Some binomial sums involving absolute values

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Abstract

We consider several families of binomial sum identities whose definition involves the absolute value function. In particular, we consider centered double sums of the form

\[ S_{\alpha,\beta}(n) := \sum_{k, \ell} \left( \binom{2n}{n+k} \binom{2n}{n+\ell} \right) |k^\alpha - \ell^\alpha|^\beta, \]

obtaining new results in the cases \( \alpha = 1, 2 \). We show that there is a close connection between these double sums in the case \( \alpha = 1 \) and the single centered binomial sums considered by Tuenter.
1 Introduction

The problem of finding a closed form for the binomial sum
\[ \sum_{k, \ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^2 - \ell^2| \]  

arises in an application of the probabilistic method to the Hadamard maximal determinant problem [7]. Because of the double-summation and the absolute value occurring in (1), it is not obvious how to apply standard techniques [10, 15, 19]. A closed-form solution
\[ 2n^2 \left( \frac{2n}{n} \right)^2 \]  
was proved by Brent and Osborn in [6], and simpler proofs were subsequently found [5, 8, 16]. In this paper we consider a wider class of binomial sums with the distinguishing feature that an absolute value occurs in the summand.

Specifically, we consider certain d-fold binomial sums of the form
\[ S(n) := \sum_{k_1, \ldots, k_d} \prod_{i=1}^{d} \binom{2n}{n+k_i} |f(k_1, \ldots, k_d)|, \]
where \( f : \mathbb{Z}^d \rightarrow \mathbb{Z} \) is a homogeneous polynomial and \( |f| \) will be called the weight function. For example, a simple case is \( d = 1, f(k) = k \). This case was considered by Best [1] in an application to Hadamard matrices. The closed-form solution is
\[ \sum_{k} \binom{2n}{n+k} |k| = n \binom{2n}{n}. \]

A generalization \( f(k) = k^r \) (for a fixed \( r \in \mathbb{N} \)) was considered by Tuenter [18], and shown to be expressible using Dumont-Foata polynomials [9]. Tuenter gave an interpretation in terms of the moments of the distance to the origin in a symmetric Bernoulli random walk. It is easy to see that this interpretation generalizes: \( 4^{-nd} S(n) \) is the expectation of \( |f(k_1, \ldots, k_d)| \) if we start at the origin and take \( 2n \) random steps \( \pm \frac{1}{2} \) in each of \( d \) dimensions, thus arriving at the point \( (k_1, \ldots, k_d) \in \mathbb{Z}^d \) with probability
\[ 4^{-nd} \prod_{i=1}^{d} \binom{2n}{n+k_i}. \]
A further generalization replaces $\binom{2n_i}{n_i+k_i}$ by $\binom{2n_i}{n_i}$, allowing the number of random steps $(2n_i)$ in dimension $i$ to depend on $i$. With a suitable modification to the definition of $S$, we could also drop the restriction to an even number of steps in each dimension.\footnote{For example, in the case $d = 1$ we could consider $\sum_k \binom{n}{k} |f(n - 2k)|$.}

We briefly consider such a generalization in \[\text{§2}\].

Tuenter’s results for the case $d = 1$ were generalized by the first author \[3\]. In this paper we concentrate on the case $d = 2$. Generalizations of some of the results to arbitrary $d$ are known. More specifically, the paper \[4\] gives closed-form solutions for the $d$-dimensional generalization of the sum \[9\] below in the cases $\alpha, \beta \in \{1, 2\}$.

There are many binomial coefficient identities in the literature, e.g. 500 are given by Gould \[11\]. Many such identities can be proved via generating functions \[12, 19\] or the Wilf-Zeilberger algorithm \[15\]. Nevertheless, we hope that the reader will find our results interesting, in part because of the applications mentioned above, and also because it is a challenge to generalize the results to higher values of $d$.

A preliminary version of this paper, with some of the results conjectural, was made available on arXiv \[5\]. All the conjectures have since been proved by Bostan, Lairez and Salvy \[2\], Krattenthaler and Schneider \[14\], Brent, Krattenthaler and Warnaar \[4\], and the present authors.

