in the numerator and denominator on the left side of (5). The function $(x^k/k!)^n$ acts as a smoothing kernel with a peak at $k \simeq x$. Since

$$H_k = \ln k + \gamma + O(1/k),$$

the result (5) is not surprising. What may be surprising is the speed of convergence.

Speed of Convergence

Brent and McMillan [2] show that

$$\sum_{k=0}^{\infty} H_k \left(\frac{x^k}{k!}\right)^n \Big/ \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)^n = \ln x + \gamma + O(e^{-c_n x}) \qquad (6)$$

as $x \to \infty$, where

$$c_n = \begin{cases} 1, & \text{if } n = 1; \\ 2n \sin^2(\pi/n), & \text{if } n \geq 2. \end{cases}$$

In the case $n = 2$, (6) has error $O(e^{-4x})$. Brent and McMillan used this case with $x \simeq 17{,}400$ to compute γ to high precision. They deduced that, if $\gamma = p/q$ is rational, then $q > 10^{15000}$. From Corollary 1, the same value of x in (2) would give less than 8-decimal place accuracy.

Another view

If you are looking for a good way to compute Euler's constant γ, you might scan Abramowitz and Stegun (or the online *Digital Library of Mathematical Functions*) looking for formulas in which γ occurs.

For example, in the chapter on Bessel functions, we find (9.6.13):

$$K_0(2x) = -(\ln(x) + \gamma)I_0(2x) + \frac{x^2}{(1!)^2} + (1 + \tfrac{1}{2})\frac{x^4}{(2!)^2} + \cdots$$

(where I replaced z by $2x$). Here $I_0(z)$ and $K_0(z)$ are *modified Bessel functions* (sometimes called *Bessel functions of imaginary argument* because we obtain them by $z \mapsto iz$ in the usual Bessel functions $J_0(z)$ etc).

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Formula (9.6.13) might be useful for evaluating γ if we had an independent way of evaluating $K_0(2x)$ and $I_0(2x)$.

Differential equation (ODE)

$I_0(z)$ and $K_0(z)$ are independent solutions of the *modified Bessel equation*

$$zw'' + w' - zw = 0.$$

This is the special case $\nu = 0$ of

$$z^2 w'' + zw' - (z^2 + \nu^2)w = 0.$$

Power series for I_0

Abramowitz and Stegun (9.6.12) gives the nice series

$$I_0(2x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(k!)^2},$$

so there is no difficulty in computing $I_0(2x)$.
(As usual, we replaced z by $2x$.)

Asymptotic series for I_0 and K_0

In the same chapter of Abramowitz and Stegun, we find the asymptotic expansions:

$$I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 + \frac{1^2}{1!(8z)} + \frac{1^2\cdot 3^2}{2!(8z)^2} + \frac{1^2\cdot 3^2\cdot 5^2}{3!(8z)^3} + \cdots\right),$$

$$K_0(z) \sim e^{-z}\sqrt{\frac{\pi}{2z}}\left(1 - \frac{1^2}{1!(8z)} + \frac{1^2\cdot 3^2}{2!(8z)^2} - \frac{1^2\cdot 3^2\cdot 5^2}{3!(8z)^3} + \cdots\right).$$

These expansions give a way of computing $I_0(z)$ and $K_0(z)$ accurately if z is sufficiently large (z is always real and positive in our applications).

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These expansions give a way of computing $I_0(z)$ and $K_0(z)$ accurately if z is sufficiently large (z is always real and positive in our applications).

The leading terms show that $K_0(z)/I_0(z) = O(e^{-2z})$ is exponentially small if z is large and positive.

