Computation of Maximal Determinants of Binary Circulant Matrices

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Abstract

We describe algorithms for computing maximal determinants of binary circulant matrices of small orders. Here "binary matrix" means a matrix whose elements are drawn from $\{0, 1\}$ or $\{-1, 1\}$. We describe efficient parallel algorithms for the search, using Duval's algorithm for generation of Lyndon words and the well-known representation of the determinant of a circulant in terms of roots of unity. Our computations extend earlier results and disprove two plausible conjectures.

1 Introduction

A circulant matrix $A = (a_{j,k})$ of order n is an $n \times n$ matrix whose elements $a_{j,k}$ depend only on $(k - j) \mod n$. Thus, an $n \times n$ circulant is a matrix of the form $A = (a_{(k-j) \mod n})_{0 \le j,k < n}$. Circulants arise in various applications in signal processing and combinatorics, and have a close connection with Fourier transforms. The set of all circulants of order n (with elements in some fixed ring R) form a commutative algebra, since the sum and product of two circulants is a circulant, and it is easy to see that multiplication of circulants is commutative.

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We write $\operatorname{circ}(a_0, a_1, \ldots, a_{n-1})$ for the circulant $(a_{(k-j) \mod n})_{0 \le j,k < n}$ whose first row is $(a_0, a_1, \ldots, a_{n-1})$.

By a *binary* matrix we mean a matrix whose elements are in one of the sets $S_{01} := \{0, 1\}$ or $S_{\pm 1} := \{-1, 1\}$. It will be clear from the context which of these two cases is being considered. A *binary circulant* is a circulant matrix whose elements are in S_{01} or $S_{\pm 1}$.

There is a natural one-to-one correspondence between the integers $\{0, 1, \ldots 2^n - 1\}$ and the binary circulant matrices of order n. More precisely, if $N \in \{0, 1, \ldots, 2^n - 1\}$ has the representation

$$N = \sum_{j=0}^{n-1} 2^{n-1-j} b_j,$$

so may be written in binary as $b_0 \dots b_{n-1}$, we associate N with $\operatorname{circ}(a_0, \dots, a_{n-1})$, where $a_j = b_j$ in the case of S_{01} , and $a_j = 2b_j - 1$ in the case of $S_{\pm 1}$.

The maximal determinant problem is concerned with the maximal value of $|\det(A)|$ for an $n \times n$ binary matrix A. The Hadamard bound [14] states that, in the case of binary matrices A over $\{\pm 1\}$, we have

$$|\det(A)| \le n^{n/2}.\tag{1}$$

Moreover, Hadamard's inequality is sharp for infinitely many n, for example powers of two (Sylvester [26]) or n of the form q+1 where q is a prime power and $q \equiv 3 \mod 4$ (Paley [22]).

There is a well-known connection between the determinants of $\{0, 1\}$ matrices of order n and $\{\pm 1\}$ -matrices of order n + 1. This implies that an $(n+1) \times (n+1) \{\pm 1\}$ -matrix always has determinant divisible by 2^n . See [18] or [21, Lemma 3.1] for details. We give an example with n = 3, starting with an $n \times n$ binary matrix B and ending with an $(n+1) \times (n+1) \{\pm 1\}$ -matrix A, with det $(A) = 2^n \det(B)$.

The doubling step is the only step where the determinant changes, and there it is multiplied by 2^n .

Thus, Hadamard's bound (1) gives the bound

$$|\det(B)| = |\det(A)|/2^n \le (n+1)^{(n+1)/2}/2^n,$$
(2)

which applies for all $\{0, 1\}$ -matrices *B* of order *n*. We shall refer to both (1) and (2) as *Hadamard's inequality*, since it will be clear from the context which inequality is intended.¹

The mapping from $\{0, 1\}$ -matrices to $\{\pm 1\}$ -matrices is reversible if we are allowed to normalise the first row and column of the $\{\pm 1\}$ -matrix by changing the signs of rows/columns as necessary.

The transformation illustrated above (or its reverse) does *not* preserve any circulant structure.

Hadamard matrices are square matrices with entries in $S_{\pm 1}$ and mutually orthogonal rows. The order of a Hadamard matrix is 1, 2, or a multiple of 4. It is not known whether a Hadamard matrix of order 4k exists for every positive integer k (this is the Hadamard conjecture).