An outline of the paper follows.

In \[\text{§2}\] we consider a special class of double sums that can be reduced to the single sums of \[3, 18\].

In \[\text{§3}\] we consider a generalization of the motivating case \[1\] described above: $f(k, \ell) = (k^\alpha - \ell^\alpha)^\beta$. In the case $\alpha = 2$ we give recurrence relations that allow such sums to be evaluated in closed form for any given positive integer $\beta$. The recurrence relations naturally split into the cases where $\beta$ is even (easy) and odd (more difficult).

Theorem \[6\] in \[\text{§4}\] gives a closed form for an analogous triple sum. In \[5, \text{Conjecture 2}\] a closed form for the analogous quadruple sum was conjectured. This conjecture has now been proved by Brent, Krattenthaler and Warnaar \[4\]; in fact they give a generalization to arbitrary positive integer $d$.

In \[\text{§5}\] we state several double sum identities that were proved or conjectured by us \[5\]. The missing proofs have now been provided by Bostan, Lairez and Salvy \[2\] and by Krattenthaler and Schneider \[14\].
Notation

The set of all integers is \( \mathbb{Z} \), and the set of non-negative integers is \( \mathbb{N} \).

The binomial coefficient \( \binom{n}{k} \) is defined to be zero if \( k < 0 \) or \( k > n \) (and hence always if \( n < 0 \)). Using this convention, we often avoid explicitly specifying upper and lower limits on \( k \) or excluding cases where \( n < 0 \).

In the definition of the weight function \( |f| \), we always interpret \( 0^0 \) as 1.

2 Some double sums reducible to single sums

Tuenter [18] considered the binomial sum

\[
S_\beta(n) := \sum_k \binom{2n}{n+k} |k|^\beta, \tag{4}
\]

and a generalization\(^2\) to

\[
U_\beta(n) := \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^\beta \tag{5}
\]

was given by the first author [3].

Tuenter showed that

\[
S_{2\beta}(n) = Q_\beta(n) 2^{2n-\beta}, \quad S_{2\beta+1}(n) = P_\beta(n) n \binom{2n}{n}, \tag{6}
\]

where \( P_\beta(n) \) and \( Q_\beta(n) \) are polynomials of degree \( \beta \) with integer coefficients, satisfying certain three-term recurrence relations, and expressible in terms of Dumont-Foata polynomials [9]. Closed-form expressions for \( S_\beta(n) \), \( P_\beta(n) \), \( Q_\beta(n) \) are known [3].

In this section we consider the double sum

\[
T_\beta(m, n) := \sum_{k, \ell} \binom{2m}{m+k} \binom{2n}{n+\ell} |k - \ell|^\beta \tag{7}
\]

and show that it can be expressed as a single sum of the form (4).

\(^2\)It is a generalization because \( S_\beta(n) = U_\beta(2n) \), but \( U_\beta(n) \) is well-defined for all \( n \in \mathbb{N} \).
Theorem 1. For all $\beta, m, n \in \mathbb{N}$, we have

\[ T_\beta(m, n) = S_\beta(m + n), \]

where $T_\beta$ is defined by (7) and $S_\beta$ is defined by (11).

Proof. If $\beta = 0$ then $T_0(m, n) = 2^{2(m+n)} = S_0(m + n)$. Hence, we may assume that $\beta > 0$ (so $0^\beta = 0$). Let $d = |k - \ell|$. We split the sum (7) defining $T_\beta(m, n)$ into three parts, corresponding to $k > \ell$, $k < \ell$, and $k = \ell$. The third part vanishes. If $k > \ell$ then $d = k - \ell$ and $k = d + \ell$; if $k < \ell$ then $d = \ell - k$ and $\ell = d + k$. Thus, we get

\[
T_\beta(m, n) = \sum_{d > 0} \sum_\ell \left( \binom{2m}{m+d+\ell} \binom{2n}{n+\ell} \right) d^\beta + \sum_{d > 0} \sum_k \left( \binom{2m}{m+k} \binom{2n}{n+k+d} \right) d^\beta \\
= \sum_{d > 0} d^\beta \sum_\ell \left( \binom{2m}{m+d+\ell} \binom{2n}{n-\ell} \right) + \sum_{d > 0} d^\beta \sum_k \left( \binom{2n}{n+k+d} \binom{2m}{m-k} \right).
\]

