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# *The Probabilistic Method*

Third Edition

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# Preface

The Probabilistic Method is one of the most powerful and widely used tools applied in combinatorics. One of the major reasons for its rapid development is the important role of randomness in theoretical computer science and in statistical physics.

The interplay between discrete mathematics and computer science suggests an algorithmic point of view in the study of the probabilistic method in combinatorics and this is the approach we tried to adopt in this book. The book thus includes a discussion of algorithmic techniques together with a study of the classical method as well as the modern tools applied in it. The first part of the book contains a description of the tools applied in probabilistic arguments, including the basic techniques that use expectation and variance, as well as the more recent applications of martingales and correlation inequalities. The second part includes a study of various topics in which probabilistic techniques have been successful. This part contains chapters on discrepancy and random graphs, as well as on several areas in theoretical computer science: circuit complexity, computational geometry, and derandomization of randomized algorithms. Scattered between the chapters are gems described under the heading *The Probabilistic Lens*. These are elegant proofs that are not necessarily related to the chapters after which they appear and can usually be read separately.

The basic Probabilistic Method can be described as follows: In order to prove the existence of a combinatorial structure with certain properties, we construct an appropriate probability space and show that a randomly chosen element in this space has the desired properties with positive probability. This method was initiated by Paul Erdös, who contributed so much to its development over a fifty year period, that it seems appropriate to call it "The Erdös Method." His contribution can be measured not only by his numerous deep results in the subject, but also by his many intriguing problems and conjectures that stimulated a big portion of the research in the area.

It seems impossible to write an encyclopedic book on the Probabilistic Method; too many recent interesting results apply probabilistic arguments, and we do not even try to mention all of them. Our emphasis is on methodology, and we thus try to describe the ideas, and not always to give the best possible results if these are too technical to allow a clear presentation. Many of the results are asymptotic, and we use the standard asymptotic notation: for two functions $f$ and $g$, we write $f = O(g)$ if $f \leq cg$ for all sufficiently large values of the variables of the two functions, where $c$ is an absolute positive constant. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If the limit of the ratio $f/g$ tends to zero as the variables of the functions tend to infinity we write $f = o(g)$. Finally, $f \sim g$ denotes that $f = (1 + o(1))g$; that is, $f/g$ tends to 1 when the variables tend to infinity. Each chapter ends with a list of exercises. The more difficult ones are marked by (*). The exercises enable readers to check their understanding of the material and also provide the possibility of using the book as a textbook.

This is the third edition of the book; it contains several improved results and covers various additional topics that developed extensively during the last few years. The additions include a modern treatment of the Erdös–Rényi phase transition discussed in Chapter 11, focusing on the behavior of the random graph near the emergence of the giant component and briefly exploring its connection to classical percolation theory. Another addition is Chapter 17, Graph Property Testing—a recent topic that combines combinatorial, probabilistic and algorithmic techniques. This chapter also includes a proof of the Regularity Lemma of Szemerédi (described in a probabilistic language) and a presentation of some of its applications in the area. Further additions are two new *Probabilistic Lenses,* several additional exercises, and a new part in Appendix A focused on lower bounds.

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