Various constructions for Hadamard matrices use circulant matrices. For example, the first Paley construction [22] uses a circulant matrix of order p, where p is a prime, $p \equiv 3 \mod 4$, to construct a Hadamard matrix of order p+1. (The Paley construction also works for prime powers, e.g. $27 = 3^3$, but does not involve circulants in such cases.) Fletcher, Gysin and Seberry [12] use two circulants and a border of width two to construct Hadamard matrices. The Williamson construction [27] requires four matrices A, \ldots, D which satisfy certain conditions, and for computational reasons these matrices are usually taken to be circulants.

It is well-known that the (unnormalised) eigenvectors of $\operatorname{circ}(a_0, \ldots, a_{n-1})$ are given by $v_j = (1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j})^T$, $0 \leq j < n$, where ω is a primitive *n*-th root of unity. For example, in \mathbb{C} we can take $\omega := \exp(2\pi i/n)$. It follows that the eigenvalues are

$$\lambda_j = a_0 + a_1 \omega^j + \dots + a_{n-1} \omega^{(n-1)j}, \ 0 \le j < n,$$
(3)

and the determinant is

$$\prod_{j=0}^{n-1} \lambda_j = \prod_{j=0}^{n-1} f(\omega^j),$$
(4)

¹In fact, Hadamard in [14] proved a more general inequality than (1), and as far as we are aware he never stated (2) explicitly. A simple proof of (1) is given by Cameron [7].

where

$$f(z) := \sum_{k=0}^{n-1} a_k z^k.$$

The polynomial f(z) is called the *associated polynomial* of the circulant.

Using (4) to compute det(A) for a circulant matrix A takes $O(n^2)$ arithmetic operations, whereas Gaussian elimination does not take advantage of the circulant structure and takes of order n^3 operations. If we are considering binary matrices, whose determinants are integers, it is necessary to perform the operations in \mathbb{C} to sufficient precision to obtain a result with absolute error less than 1/2, so that the correct result can be found by rounding to the nearest integer. From the Hadamard bounds (1)–(2), this means we have to work with of order $n \log n$ bits of precision.

To avoid the problem of rounding errors altogether, we can work over a finite field. If p is a prime such that $p \equiv 1 \mod n$, and ρ is a primitive root (mod p), then²

$$\omega = \rho^{(p-1)/n} \bmod p$$

is a primitive *n*-th root of unity in the finite field F_p , and we can use (4) to compute det(A) mod p. If U is an upper bound on $|\det(A)|$, and $p \ge 2U+1$, then the result mod p is sufficient to determine det(A). Thus, if we use a Hadamard bound for U, the prime p should have of order $n \log n$ bits. Alternatively, we could use several smaller primes with a sufficiently large product, and reconstruct the result using the Chinese Remainder Theorem.³

2 Lyndon words

The usual definition of a Lyndon word is a nonempty string that is strictly smaller in lexicographic order than all of its proper rotations. Thus, the first six Lyndon words over S_{01} are 0, 1, 11, 101, 111, and 1111. Lyndon

²It is not necessary to know a primitive root (mod p). We can choose a random a, compute $\omega = a^{(p-1)/n}$, and check if $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are distinct (mod p). If not, reject ω and repeat with another random a. In this way we work in a (small) group of order n, instead of a (large) group of order p-1, and there is no need to factor p-1. The expected number of iterations is $n/\phi(n) = O(\log \log n)$.

³Tests indicate that for $n \leq 50$ it is faster to use a single prime. One reason for this is that the value det(A) needs to be reconstructed for each circulant A, so the cost of the reconstruction steps is not negligible.

words were introduced by Shirshov [24] (who called them "regular words") and Lyndon [17] (who called the "standard lexicographic sequences").

Since we consider words of a fixed length n, it is convenient to use a slightly different definition. We say that $w = w_0 \dots w_{n-1}$ is a Lyndon word of length n if w is not larger (in the lexicographic order) than any of its rotations. This corresponds to Duval's "representative of a class of words of length n" [9, (3) on pg. 258], where two words are said to be in the same class if one is a rotation of the other.

For example, according to our definition, the Lyndon words of length 4 over S_{01} are 0000, 0001, 0011, 0101, 0111, and 1111. It can be seen that the two definitions only differ in whether leading zeros are excluded or not.

The number L(n) of Lyndon words of length n over a binary alphabet is

$$L(n) = \frac{1}{n} \sum_{d|n} 2^{n/d} \phi(d) \sim 2^n/n,$$
(5)

where ϕ is Euler's phi function. L(n) is tabulated in OEIS A000031 [25].

If A is a circulant, then $|\det(A)|$ is invariant under rotations of the first row (a_0, \ldots, a_{n-1}) . Thus, when searching for circulants of order n with maximal determinants, it is sufficient to consider circulants whose first row is a Lyndon word of length n. From (5), this saves a factor of approximately n.