By Vandermonde’s identity, the inner sums over $k$ and $\ell$ are both equal to \(\binom{2m+2n}{m+n+d}\). Thus,

\[
T_\beta(m, n) = 2 \sum_{d > 0} \left( \binom{2m+2n}{m+n+d} \right) d^\beta = \sum_d \left( \binom{2m+2n}{m+n+d} \right) |d|^\beta = S_\beta(m + n).
\]

Remark 1. If $m = n$ then, by the shift-invariance of the weight $|k - \ell|^\beta$, we have

\[
T_\beta(n, n) = \sum_{k, \ell} \left( \binom{2n}{k} \binom{2n}{\ell} \right) |k - \ell|^\beta = S_\beta(2n). \tag{8}
\]

There is no need for the upper argument of the binomial coefficients to be even in (8). We can adapt the proof of Theorem 1 to show that, for all $n \in \mathbb{N}$,

\[
\sum_{k, \ell} \left( \binom{n}{k} \binom{n}{\ell} \right) |k - \ell|^\beta = S_\beta(n).
\]
3 Centered double sums

In this section we consider the centered double binomial sums defined by

\[ S_{\alpha,\beta}(n) := \sum_{k, \ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^\alpha - \ell^\beta|. \] (9)

Note that \( S_{1,\beta}(n) = T_{\beta}(n, 0) \), so the case \( \alpha = 1 \) is covered by Theorem 1. Thus, in the following we can assume that \( \alpha \geq 2 \). Since we mainly consider the case \( \alpha = 2 \), it is convenient to define

\[ W_{\beta}(n) := S_{2,\beta}(n) = \sum_{k, \ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^2 - \ell^2|^\beta. \] (10)

Remark 2. The sequences \( (S_{\alpha,\beta}(n))_{n \geq 1} \) for \( \alpha \in \{1, 2\} \) and \( 1 \leq \beta \leq 4 \) are in the OEIS [17]. Specifically, \( (S_{1,1}(n))_{n \geq 1} \) is a subsequence of A166337 (the entry corresponding to \( n = 0 \) must be discarded). \( (S_{2,1}(n))_{n \geq 0} \) is A254408, and \( (S_{\alpha,\beta}(n))_{n \geq 0} \) for \( (\alpha, \beta) = (1,2), (2,2), (1,3), (2,3), (1,4), (2,4) \) are A268147, A268148, ..., A268152 respectively.

3.1 \( W_{\beta} \) for odd \( \beta \)

The analysis of \( W_{\beta}(n) \) naturally splits into two cases, depending on the parity of \( \beta \). We first consider the case that \( \beta \) is odd. A simpler approach is possible when \( \beta \) is even, as we show in §3.3.

As mentioned in §1, the evaluation of \( W_1(n) \) was the motivation for this paper, and is given in the following theorem.

Theorem 2 (Brent and Osborn).

\[ W_1(n) = \sum_{k, \ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^2 - \ell^2| = 2n^2 \binom{2n}{n}^2. \]

Numerical evidence suggested the following generalization of Theorem 2. It was conjectured by the present authors [5, Conjecture 2], and proved by Krattenthaler and Schneider [14].

\footnote{The double sum \( S_{\alpha,\beta}(n) \) should not be confused with the single sum \( S_{\alpha}(n) \) of \[2\].}
**Theorem 3** (Krattenthaler and Schneider). For all \( m, n \in \mathbb{N} \),

\[
\sum_{k, \ell} \binom{2m}{m+k}\binom{2n}{n+\ell}|k^2-\ell^2| \geq 2mn\binom{2m}{m}\binom{2n}{n},
\]

with equality if and only if \( m = n \).

