A NOTE ON PÓLYA’S OBSERVATION CONCERNING
LIOUVILLE’S FUNCTION

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Dedicated to Herman J. J. te Riele on the occasion of his retirement from the
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Abstract. We show that a certain weighted mean of the Liouville function \( \lambda(n) \) is negative. In this sense, we can say that the Liouville function is negative “on average”.

1. Introduction

For \( n \in \mathbb{N} \) let \( n = \prod_{p|n} p^{e_p(n)} \) be the canonical prime factorization of \( n \) and let \( \Omega(n) := \sum_{p|n} e_p(n) \). Here (as always in this paper) \( p \) is prime. Thus, \( \Omega(n) \) is the total number of prime factors of \( n \), counting multiplicities. For example: \( \Omega(1) = 0, \Omega(2) = 1, \Omega(4) = 2, \Omega(6) = 2, \Omega(8) = 3, \Omega(16) = 4, \Omega(60) = 4 \), etc.

Define Liouville’s multiplicative function \( \lambda(n) = (-1)^{\Omega(n)} \). For example \( \lambda(1) = 1, \lambda(2) = -1, \lambda(4) = 1 \), etc. The Möbius function \( \mu(n) \) may be defined to be \( \lambda(n) \) if \( n \) is square-free, and 0 otherwise.

It is well-known, and follows easily from the Euler product for the Riemann zeta-function \( \zeta(s) \), that \( \lambda(n) \) has the Dirichlet generating function

\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}
\]

for Re \( (s) > 1 \). This provides an alternative definition of \( \lambda(n) \).

Let \( L(n) := \sum_{k \leq n} \lambda(k) \) be the summatory function of the Liouville function; similarly \( M(n) := \sum_{k \leq n} \mu(k) \) for the Möbius function.

The topic of this note is closely related to Pólya’s conjecture [12, 1919] that \( L(n) \leq 0 \) for \( n \geq 2 \).

Pólya verified this for \( n \leq 1500 \) and Lehmer [9, 1956] checked it for \( n \leq 600\,000 \). However, Ingham [5, 1942] cast doubt on the plausibility of Pólya’s conjecture by showing that it would imply not only the Riemann Hypothesis and simplicity of the zeros of \( \zeta(s) \), but also the linear dependence over the rationals of the imaginary parts of the zeros.
ρ of ζ(s) in the upper half-plane. Ingham cast similar doubt on the Mertens conjecture |M(n)| ≤ √n, which was subsequently disproved in a remarkable tour de force by Odlyzko and te Riele [11, 1985]. More recent results and improved bounds were given by Kotnik and te Riele [7, 2006]; see also Kotnik and van de Lune [6, 2004].

In view of Ingham’s results, it was no surprise when Haselgrove showed [2, 1958] that Pólya’s conjecture is false. He did not give an explicit counter-example, but his proof suggested that L(u) might be positive in the vicinity of u ≈ 1.8474 × 10^{361}.

Sherman Lehman [8, 1960] gave an algorithm for calculating L(n) similar to Meissel’s [10, 1885] formula for the prime-counting function π(x), and found the counter-example L(906 180 359) = +1.

Tanaka [14, 1980] found the smallest counter-example L(n) = +1 for n = 906 150 257. Walter M. Lioen and Jan van de Lune [circa 1994] scanned the range n ≤ 2.5 × 10^{11} using a fast sieve, but found no counter-examples beyond those of Tanaka. More recently, Borwein, Ferguson and Mossinghoff [1, 2008] showed that L(n) = +1 160 327 for n = 351 753 358 289 465.

Humphries [3, 4] showed that, under certain plausible but unproved hypotheses (including the Riemann Hypothesis), there is a limiting logarithmic distribution of L(n)/√n, and numerical computations show that the logarithmic density of the set \{n ∈ N|L(n) < 0\} is approximately 0.99988. Humphries’ approach followed that of Rubinstein and Sarnak [13], who investigated “Chebyshev’s bias” in prime “races”.

Here we show in an elementary manner, and without any unproved hypotheses, that λ(n) is (in a certain sense) “negative on average”. To prove this, all that we need are some well-known facts about Mellin transforms, and the functional equation for the Jacobi theta function (which may be proved using Poisson summation). Our main result is:

**Theorem 1.** There exists a positive constant c such that for every (fixed) N ∈ N

\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{\pi x} + 1} = -\frac{c}{\sqrt{\pi}} + \frac{1}{2} + O(x^N) \quad \text{as} \quad x \downarrow 0.
\]

Thus, a weighted mean of \{λ(n)\}, with positive weights initially close to a constant (1/2) and becoming small for n ≫ 1/x, is negative for x < x_0 and tends to −∞ as x ↓ 0.

In the final section we mention some easy results on the Möbius function µ(n) to contrast its behaviour with that of λ(n).
2. Proof of Theorem 1

We prove Theorem 1 in three steps.