In our computations we use two nontrivial algorithms related to Lyndon words. One is the algorithm of Booth [5], which determines in linear time if a word $w = w_0 \dots w_{n-1}$ is in fact a Lyndon word.⁴ Booth's algorithm is closely related to the initial phase of the Knuth, Morris and Pratt fast pattern-matching algorithm [15].

The other algorithm that we use is Duval's algorithm [9] which, given a Lyndon word of length n, returns the next Lyndon word (of length n) in lexicographic order, in amortised (i.e. average) constant time, see [4]. Using Duval's algorithm we can cycle through all Lyndon words of length n in time $O(2^n/n)$.

Other algorithms could be used. For example, Shiloach [23] gives an algorithm that reduces the number of comparisons used by Booth's algorithm. We chose Booth's algorithm because it was sufficient for our purposes, and simpler to implement than Shiloach's algorithm. The overall complexity of

⁴We use a simplified version of Booth's algorithm since we do not need to know the rotation that would convert w into a Lyndon word.

our algorithms is dominated by the time required to evaluate determinants using (4), not by the time required to check or enumerate Lyndon words.

3 Fast evaluation of circulant determinants

Standard algorithms of linear algebra, such as Gaussian elimination, require of order n^3 operations to evaluate the determinant of an $n \times n$ matrix A. Using formula (4), this can be reduced to order n^2 if A is a circulant. In fact, using the fast Fourier transform (FFT), $O(n \log n)$ operations suffice.

However, in our application we can do even better. Because Duval's algorithm takes constant time (on average), the number of symbols that are changed as we go from one Lyndon word to the next is O(1) on average.⁵ Thus, each λ_j value given by (3) can be updated in O(1) operations (on average), and the determinant, given by (4), can be updated with O(n) operations (on average). Since there are $\tilde{2}^n/n$ Lyndon words of length n, the computation of all the relevant determinants can be done with $O(2^n)$ operations. The cost of precomputing a table of powers ω^{jk} ($0 \leq j, k < n$), is negligible.

Note that we used the term "operations" rather than "time", because the arithmetic operations need to be performed using of order $n \log n$ bits of precision, as noted above. Thus, the overall complexity is $O(2^n M(n \log n))$, where M(N) is the time required to multiply N-bit numbers.

In theory, a slightly better complexity can be attained by using several small primes and reconstructing the result via the Chinese Remainder Theorem. However, the cost of $O(2^n/n)$ reconstructions must be taken into account. In practice, n can not be very large, because of the exponentially growing factor 2^n in the complexity, so the difference between the two approaches is essentially an implementation-dependent constant factor.

4 Parallel algorithms

Suppose we wish to use $P \ge 1$ processors in parallel. If the $K \sim 2^n/n$ Lyndon words of length n are $W_0 = 0 \dots 0, W_1, W_2, \dots, W_{K-1} = 1 \dots 1$, we

⁵We find experimentally that the mean number of symbols changed is $2 + O(n/2^n)$ as $n \to \infty$. The limiting value 2 is the same as the mean number of bits changed when counting up in binary.

would like to ask processor q $(0 \le q < P)$ to compute the determinants corresponding to Lyndon words $W_{\lfloor qK/P \rfloor}, \ldots, W_{\lfloor (q+1)K/P \rfloor - 1}$. The problem is how to determine the starting point $W_{\lfloor qK/P \rfloor}$ for processor q, without enumerating $W_1, W_2, \ldots, W_{\lfloor qK/P \rfloor}$. A polynomial-time algorithm for this problem is claimed in [16], but it is very complicated. We preferred to adopt a simpler approach which is much easier to implement and sufficient in practice.

The idea is to take a random sample of (say) $T := 4000P^2$ Lyndon words (each of length *n*). Sort the sample, and then divide it into *P* equal-sized segments. Modify the initial segment to start with $W_0 = 0...0$ and the final segment to end with $W_{K-1} = 1...1$. Thus, each processor has the same number $\lfloor K/P \rfloor$ words to process, apart from a small sampling error which is negligible in practice. Also, we know the Lyndon word starting each segment, so we can use Duval's algorithm to enumerate all Lyndon words in a segment.

We describe how to randomly sample the set of all Lyndon words of length n in such a manner that each Lyndon word occurs in the sample with equal probability. Generate a random binary string of length n, and test (using Booth's algorithm) if it corresponds to a Lyndon word. If so, the string is accepted. Otherwise, the string is rejected and we try again. The process is repeated until we have the desired number T of Lyndon words (not necessarily distinct). Clearly each Lyndon word is equally likely to appear in the final list of Lyndon words. Since the probability that a random binary string is a Lyndon word is close to 1/n, the number of random binary strings that are needed is of order nT. What we have described is, in fact, a simple example of Von Neumann's *rejection method*, first described by Forsythe in [19]. Other examples may be found in Devroye's book [8].