Asymptotic series for $I_0 K_0$

While on the subject of asymptotic expansions, note that if we multiply the asymptotic expansions for $I_0(z)$ and $K_0(z)$, then half the terms vanish, and we obtain (at least formally)

$$I_0(z)K_0(z) \sim \frac{1}{2z}\left(1 + \frac{1^3}{1!(8z^2)} + \frac{1^3 \cdot 3^3}{2!(8z^2)^2} + \frac{1^3 \cdot 3^3 \cdot 5^3}{3!(8z^2)^3} + \cdots\right).$$

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To prove the formula for the general term, we could use the Wilf-Zeilberger (WZ) method. Easier is to use the ODE

$$z^3 f''' + z(1-4z^2)f' - f = 0$$

satisfied by $f(z) = zI_0(z)K_0(z)$. It is straightforward to deduce a recurrence relation for the coefficients in the asymptotic expansion from this ODE.

Error bounds for the asymptotic expansions

Suppose the asymptotic expansions for I_0, K_0 or $I_0 K_0$ are written as

$$F(z) \sim a_0(z) + a_1(z) + a_2(z) + \cdots$$

(where the $a_j(z)$ are not identically zero), and the error $E_n(z)$ is defined by

$$F(z) = a_0(z) + a_1(z) + \cdots + a_{n-1}(z) + E_n(z).$$

Then, provided z is real, $z \geq 1$, and $n > 0$, we can show that

$$|E_n(z)| = O(\sqrt{n}|a_n(z)|),$$

and even give an explicit constant in the "O" result (e.g. 4). In the case of K_0, the errors alternate in sign and there is a sharper bound

$$|E_n(z)| < |a_n(z)|.$$

Proofs of Error Bounds

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The proof for $I_0 K_0$ does not seem to have been published, and was stated as a conjecture in Brent and McMillan (1980). It is possible to deduce it from bounds for the K_0 and I_0 expansions.

Deducing γ

Rearranging (9.6.13) and using the power series for I_0 gives

$$\gamma + \ln(x) = \sum_{k=0}^{\infty} H_k \left(\frac{x^k}{k!}\right)^2 \Big/ \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)^2 - \frac{K_0(2x)}{I_0(2x)},$$

but the last term is $O(e^{-4x})$ so can be neglected if x is large. This is essentially Algorithm B1 of Brent & McMillan.

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We have just recovered (6) in the case $n = 2$, with an explicit error term $K_0(2x)/I_0(2x)$.

Approximating the error term

To get a more accurate algorithm (with the same x) we can try to approximate the error term $K_0(2x)/I_0(2x)$. Since $I_0(2x)$ has already been computed (denominator of the main term), we only need to approximate $K_0(2x)$. This can be done with relative error $O(e^{-4x})$ by taking $z = 2x$ and $\lceil 4x \rceil$ terms in the asymptotic expansion for $K_0(z)$.

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A faster way is to take $\lceil 2x \rceil$ terms in the asymptotic expansion for $I_0(2x)K_0(2x)$, and divide the result by $I_0(2x)^2$.

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A faster way is to take $\lceil 2x \rceil$ terms in the asymptotic expansion for $I_0(2x)K_0(2x)$, and divide the result by $I_0(2x)^2$.

In this way we get an algorithm for γ with error $O(e^{-8x})$ (Algorithm B3 of Brent & McMillan).

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It is easy to use this technique to sum the power series for $\exp(z)$ or for $I_0(z)$ when z is rational. The complexity for d digits is reduced from $O(d^2)$ to $O(d(\log d)^c)$ for some (small) constant c.

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It is trickier to implement binary splitting for the series involving Harmonic numbers H_k, but it can be done.

Binary Splitting – 1D case

The recursive procedure

$$S_1(f,j,\ell) = \begin{cases} 0 \text{ if } \ell \le 0, \\ f_j \text{ if } \ell = 1, \\ S_1(f,j,\lfloor \ell/2 \rfloor) + S_1(f,j+\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil) \text{ otherwise} \end{cases}$$

returns the sum

$$\sum_{0 \le k < \ell} f_{j+k} \, .$$

It is easy to modify S_1 to compute the polynomial

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(details left as an exercise).