### 3.2 Recurrence relations for the odd case

Theorem 2 gives \( W_1(n) \). We show how \( W_{2k+1}(n) \), \( W_{2k+3}(n) \), \( W_{2k+5}(n) \), \ldots can be computed using recurrence relations. More precisely, we express the double sums \( W_{2k+1}(n) \) in terms of certain single sums \( G_k(n, m) \), and give a recurrence for the \( G_k(n, m) \). We then show that \( W_{2k+1}(n) \) is a linear combination of \( P_k(n) \), \ldots, \( P_{2k}(n) \), where the polynomials \( P_m(n) \) are as in (6), and the coefficients multiplying these polynomials satisfy another recurrence relation.

Define

\[
f_q = \begin{cases} 
1 & \text{if } q \neq 0; \\
\frac{1}{2} & \text{if } q = 0.
\end{cases}
\]

Using symmetry and the definition (10) of \( W_k(n) \), we have

\[
W_{2k+1}(n) = 8 \sum_{q=0}^{n} \sum_{p=q}^{n} \binom{2n}{n+p}\binom{2n}{n+q}(p^2 - q^2)^{2k+1}f_q; \tag{11}
\]

the factor \( f_q \) allows for terms which would otherwise be counted twice.

Let \( m = p - q \). Since \( p^2 - q^2 = m(m + 2q) \), we can write the double sum \( W_{2k+1}(n)/8 \) in (11) as

\[
\sum_{q=0}^{n} \sum_{p=q}^{n} \binom{2n}{n+p}\binom{2n}{n+q}(p^2 - q^2)^{2k+1}f_q = \sum_{m \geq 0} m^{2k+1}G_k(n, m), \tag{12}
\]

where

\[
G_k(n, m) := \sum_{q \geq 0} \binom{2n}{n+m+q}\binom{2n}{n+q}(m + 2q)^{2k+1}f_q. \tag{13}
\]

Observe that \( G_k(0, m) = 0 \). For convenience we define \( G_k(-1, m) = 0 \). We observe that \( G_k(n, m) \) satisfies a recurrence relation, as follows.
Lemma 1. For all $k, m, n \in \mathbb{N}$,

$$
G_{k+2}(n, m) = 2(4n^2 + m^2)G_{k+1}(n, m) - (4n^2 - m^2)^2 G_k(n, m) \\
+ 64n^2(2n - 1)^2 G_k(n - 1, m).
$$

(14)

Proof. If $n = 0$ the proof of (14) is trivial, since $G_k(0, m) = G_k(-1, m) = 0$. Hence, suppose that $n > 0$. We observe that

$$
[(m + 2q)^4 - 2(4n^2 + m^2)(m + 2q)^2 + (4n^2 - m^2)^2] \binom{2n}{n+m+q} \binom{2n}{n+q} \\
= 16(n + m + q)(n-m-q)(n+q)(n-q) \binom{2n}{n+m+q} \binom{2n}{n+q} \\
= 64n^2(2n - 1)^2 \binom{2n-2}{n-1+m+q} \binom{2n-2}{n-1+q}.
$$

Now multiply each side by $(m + 2q)^{2k+1} f_q$ and sum over $q \geq 0$. \hfill \Box

The recurrence (14) may be used to compute $G_k(n, m)$ for given $(n, m)$ and $k = 0, 1, 2, \ldots$, using the initial values

$$G_0(n, m) = \frac{n}{2} \binom{2n}{n} \binom{2n}{n+m}
$$

and

$$G_1(n, m) = \frac{4n^2 + (2n - 5)m^2}{2n - 1} G_0(n, m).
$$

These initial values may be verified from the definition (13) by standard methods [15] – we omit the details.