5 Computational results

In Tables 1–2 we give computational results for the maximal determinants $D_{01}(n)$ of $\{0, 1\}$ -circulants of order $n \leq 47$. The third column of each table gives the ratio $D_{01}(n)/U_{01}(n)$, where $D_{01}(n)$ is the maximum of $|\det(B)|$ for $\{0, 1\}$ -circulants B of order n, and $U_{01}(n)$ is an upper bound on $D_{01}(n)$.

Similarly, in Tables 3–4 we give computational results for the maximal determinants $D_{\pm 1}(n)$ of $\{\pm 1\}$ -circulants of order $n \leq 47$. Here the third column is the ratio $D_{\pm 1}(n)/U_{\pm 1}(n)$, where $U_{\pm 1}(n)$ is an upper bound on $D_{\pm 1}(n)$. In Tables 3–4 we scale the determinants of $\{\pm 1\}$ -circulants by dividing by

the known factor 2^{n-1} . In the last column of Table 3, "-" and "+" are used as abbreviations for -1 and +1 respectively.

The bounds $U_{01}(n)$ and $U_{\pm 1}(n)$ are defined as follows. Let

$$HBE(n) := \begin{cases} n^{n/2} \text{ if } n \equiv 0 \mod 4, \\ 2(n-1) (n-2)^{(n-2)/2} \text{ if } n \equiv 2 \mod 4, \\ (2n-1)^{1/2} (n-1)^{(n-1)/2} \text{ otherwise.} \end{cases}$$
(6)

Then HBE(n) is an upper bound on $|\det(A)|$ for $\{\pm 1\}$ -matrices A of order n. The case $n \equiv 0 \mod 4$ is due to Hadamard [14]; the case $n \equiv 2 \mod 4$ is due to Ehlich [10] and Wojtas [28]; and the remaining case (n odd) is due to Barba [3], Ehlich [10], and Wojtas [28]. We do not use Ehlich's slightly sharper, but more complicated, bound that applies when $n \equiv 3 \mod 4$. For this bound, see Ehlich [11] or Orrick [20].

In view of the discussion in $\S1$, we take

$$U_{\pm 1}(n) := 2^{n-1} \lfloor HBE(n)/2^{n-1} \rfloor$$

and

$$U_{01}(n) := \lfloor HBE(n+1)/2^n \rfloor.$$

It is an open question whether $D_{\pm 1}(n)$ attains the bound $U_{\pm 1}(n)$ for any n > 13. (If we restrict attention to the cases $n \equiv 0 \mod 4$, this is the *circulant Hadamard* problem.) On the other hand, $D_{01}(p) = U_{01}(p)$ for all primes $p \equiv 3 \mod 4$. This follows from the first *Paley construction* [22], which constructs a Hadamard matrix of order p+1 with a circulant submatrix of order p. Inspection of Tables 1–2 reveals that $D_{01}(n) = U_{01}(n)$ in some other cases, specifically $n \in \{1, 2, 4, 15, 35\}$.

Table 2 extends the list of $D_{01}(n)$ values given for $n \leq 37$ in OEIS A086432 and the associated b-file [1]. Table 4 extends the list of $D_{\pm 1}(n)/2^{n-1}$ values given for $n \leq 28$ in OEIS A215897 [2]. This implies a corresponding extension for OEIS A215723, which lists the unscaled values $D_{\pm 1}(n)$.

As an indication of the time required to compute the tables, we note that the computation of $D_{01}(46)$ using our parallel program (implemented in C using GMP [13]) took 1394 processor-hours (87.1 hours × 16 processors) using a 2.6 GHz Intel Xeon E5-2697A. The computation times for other orders n may be estimated as they are roughly proportional to 2^n . For verification, all the values given in the tables for orders $n \leq 44$ were computed at least twice, using different programs and/or different prime moduli p.

6 Some conjectures

In this section we discuss, and disprove, some plausible conjectures.