Binary Splitting – 2D case

Similarly, to compute the sum

$$\sum_{0 \le p+q < \ell} f_{j+p} g_{k+q}$$

we can use the recursive procedure

$$S_2(j,k,\ell) = \begin{cases} 0 \text{ if } \ell \le 0, \\ f_j g_k \text{ if } \ell = 1, \\ S_2(j + \lfloor \ell/2 \rfloor, k, \lceil \ell/2 \rceil) + S_2(j, k + \lceil \ell/2 \rceil, \lfloor \ell/2 \rfloor) + \\ \quad S_1(f, j, \lfloor \ell/2 \rfloor) S_1(g, k, \lceil \ell/2 \rceil) \text{ otherwise.} \end{cases}$$

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This is essentially the "short product" algorithm of Mulders (2000).

Binary Splitting cont.

We can use the recursive procedure S_2 to compute sums such as

$$\sum_{0<k<n} H_k b_k = \sum_{0<j\le k<n} \frac{b_k}{j}$$

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Record Computation of γ

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For more details, history, and other constants, see `http://numberworld.org/nagisa_runs/computations.html`.

Who was McMillan?

I met Ed McMillan when I was on sabbatical leave in Berkeley in 1977/8. At that time he had recently retired from Lawrence Berkeley Laboratory but still had an office there. He had seen my (first) paper on Euler's constant in *Math. Comp. 31 (1977)* and wanted to talk to me about possible improvements. Thus, I walked up the hill to LBL to talk to him, and our collaboration started.

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Luckily I knew nothing about him at the time, or I might have been intimidated. He seemed to be just a scholarly old gentleman who was interested in Bessel functions.

McMillan's notes 17 Nov 1977, page 1

A Suggested Method for Computing Euler's Constant

Edwin M. McMillan
Nov. 17, 1977

The most recent computation[1] of Euler's constant γ used a series expansion for the exponential integral, which was set equal to an asymptotic expression for the same integral:

$$\int_n^\infty \frac{e^{-x}}{x}\,dx = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{n^k}{k \cdot k!} - \ln n - \gamma \approx \frac{e^{-n}}{n}\left(1 - \frac{1}{n} + \frac{2!}{n^2} - \frac{3!}{n^3} \cdots\right) \quad ($$

The asymptotic series is only semi-convergent; if the sum is terminated just before the smallest term, the error is less than the smallest term. In this case the smallest term is $n!/n^n \approx \sqrt{2\pi n}\, e^{-n}$, and the resulting uncertainty in the value of γ is $\approx \sqrt{2\pi/n}\, e^{-2n}$.

Therefore, if one desires γ to d decimal places, n should be chosen to be $\approx \frac{1}{2} d \ln 10$.

The other series converges, but to match the precision of the asymptotic expression it needs to be summed to $\approx 4.32\, n$ terms. It is an alternating series with a largest

(1) R. P. Brent, Math. Comp. 31, 771–777 (1977)

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term of order e^{-n}, therefore the summation must be carried out to $\approx 3/2\, d$.

A non-alternating series would have the advantage that the precision you use is the precision you get. The purpose of this note is to suggest the use of the modified Bessel function $K_0(x)$, which involves non-alternating series, with the additional advantage that the denominator is $(k!)^2$, making the series converge very rapidly.

$$K_0(x) = -\left(\gamma + \ln \frac{x}{2}\right) I_0(x) + S_0(x) \qquad (2)$$

$$I_0(x) = \sum_{k=0}^{\infty} \left(\frac{\left(\frac{x}{2}\right)^k}{k!} \right)^2 \qquad (3)$$

Double → (can get over it)

$$S_0(x) = \sum_{k=1}^{\infty} \left(\frac{\left(\frac{x}{2}\right)^k}{k!} \right)^2 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{k}\right) \qquad (4)$$

Asymptotic expressions:

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}} \left(1 + \frac{1}{8x} + \cdots \right) \qquad (5)$$

$$K_0(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[1 - \frac{1}{8x} + \frac{(1 \cdot 3)^2}{2!\,(8x)^2} - \frac{(1 \cdot 3 \cdot 5)^2}{3!\,(8x)^3} + \cdots \right] \qquad (6)$$

$\frac{1}{2}$? (see Dwight 81)

Letting $A_0(x)$ represent the asymptotic expression (6) summed up to its smallest term,

$$\gamma + \ln \frac{x}{2} \approx \frac{S_0(x) - A_0(x)}{I_0(x)} \qquad (7)$$

The general term ~~of (6)~~ the series in (6) can be written $\left[\frac{(2m)!}{2^m m!}\right]^2 \frac{(-1)^m}{m!\,(8x)^m}$.

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The smallest term is at $m = 2X+1$, and has the value $\approx \sqrt{1/\pi X}\, e^{-2X}$. The resulting uncertainty in γ is $\approx 2\sqrt{\pi/X}\, e^{-4X}$.

Therefore, to get γ to d decimal places, X should be chosen to be $\approx 1/4\, d \ln 10$, i.e. half as large as the value of n used in the exponential integral method. Both methods need the same number of places ($1/2\, d$) and the same number of terms ($1/2\, d \ln 10$) in the evaluation of the asymptotic series. Both methods require evaluation of e^{x} (or e^{n}) in the factors multiplying the asymptotic series to $1/2\, d$, and the Bessel function method requires in addition $\sqrt{\pi}$ (and $\sqrt{2/n}$ unless $n/2$ is a perfect square) to ~~1/2 d~~ $3d/4$.

In order to match the precision allowed by the uncertainty in $A_0(X)$, the series for $I_0(X)$ and $S_0(X)$ must be summed to k terms, where:

$$\frac{2k}{X} \ln \frac{2k}{X} - \frac{2k}{X} \approx 3,$$

with the solution $k \approx 2.79X$.

Therefore these series must be summed to $\approx 0.62\, d \ln 10$

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terms, to d decimal places.

A comparison of the two methods:

	decimals needed	number of terms	[illegible]
exponential integral	$1.5\,d$	$2.16\,d\,\ln 10$	$3.24\,d^2 \ln 10$ $\times 3$ (mult/div)
Bessel function	d	$0.32\,d\,\ln 10$	$0.62\,d^2 \ln 10$ $\times 8$ (mult/div)

This great advantage is somewhat compensated by the fact that the Bessel function method requires the summation of the two series $I_0(x)$ and $S_0(x)$, but these have in common the factor $(x/2)^{2k}/(k!)^2$, so the summations can be carried on in parallel. The factor $\left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{k}\right)$ can be stored, with an additional reciprocal added at each step. [illegible] 8/3/94

but MP mult. of each term involved $\therefore O(d^3)$ method

(But can get around it – see 771213/1 notes)

Results with a hand computer:

x	γ
1	0.59
2	0.5774
3	0.577223
true value	0.577215

McMillan's other life

Later I discovered that McMillan had worked at Los Alamos during the second world war, and was famous for discovering the first trans-uranium element (neptunium) in 1940, soon followed by the discovery of plutonium with Seaborg.

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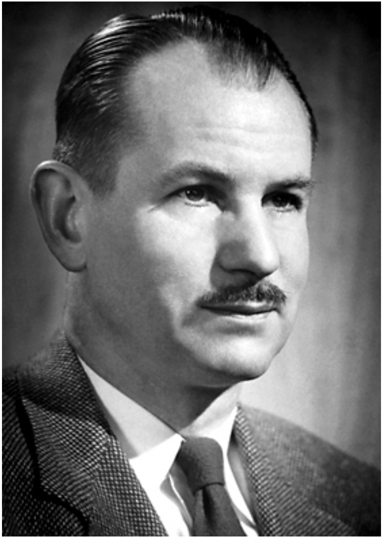
According to the obituary by Jackson and Panofsky, McMillan's last published paper was the one that he wrote with me on the computation of Euler's constant. It was published in *Math. Comp.* 34 (1980).