Step 1. For $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} - 1} = \phi(x) = \frac{\theta(x) - 1}{2},$$

where

$$\phi(x) := \sum_{k=1}^{\infty} e^{-k^2 \pi x}, \quad \theta(x) := \sum_{k \in \mathbb{Z}} e^{-k^2 \pi x}.$$

Step 2. For $x > 0$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = \phi(x) - 2\phi(2x).$$

Step 3. Theorem 1 now follows from the functional equation

$$\theta(x) = \frac{1}{\sqrt{x}} \theta \left( \frac{1}{x} \right)$$

for the Jacobi theta function $\theta(x)$.

Proof of Theorem 1.

(1) In the following, we assume that $\text{Re}(s) > 1$, so the Dirichlet series and integrals are absolutely convergent, and interchanging the orders of summation and integration is easy to justify.

As mentioned above, it is well-known that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{(1 + p^{-s})^{-1}} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}.$$

Define

$$f(x) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{nx} - 1}, \quad (x > 0).$$

We will use the well known fact that if two sufficiently well-behaved functions (such as ours below) have the same Mellin transform then the functions are equal.
The Mellin transform of \( f(x) \) is
\[
F(s) := \int_0^\infty f(x)x^{s-1} \, dx = \int_0^\infty \sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{nx} - 1} x^{s-1} \, dx
\]
\[
= \sum_{n=1}^{\infty} \lambda(n) \int_0^\infty \frac{x^{s-1}}{e^{nx} - 1} \, dx = \left( \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \right) \times \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx
\]
\[
= \frac{\zeta(2s)}{\zeta(s)} \times \zeta(s) \Gamma(s) = \zeta(2s) \Gamma(s).
\]

We also have
\[
\int_0^\infty \phi \left( \frac{x}{\pi} \right) x^{s-1} \, dx = \int_0^\infty \left( \sum_{n=1}^{\infty} e^{-n^2x} \right) x^{s-1} \, dx
\]
\[
= \left( \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right) \times \int_0^\infty e^{-x} x^{s-1} \, dx = \zeta(2s) \Gamma(s),
\]
so the Mellin transforms of \( f(x) \) and of \( \phi(x/\pi) \) are identical. Thus \( f(x) = \phi(x/\pi) \).
Replacing \( x \) by \( \pi x \), we see that
\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} - 1} = \sum_{k=1}^{\infty} e^{-k^2\pi x},
\]
completing the proof of step (1). (2) Observe that
\[
\frac{1}{e^{n\pi x} + 1} = \frac{1}{e^{n\pi x} - 1} - \frac{2}{e^{2n\pi x} - 1},
\]
so, from step (1),
\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x} + 1} = \phi(x) - 2\phi(2x).
\]

(3) Using the functional equation for \( \theta(x) \), we easily find that
\[
\phi(x) - 2\phi(2x) = -\frac{c}{\sqrt{x}} + \frac{1}{2} + \frac{1}{\sqrt{x}} \left( \phi \left( \frac{1}{x} \right) - \sqrt{2} \phi \left( \frac{1}{2x} \right) \right)
\]
with \( c = (\sqrt{2} - 1)/2 > 0 \), proving our claim, since the “error” term is bounded by \( \phi(1/x)/\sqrt{x} \sim \exp(-\pi/x)/\sqrt{x} = O(x^N) \) as \( x \downarrow 0 \) (for any fixed exponent \( N \)).
3. Remarks on the Möbius function

We give some further applications of the identity

\((*)\)

\[
\frac{1}{z + 1} = \frac{1}{z - 1} - \frac{2}{z^2 - 1}
\]

that we used (with \(z = e^{n\pi x}\)) in proving step (2) above.

**Lemma 2.** For \(|x| < 1\), we have

\[
\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n + 1} = x - 2x^2.
\]

**Proof.** Assume that \(|x| < 1\). It is well known that

\[
\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1 - x^n} = x,
\]

in fact this “Lambert series” identity is equivalent to the Dirichlet series identity \(\sum \mu(n)/n^s = 1/\zeta(s)\). Writing \(y = 1/x\), we have

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n - 1} = 1/y.
\]

If follows on taking \(z = y^n\) in our identity \((*)\) that

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{y^n + 1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{y^n - 1} - 2 \sum_{n=1}^{\infty} \frac{\mu(n)}{y^{2n} - 1} = y^{-1} - 2y^{-2}.
\]

Replacing \(y\) by \(1/x\) gives the result. \(\Box\)

**Corollary 3.**

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{2^n + 1} = 0.
\]

**Proof.** Take \(x = 1/2\) in Lemma 2. \(\Box\)

If follows from Lemma 2 that

\[
\lim_{x \uparrow 1} \sum_{n=1}^{\infty} \mu(n) \frac{x^n}{x^n + 1} = -1,
\]

so that one might say that in this sense \(\mu(n)\) is negative on average. However, this is much weaker than what we showed in Theorem 1 for \(L(n)\), where the corresponding sum tends to \(-\infty\). The “complex-analytic” reason for this difference is that \(\zeta(2s)/\zeta(s)\) has a pole (with negative residue) at \(s = 1/2\), but \(1/\zeta(s)\) is regular at \(s = 1\).
References


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