Conjecture A

From the third column of Table 1, the determinant of a $\{0, 1\}$ -circulant can attain the upper bound $U_{01}(n)$ in the cases $n \in \{1, 2, 3, 4, 7, 11, 15, 19\}$. The Paley construction explains this for n = 3, 7, 11, 19, and larger cases where n is a prime and $n \equiv 3 \mod 4$. However, it does not explain the case n = $15 = 3 \times 5$. Also, the upper bound is not attained for $n = 27 = 3^3$. Thus, a plausible conjecture is that the upper bound can be attained whenever $n \equiv 3 \mod 4$ is square-free. A weaker conjecture would replace "square-free" by "product of at most two distinct primes". Some support is provided by the computation for $n = 35 = 5 \times 7$, where we find that $D_{01}(35) = U_{01}(35)$.

Our computation for n = 39 disproves these conjectures, since $39 = 3 \times 13$ is a product of two distinct primes, but $D_{01}(39) < U_{01}(39)$.

Conjecture B, case [0,1]

When considering maximal determinants of matrices with real elements in the interval [0, 1], we can see that the maximum occurs at extreme points of the polytope.⁶ To prove this, we need only note that the determinant det(A) of a square matrix $A = (a_{j,k})$ is a linear function of each variable $a_{j,k}$ considered separately. Thus, if a local maximum of det(A) occurs for some $a_{j,k} \in (0, 1)$, we can replace $a_{j,k}$ by (at least one of) 0 or 1 without decreasing det(A).

This argument does not apply if A is restricted to be a circulant of order n > 1, because then the free parameters are just the elements a_0, \ldots, a_{n-1} of the first row of A, and det(A) is not a linear function of each a_j considered separately. For example, if n = 2 we have det $(A) = a_0^2 - a_1^2$. Nevertheless, inspection of small cases suggests the conjecture that the maximum of $|\det(A)|$ occurs at extreme points of the *n*-dimensional polytope.

We were unable to prove the conjecture, so wrote a program to check it numerically, and found that, in general, the conjecture is false.

Our program works as follows. We consider all possible circulants A of order n with entries in $\{0, 1\}$. (The first row of A is a rotation of a Lyndon

⁶This is already implicit in Hadamard [14].

word of length n.) If det $(A) = \pm D_{01}(n)$, we check if a small perturbation of a_0 towards the interior of the polytope would increase $|\det(A)|$. Although such behaviour is rare, it does occur.

The smallest examples occur for n = 9. Consider $A = \operatorname{circ}(a_0, \ldots, a_8)$ with $(a_0, \ldots, a_8) = (0, 0, 0, 1, 1, 1, 1, 0, 1)$. We have $\det(A) = 95 = D_{01}(9)$, but $\partial \det(A)/\partial a_0 = 9$. If $a_0 = \varepsilon$ for some small ε , then $|\det(A(\varepsilon))| =$ $95 + 9\varepsilon + O(\varepsilon^2)$, so $|\det(A(\varepsilon))| > 95$ for sufficiently small $\varepsilon > 0$. In fact, $|\det(A(0.241))| > 96.757$.

For n = 10, an example is A = circ(0, 0, 1, 0, 0, 1, 1, 1, 1, 0), $\det(A) = 275$. Replacing a_0 by $\varepsilon = 0.112$, we obtain $\det(A(\varepsilon)) > 279.4$.

We found examples of such behaviour for n = 9, 10 and no other n < 48. However, our search for interior extrema was not exhaustive, so there may be other n < 48 for which the maximum determinant does not occur at an extreme point of $[0, 1]^n$.

Conjecture B, case [-1,1]

Replacing [0,1] by [-1,1], we find similar behaviour for n = 2, 9, 10, 11, 18, 22 and no other n < 48. The case n = 2 is trivial because, for circulants of order 2 over $S_{\pm 1}$, we necessarily have $\det(A) = 0$ at the extreme points $(a_0, a_1) = (\pm 1, \pm 1)$.

The other cases are non-trivial. For example, if n = 9, consider

$$A(\varepsilon) := \operatorname{circ}(1 - \varepsilon, 1, -1, 1, -1, -1, 1, 1, 1).$$

We find that

 $\det(A(\varepsilon)) = 6912 + 4608\varepsilon + O(\varepsilon^2),$

so sufficiently small $\varepsilon > 0$ gives $\det(A(\varepsilon)) > 6912 = U_{\pm 1}(9)$. Indeed, we can take $\varepsilon = 1$, as $\det(A(1)) = 8582 > 6912$.