Write $g_k(n, m) = 0$ if $G_k(n, m) = 0$, and otherwise define $g_k(n, m)$ by

$$G_k(n, m) = \binom{2n}{n} \binom{2n}{n+m} g_k(n, m).
$$

The recurrence (14) for $G_k$ gives a corresponding recurrence for $g_k$:

$$g_{k+2}(n, m) = 2(4n^2 + m^2)g_{k+1}(n, m) - (4n^2 - m^2)^2 g_k(n, m) \\
+ 16n^2(n^2 - m^2) g_k(n - 1, m),
$$

(15)

with initial values

$$g_0(n, m) = \frac{n}{2}, \quad g_1(n, m) = \frac{4n^2 + (2n - 5)m^2}{2n - 1} g_0(n, m).$$
Note that the \( g_k(n, m) \) are rational functions in \( n \) and \( m \); if computation with bivariate polynomials over \( \mathbb{Z} \) is desired then \( g_k(n, m) \) can be multiplied by \( (2^n - 1)(2^n - 3) \cdots (2^n - (2k - 1)) \). If \( n \) is fixed, then \( g_k(n, m) \) is an even polynomial in \( m \) and, from the recurrence (15), the degree is \( 2k \). This suggests that we should define rational functions \( \gamma_{k,j}(n) \) by

\[
g_k(n, m) = \sum_{j=0}^{k} \gamma_{k,j}(n) m^{2j}.
\]

For \( j < 0 \) or \( j > k \) we define \( \gamma_{k,j}(n) = 0 \). From the recurrence (15), we obtain the following recurrence for the \( \gamma_{k,j}(n) \):

\[
\gamma_{k+2,j}(n) = 8n^2 \gamma_{k+1,j}(n) + 2 \gamma_{k+1,j-1}(n) - 16n^4 \gamma_{k,j}(n) + 8n^2 \gamma_{k,j-1}(n) - \gamma_{k,j-2}(n) + 16n^4 \gamma_{k,j}(n-1) - 16n^2 \gamma_{k,j-1}(n-1).
\]

(16)

The \( \gamma_{k,j}(n) \) can be computed from (16), using the initial values

\[
\begin{align*}
\gamma_{0,0}(n) &= n/2, \\
\gamma_{1,0}(n) &= 2n^3/(2n - 1), \\
\gamma_{1,1}(n) &= n(2n - 5)/(4n - 2).
\end{align*}
\]

(17)

Using the definition of \( \gamma_{k,j}(n) \) and (11)–(13), we obtain

\[
W_{2k+1}(n) = 4 \left( \frac{2n}{n} \right) \sum_{j=0}^{k} \gamma_{k,j}(n) S_{2k+2j+1}(n).
\]

Since \( S_{2r+1}(n) = P_r(n)(\binom{2n}{n}) \), we obtain the following theorem, which shows that the double sums \( W_{2k+1}(n) \) may be expressed in terms of the same polynomials \( P_m(n) \) that occur in expressions for the single sums of [3, 18].

**Theorem 4.**

\[
W_{2k+1}(n) = 4n \sum_{j=0}^{k} \gamma_{k,j}(n) P_{k+j}(n) \cdot \left( \frac{2n}{n} \right)^2,
\]

(18)

where the polynomials \( P_{k+j}(n) \) are as in (6), and the \( \gamma_{k,j}(n) \) may be computed from the recurrence (16) and the initial values given in (17).
The factor before the binomial coefficient in (18) is a rational function $\omega_k(n)$ with denominator $(2n - 1)(2n - 3) \cdots (2n - 2\lceil k/2 \rceil + 1)$. Thus, we have the following corollary of Theorem 4.