Edwin Mattison McMillan 1907–1991 (about 1950)



Another photo (much as I remember him in 1977)

Courtesy of the Lawrence Berkeley Laboratory

Prehistory - Riemann 1855

$$\int_1^\infty \frac{e^{-2qt}\,dt}{\sqrt{tt-1}} \text{ durch } f(q), \quad \int_{-1}^1 \frac{e^{-2qt}\,dt}{\sqrt{1-tt}} \text{ durch } \varphi(q)$$

bezeichnet:

$$b_n = c_n f\left(n\frac{\pi}{4\beta}r\right) + \gamma_n \varphi\left(n\frac{\pi}{4\beta}r\right).$$

Die Entwicklung nach steigenden Potenzen von q giebt

$$f(q) = \sum_{0,\infty} \frac{q^{2m}}{m!\,m!}(\Psi(m) - \log q)$$

$$\varphi(q) = \pi \sum_{0,\infty} \frac{q^{2m}}{m!\,m!}; \qquad (^3)$$

es wird also $f(q)$ für $q = 0$ unendlich und damit u'' für $r = 0$ stetig bleibe, muss $c_n = 0$ sein; γ_n ergiebt sich dann aus (4) gleich

$$-\frac{4\sin n\frac{\pi}{2\beta}\alpha}{\beta}\,\frac{f'\left(n\frac{\pi}{4\beta}c\right)}{\varphi'\left(n\frac{\pi}{4\beta}c\right)},$$

mithin

$$u = \Sigma^n \sin n\frac{\pi}{2\beta}z\,\frac{4\sin n\frac{\pi}{2\beta}\alpha}{\beta}\left\{f\left(n\frac{\pi}{4\beta}r\right) - \varphi\left(n\frac{\pi}{4\beta}r\right)\frac{f'\left(n\frac{\pi}{4\beta}c\right)}{\varphi'\left(n\frac{\pi}{4\beta}c\right)}\right\},$$

über alle positiven ungeraden Werthe von n ausgedehnt.

Zur Berechnung von $f(q)$ und $\varphi(q)$ können für grosse Werthe von q die halbconvergenten Reihen

$$f(q) = e^{-2q}\sqrt{\frac{\pi}{4q}} \sum_{m<4q+1} (-1)^m \frac{(1\,.\,3\ldots\overline{2m-1})^2}{m!\,(16q)^m},$$

$$\varphi(q) = e^{2q}\sqrt{\frac{\pi}{4q}} \sum_{m<4q+1} \frac{(1\,.\,3\ldots\overline{2m-1})^2}{m!\,(16q)^m} \qquad (^4)$$

benutzt werden, welche indess ihren Werth nur bis auf Bruchtheile von der Ordnung der Grösse e^{-4q} geben; genügt diese Genauigkeit nicht, so ist es wohl am zweckmässigsten die Entwicklungen nach steigenden Potenzen von q anzuwenden.

Für hinreichend grosse Werthe von $\frac{r}{\beta}$ erhält man also mit Vernachlässigung von Grössen von der Ordnung der Grösse $e^{-3\frac{\pi}{2\beta}r}$

Conclusion

We did not succeed in proving that γ is irrational, but the quest was worthwhile because it provided the motivation:

- to read Ramanujan's papers and Notebooks;
- to meet and collaborate with Ed McMillan;
- to learn more about Bessel functions (a topic of classical mathematics that should be better known).

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Finally, here is a nice integral for $K_0(x)$, $x > 0$:

$$K_0(x) = \int_0^\infty \frac{\cos(xt)}{\sqrt{1+t^2}}\,\mathrm{d}t.$$

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We could use this with numerical quadrature to compute γ, but it would be unlikely to give a fast algorithm.

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