If n = 10, we find that

$$\det(\operatorname{circ}(1-\varepsilon,-1,1,1,-1,-1,-1,-1,-1,-1)) = -(22528+2560\varepsilon+O(\varepsilon^2)),$$

and

$$\det(\operatorname{circ}(-1+\varepsilon,-1,-1,1,-1,1,-1,-1,-1)) = 22528 + 7680\varepsilon + O(\varepsilon^2),$$

so in both cases a sufficiently small $\varepsilon > 0$ disproves the conjecture. A different type of exceptional case is illustrated by

$$A(x) := \operatorname{circ}(x, -1, 1, -1, 1, 1, -1, -1, -1, -1),$$

where we find that det(A(x)) is an even polynomial in x, and

 $-\det(A(0)) = 33489 > -\det(A(\pm 1)) = 22528 = U_{\pm 1}(10).$

Similarly, for order 22, consider

Then

 $-\det(A(0)) = 216409254831025 > -\det(A(\pm 1)) = 215055782117376.$

Since $215055782117376 = U_{\pm 1}(22) = 2^{21} \times 102546588$ (see Table 4), we have $|\det(A(0))| > U_{\pm 1}(22)$.

As before, our search was not exhaustive, so there may be other n < 48 for which the maximum determinant does not occur at an extreme point of $[-1,1]^n$.

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References

- J. Arndt, Y. Dekel, H. Havermann, V. Jovivic and H. Yamanouchi, The On-Line Encyclopedia of Integer Sequences, A086432: Maximum of |det(A)| where A is an n×n circulant (0,1) matrix over the integers, https://oeis.org/A086432/, Dec. 16, 2016.
- [2] J. Arndt, W. Smith *et al*, The On-Line Encyclopedia of Integer Sequences, A215897: $a(n) = A215723(n)/2^{(n-1)}$, https://oeis.org/A215897, Aug. 26, 2012.
- [3] G. Barba, Intorno al teorema di Hadamard sui determinanti a valore massimo, *Giorn. Mat. Battaglini* 71 (1933), 70–86.

- [4] J. Berstel and M. Pocchiola, Average cost of Duval's algorithm for generating Lyndon words, *Theoretical Computer Science* 132 (1994), 415– 425.
- [5] K. S. Booth, Lexicographically least circular substrings, Information Processing Letters 10 (1980), 240–242.
- [6] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235–265.
- P. J. Cameron, Hadamard Matrices, chapter in *Encyclopedia of Design Theory*, http://www.maths.qmul.ac.uk/~lsoicher/designtheory. org/\library/encyc/topics/.
- [8] L. Devroye, Non-Uniform Random Variate Generation, Springer-Verlag, New York, 1986, §II.3 Available from http://luc.devroye.org/ rnbookindex.html.
- [9] J-P. Duval, Génération d'une section des classes de conjugaison et arbre des mots de Lyndon de longueur bornée, *Theoretical Computer Science* 60 (1988), 255–383.
- [10] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, Math. Z.
 83 (1964), 123–132.
- [11] H. Ehlich, Determinantenabschätzungen für binäre Matrizen mit $n \equiv 3 \mod 4$, Math. Z. 84 (1964), 438–447.
- [12] R.J. Fletcher, M. Gysin, and J. Seberry, Application of the discrete Fourier transform to the search for generalised Legendre pairs and Hadamard matrices, *Australasian J. Combinatorics* 23 (2001), 75–86.
- [13] T. Granlund *et al*, *The GNU MP Bignum Library*, https://gmbplib.org/.
- [14] J. Hadamard, Résolution d'une question relative aux déterminants, Bull. des Sci. Math. 17 (1893), 240–246. Reprinted in Oeuvres de Jacques Hadamard, Tome 1, CNRS, Paris, 1968, 239–245.
- [15] D. E. Knuth, J. H. Morris and V. Pratt, Fast pattern matching in strings, SIAM J. on Computing 6 (1977), 323–350.

- [16] T. Kociumaka, J. Radoszewski and W. Rytter, Computing k-th Lyndon word and decoding lexicographically minimal de Bruijn sequence, CPM 2014, LNCS 8486 (2014), 202–211.
- [17] R. C. Lyndon, On Burnside's problem, Trans. Amer. Math. Soc. 77 (1954), 202–215.
- [18] M. G. Neubauer and A. J. Radcliffe, The maximum determinant of {±1}-matrices, *Linear Algebra Appl.* 257 (1997), 289-306. Also http: //www.math.unl.edu/%7Earadcliffe1/Papers/maxdet.pdf
- [19] J. von Neumann, Various techniques used in connection with random digits, in *Monte Carlo Method*, Appl. Math. Series **12**, US Nat. Bureau of Standards, 1951, 36–38 (summary written by G. E. Forsythe); reprinted in *John von Neumann Collected Works* (ed. A. H. Taub), **5**, Pergamon Press, New York, 1963, 768–770.
- [20] W. Orrick, The Hadamard maximal determinant problem, http://www. indiana.edu/~maxdet/.
- [21] J. H. Osborn, The Hadamard Maximal Determinant Problem, thesis, Univ. of Melbourne, 2003.
- [22] R. E. A. C. Paley, On orthogonal matrices, J. Mathematics and Physics 12 (1933), 311–320.
- [23] Y. Shiloach, Fast canonization of circular strings, Journal of Algorithms 2 (1981), 107–121.
- [24] A. I. Shirshov, Subalgebras of free Lie algebras, Mat. Sbornik N.S. 33 (75), 441–452.
- [25] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, A000031: Number of n-bead necklaces with 2 colors when turning over is not allowed; also number of output sequences from a simple n-stage cycling shift register; also number of binary irreducible polynomials whose degree divides n, https://oeis.org/A000031/, Dec. 27, 2017.
- [26] J. J. Sylvester, Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tesselated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of