**Corollary 1.** If $k \in \mathbb{N}$ and $W_k(n)$ is defined by (10), then

$$W_{2k+1}(n) = \omega_k(n) \left(\frac{2n}{n}\right)^2,$$

where

$$\omega_k(n) \prod_{j=1}^{\lceil k/2 \rceil} (2n - 2j + 1)$$

is a polynomial of degree $2k + \lceil k/2 \rceil + 2$ over $\mathbb{Z}$. The first four cases are:

- $\omega_0(n) = 2n^2$,
- $\omega_1(n) = \frac{2n^3(8n^2 - 12n + 5)}{2n - 1}$,
- $\omega_2(n) = \frac{2n^3(128n^4 - 512n^3 + 800n^2 - 568n + 153)}{2n - 1}$, and
- $\omega_3(n) = \frac{2n^3 \overline{\omega}_3(n)}{(2n - 1)(2n - 3)}$, where

$$\overline{\omega}_3(n) = 9216n^7 - 86016n^6 + 350464n^5 - 802304n^4 + 1106856n^3 - 914728n^2 + 417358n - 80847.$$

### 3.3 $W_\beta$ for even $\beta$

Now we consider $W_\beta(n)$ for even $\beta$. This case is easier than the case of odd $\beta$ because the absolute value in the definition (10) has no effect when $\beta$ is even. Theorem 5 shows that $W_{2r}(n)$ can be expressed in terms of the single sums $S_0(n), S_2(n), \ldots, S_{4r}(n)$ or, equivalently, in terms of the polynomials $Q_0(n), Q_1(n), \ldots, Q_{2r}(n)$. It follows that $2^{2r-4n}W_{2r}(n)$ is a polynomial over $\mathbb{Z}$ of degree $2r$ in $n$.

**Theorem 5.** For all $n \in \mathbb{N}$,

$$W_{2r}(n) = \sum_k (-1)^k \binom{2r}{k} S_{2k}(n) S_{4r-2k}(n)$$

$$= 2^{4n-2r} \sum_k (-1)^k \binom{2r}{k} Q_k(n) Q_{2r-k}(n).$$
where $Q_r(n)$ and $S_r(n)$ are as (4)–(6) of §2, and $W_\beta(n)$ is defined by (10).

Proof. From the definition of $W_{2r}(n)$ we have

$$W_{2r}(n) = \sum_i \sum_j \binom{2n}{n+i} \binom{2n}{n+j} (i^2 - j^2)^{2r}. $$

Write

$$(i^2 - j^2)^{2r} = \sum_k (-1)^k \binom{2r}{k} i^{4r-2k} j^{2k},$$

change the order of summation in the resulting triple sum, and observe that the inner sums over $i$ and $j$ separate, giving $S_{4r-2k}(n)S_{2k}(n)$. This proves the first part of the theorem. The second part follows from (6).

For example, the first four cases are

$\begin{align*}
W_0(n) &= 2^{4n}, \\
W_2(n) &= 2^{4n-1} n(2n-1), \\
W_4(n) &= 2^{4n-2} n(2n-1)(18n^2 - 33n + 17), \\
W_6(n) &= 2^{4n-3} n(2n-1)(900n^4 - 4500n^3 + 8895n^2 - 8055n + 2764).
\end{align*}$

It follows from Theorem 5 that the coefficients of $2^{2r-4n}W_{2r}(n)$ are in $\mathbb{Z}$, but it is not obvious how to prove the stronger result, suggested by the cases above, that the coefficients of $2^{r-4n}W_{2r}(n)$ are in $\mathbb{Z}$. We leave this as a conjecture.

4 A triple sum

In Theorem 6 we give a triple sum that is analogous to the double sum of Theorem 2. A straightforward but tedious proof is given in [5, Appendix]. The result also follows from the case $d = 3$ of a more general result proved in [4, Proposition 1.1] for the analogous $d$-fold sum, where the weight function is generalized to the absolute value of a Vandermonde $|\Delta(i^2, i_2^2, \ldots, i_d^2)|$.

**Theorem 6.** For all $n \in \mathbb{N}$,

$$\sum_{i,j,k} \left( \binom{2n}{n+i} \binom{2n}{n+j} \binom{2n}{n+k} \right) [(i^2 - j^2)(i^2 - k^2)(j^2 - k^2)]
\begin{align*}
&= 3n^3(n-1) \left( \binom{2n}{n} \right)^2 2^{2n-1}.
\end{align*}$$
5 Further identities

In this section we give various identities that were stated in [5]. Of these, (25), (26), (27), (30) and (32) were conjectural. The conjectures have since been proved by Bostan, Lairez and Salvy [2, §7.3.2].