numbers, London Edinburgh and Dublin Philos. Mag. and J. Sci. **34** (1867), 461–475.

- [27] J. Williamson, Hadamard's determinant theorem and the sum of four squares, *Duke Math. J.* 11 (1944), 65–81.
- [28] M. Wojtas, On Hadamard's inequality for the determinants of order non-divisible by 4, Collog. Math. 12 (1964), 73–83.

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| order | maximal | ratio to | lex-least word | lex-least word |
|-------|-------------|-------------|----------------|--------------------------|
| n | determinant | upper bound | (decimal) | $(\text{over } \{0,1\})$ |
| 1 | 1 | 1.0000 | 1 | 1 |
| 2 | 1 | 1.0000 | 1 | 01 |
| 3 | 2 | 1.0000 | 3 | 011 |
| 4 | 3 | 1.0000 | 7 | 0111 |
| 5 | 4 | 0.8000 | 15 | 01111 |
| 6 | 9 | 0.7500 | 11 | 001011 |
| 7 | 32 | 1.0000 | 23 | 0010111 |
| 8 | 45 | 0.6923 | 47 | 00101111 |
| 9 | 95 | 0.6597 | 47 | 000101111 |
| 10 | 275 | 0.6152 | 55 | 0000110111 |
| 11 | 1458 | 1.0000 | 183 | 00010110111 |
| 12 | 2240 | 0.6145 | 439 | 000110110111 |
| 13 | 6561 | 0.6923 | 1527 | 0010111110111 |
| 14 | 19952 | 0.5759 | 751 | 00001011101111 |
| 15 | 131072 | 1.0000 | 2479 | 000100110101111 |
| 16 | 214245 | 0.5691 | 2935 | 0000101101110111 |
| 17 | 755829 | 0.6784 | 2935 | 00000101101110111 |
| 18 | 2994003 | 0.6505 | 9903 | 000010011010101111 |
| 19 | 19531250 | 1.0000 | 22427 | 0000101011110011011 |
| 20 | 37579575 | 0.6010 | 28023 | 00000110110101110111 |

Appendix – Tables of Maximal Determinants

Table 1: Maximal determinants of $\{0,1\}$ -circulants of order $n \leq 20$.

| order | maximal | ratio to | lex-least word |
|-------|-----------------------------|-------------|----------------|
| n | determinant | upper bound | (decimal) |
| 21 | 134534444 | 0.6560 | 45999 |
| 22 | 577397064 | 0.6178 | 117623 |
| 23 | 4353564672 | 1.0000 | 340831 |
| 24 | 10757577600 | 0.7060 | 843119 |
| 25 | 31495183733 | 0.5787 | 638287 |
| 26 | 154611524732 | 0.5744 | 957175 |
| 27 | 738139162166 | 0.5442 | 1796839 |
| 28 | 3124126889325 | 0.6101 | 5469423 |
| 29 | 11937232425585 | 0.6069 | 6774063 |
| 30 | 65455857159975 | 0.6271 | 37463883 |
| 31 | 562949953421312 | 1.0000 | 77446231 |
| 32 | 1395230053365015 | 0.6148 | 47828907 |
| 33 | 5687258414265018 | 0.6123 | 196303815 |
| 34 | 30551195956571643 | 0.5827 | 95151003 |
| 35 | 300189270593998242 | 1.0000 | 1324935477 |
| 36 | 809028975189744400 | 0.6309 | 1822895095 |
| 37 | 3198686446402685263 | 0.5760 | 430812063 |
| 38 | 19288701806345611347 | 0.5825 | 2846677239 |
| 39 | 103227456252120723684 | 0.5161 | 10313700815 |
| 40 | 529663503370085366373 | 0.5885 | 6269629671 |
| 41 | 2311393009109010944326 | 0.5638 | 26764629467 |
| 42 | 15469925980869995489631 | 0.6023 | 22992859983 |
| 43 | 162805498773679522226642 | 1.0000 | 92035379515 |
| 44 | 402826140168935435652453 | 0.5245 | 162368181483 |
| 45 | 2268175963362305735661143 | 0.6192 | 226394696439 |
| 46 | 12738408112895861486972391 | 0.5307 | 631304341299 |
| 47 | 158993694406781688266883072 | 1.0000 | 4626135339999 |

Table 2: Maximal determinants of $\{0, 1\}$ -circulants, $20 < n \le 47$.

| order | maximal | ratio to | lex-least word | lex-least word |
|-------|------------------|-------------|----------------|-----------------------------------|
| n | $ \det /2^{n-1}$ | upper bound | (decimal) | $(over \{-,+\})$ |
| 1 | 1 | 1.0000 | 0 | _ |
| 2 | 0 | 0.0000 | 0 | |
| 3 | 1 | 1.0000 | 1 | + |
| 4 | 2 | 1.0000 | 1 | + |
| 5 | 3 | 1.0000 | 1 | + |
| 6 | 4 | 0.8000 | 1 | + |
| 7 | 8 | 0.6667 | 11 | +++ |
| 8 | 18 | 0.5625 | 11 | +++ |
| 9 | 27 | 0.4154 | 11 | ++ |
| 10 | 44 | 0.3056 | 11 | ++ |
| 11 | 267 | 0.5973 | 39 | +++ |
| 12 | 1024 | 0.7023 | 83 | ++ |
| 13 | 3645 | 1.0000 | 83 | +-++ |
| 14 | 6144 | 0.6483 | 83 | +-+++ |
| 15 | 23859 | 0.6886 | 359 | ++++++ |
| 16 | 50176 | 0.3828 | 691 | ++ |
| 17 | 187377 | 0.4977 | 1643 | +++++++++++++++++++++++++++++++++ |
| 18 | 531468 | 0.4770 | 2215 | ++++++ |
| 19 | 3302697 | 0.7176 | 9895 | +-++-+-+++++ |
| 20 | 1061683 | 0.5436 | 6483 | +++-+-+++++++++++++++++++++++++++ |

Table 3: Scaled maximal determinants of $\{\pm 1\}$ -circulants of order $n \leq 20$.

| order | maximal | ratio to | lex-least word |
|-------|--------------------------------|-------------|----------------|
| n | $ \text{determinant} /2^{n-1}$ | upper bound | (decimal) |
| 21 | 39337984 | 0.6291 | 67863 |
| 22 | 102546588 | 0.5000 | 21095 |
| 23 | 568833245 | 0.6087 | 72519 |
| 24 | 3073593600 | 0.7060 | 144791 |
| 25 | 8721488875 | 0.5724 | 108199 |
| 26 | 32998447572 | 0.6064 | 355463 |
| 27 | 164855413835 | 0.6125 | 604381 |
| 28 | 572108938470 | 0.4218 | 1289739 |
| 29 | 2490252810073 | 0.4863 | 1611219 |
| 30 | 10831449635712 | 0.5507 | 1680711 |
| 31 | 68045615234375 | 0.6520 | 6870231 |
| 32 | 282773291271138 | 0.5023 | 12817083 |
| 33 | 1592413932070703 | 0.7017 | 18635419 |
| 34 | 5234078743146888 | 0.5635 | 55100887 |
| 35 | 33374247484277975 | 0.6366 | 149009085 |
| 36 | 198124573871046186 | 0.6600 | 160340631 |
| 37 | 787413957917252603 | 0.6140 | 415804239 |
| 38 | 3195257068570067448 | 0.5754 | 829121815 |
| 39 | 22999238901574021485 | 0.6946 | 4737823097 |
| 40 | 117140061677844350646 | 0.5857 | 1446278811 |
| 41 | 536469708946538168543 | 0.5961 | 3001209959 |
| 42 | 2417648227367853639168 | 0.5897 | 19153917469 |
| 43 | 14611334654738350617599 | 0.5689 | 52222437727 |
| 44 | 65738632907943707712320 | 0.4038 | 20159598251 |
| 45 | 438910341492340511320163 | 0.5715 | 166482220965 |
| 46 | 2010768410464246499566152 | 0.5489 | 90422521191 |
| 47 | 12779930756727248097293989 | 0.5324 | 115099593371 |

Table 4: Scaled maximal determinants of $\{\pm 1\}$ -circulants, $20 < n \leq 47$.