Centered double sums

Recall that, from the definition (9), we have

\[
S_{\alpha,1}(n) = \sum_{i,j} \left( \frac{2n}{n+i} \right) \left( \frac{2n}{n+j} \right) |i^\alpha - j^\alpha|.
\] (19)

We give closed-form expressions for \( S_{\alpha,1}(n) \), \( 1 \leq \alpha \leq 8 \). Observe that (24) follows from Theorem 1 since \( S_{1,1}(n) = T_1(n, n) \), and (20) is equivalent to Theorem 2. It appears that, for even \( \alpha \), \( S_{\alpha,1}(n) \) is a rational function of \( n \) multiplied by \( (2n)^2 \), but for odd \( \alpha \), it is a rational function of \( n \) multiplied by \( (4n) \). This was conjectured in [5], and has been proved by Krattenthaler and Schneider [14].

\[
S_{2,1}(n) = 2n^2 \left( \frac{2n}{n} \right)^2,
\] (20)

\[
S_{4,1}(n) = \frac{2n^3(4n - 3)}{2n - 1} \left( \frac{2n}{n} \right)^2,
\] (21)

\[
S_{6,1}(n) = \frac{2n^3(11n^2 - 15n + 5)}{2n - 1} \left( \frac{2n}{n} \right)^2,
\] (22)

\[
S_{8,1}(n) = \frac{2n^3(80n^4 - 306n^3 + 428n^2 - 266n + 63)}{(2n - 1)(2n - 3)} \left( \frac{2n}{n} \right)^2,
\] (23)

\[
S_{1,1}(n) = 2n \left( \frac{4n}{2n} \right),
\] (24)

\[
S_{3,1}(n) = \frac{4n^2(5n - 2)}{4n - 1} \left( \frac{4n - 1}{2n - 1} \right),
\] (25)
\begin{align*}
S_{5,1}(n) &= \frac{8n^2(43n^3 - 70n^2 + 36n - 6)}{(4n - 2)(4n - 3)} \binom{4n - 2}{2n - 2}, \\
S_{7,1}(n) &= \frac{16n^2 P_{7,1}(n)}{(4n - 3)(4n - 4)(4n - 5)} \binom{4n - 3}{2n - 3}, \quad n \geq 2, \text{ where} \\
P_{7,1}(n) &= 531n^5 - 1960n^4 + 2800n^3 - 1952n^2 + 668n - 90, \quad (27) \\
(S_{7,1}(1) = 12 \text{ is a special case}).
\end{align*}

Following are some similar identities. We observe that, since \(i^4 - j^4 = (i^2 + j^2)(i^2 - j^2)\), (28) is easily seen to be equivalent to (21). Similarly, since \(i^6 - j^6 = (i^4 + i^2 j^2 + j^4)(i^2 - j^2)\), any two of (22), (29) and (31) imply the third. Higher-dimensional generalizations of (30)–(31) are known \[4\].

\begin{align*}
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^2 j^2(i^2 - j^2)| &= \frac{n^3(4n - 3)}{2n - 1} \binom{2n}{n}^2, \quad (28) \\
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^4 j^2(i^2 - j^2)| &= \frac{n^3(10n^2 - 14n + 5)}{2n - 1} \binom{2n}{n}^2, \quad (29) \\
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |ij(i^2 - j^2)| &= \frac{2n^3(n - 1)}{2n - 1} \binom{2n}{n}^2, \quad (30) \\
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^2 j^2(i^2 - j^2)| &= \frac{2n^4(n - 1)}{2n - 1} \binom{2n}{n}^2, \quad (31) \\
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^3 j^3(i^2 - j^2)| &= \frac{2n^4(n - 1)(3n^2 - 6n + 2)}{(2n - 1)(2n - 3)} \binom{2n}{n}^2. \quad (32)
\end{align*}

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References


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