The Geometry of Differential Equations
Titles and Abstracts

Spyros Alexakis Integral Kähler invariants and the Tian-Yau-Zelditch expansion

Motivated by the problem of understanding the local structure of the terms in the Tian-Yau-Zelditch expansion, we study the more general question of what are the possible geometric scalars defined over Kähler manifolds whose integrals remain invariant under variations of the underlying metric which preserve the Kähler class. The natural candidates are linear combinations of divergences and of the Chern integrands. An affirmative answer to this question then sheds some light on the algebraic structure of the terms in the Tian-Yau-Zelditch expansion; this can be thought of as a local version of the classical Riemann-Roch expansion which asymptotically describes the dimension of the space of holomorphic sections of an ample line bundle over a complex manifold. Joint work with Kengo Hirachi.

Paul Baird Critical points of semi-conformal mappings

A semi-conformal mapping is a mapping between conformal manifolds with the property that at each point the derivative is either the zero mapping or is surjective and conformal on the compliment of its kernel. In a neighbourhood of a regular point, a semi-conformal mapping determines a conformal foliation. In dimension 3, this latter object, when foliating by curves, is particularly interesting. In the analytic category, a conformal foliation by curves of a domain of 3-dimensional Euclidean space determines either an integrable Hermitian structure or a shear-free ray congruence, depending on whether the ambient 4-dimensional space is Euclidean or Minkowski. But what happens in the neighbourhood of a critical point of a semi-conformal mapping? Does the foliation extend? In dimension 3, if in addition the mapping is harmonic, then the answer is: yes. To show this is quite involved, requiring a Bernstein type theorem for harmonic semi-conformal mappings and a removable singularity result for elliptic PDEs. If we no longer have harmonicity, then the answer is not known. My talk will focus around this problem.

Robert Bryant A visit to the Finsler world (Colloquium)

Many people aren't aware that, when Bernard Riemann revolutionized the study of differential geometry in his famous 1854 lecture "On the hypotheses that lie at the foundations of geometry", he had in mind a wider concept of geometry than what we now call Riemannian geometry. His original vision included a kind of geometry that could be used to study a wide range of problems in the calculus of variations and optimization, but the development of this more general geometry, nowadays called Finsler geometry, had to wait many more years before being explored in depth.

It is now in a vigorous phase of development, and, in this lecture, I want to (re-)introduce Riemann's original idea and sketch some of the new developments. I'll emphasize how the Finsler world compares and contrasts with the more familiar Riemannian world and illustrate some of its applications and surprising connections with other areas of mathematics.

Andreas Čap Normal Weyl structures and solutions of first BGG operators

This talk reports on joint projects with R. Gover (Auckland) and M. Hammerl (Vienna) on the one hand and with K. Melnick (Maryland) on the other hand. The machinery of BGG sequences provides a uniform construction of overdetermined systems of PDEs which are intrinsic to a parabolic geometry. It also gives rise to a subclass of solutions (called "normal solutions" in the literature) of these systems. In my talk, I want to show how tools from the theory of parabolic geometries, in particular normal Weyl structures and the corresponding normal coordinates, can be used to study these special solutions and their zero sets. In the special case of the infinitesimal automorphism equation, one can extend parts of the study to general solutions by employing ideas from dynamics. Finally, there is also a nice extension of these ideas to holonomy reductions of Cartan geometries.

Boris Doubrov Local geometry of integral curves in parabolic geometries

We provide an algorithm for constructing canonical moving frame for arbitrary non-parametrized curves integral to the canonical vector distribution on a parabolic geometry. The algorithm is based on algebraic model for distinguished curves in parabolic geometries and includes the construction of the fundamental set of invariants and the criteria on the existence of the canonical projective parameter. Examples of curves in flag varieties and the generalizations to integral submanifolds of higher dimension are discussed.

Colin Guillarmou The Chern-Simons line bundle on Teichmüller space

We define Chern-Simons invariants on convex co-compact hyperbolic 3-manifolds, and study its structure on the deformation space of such manifolds (which is essentially the Teichmüller space of their conformal boundary). We recover and extend several results known in Teichmüller theory using this Chern-Simons tool.

Matthias Hammerl A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinors

I will discuss a non-normal Fefferman-type construction based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. The construction associates a split signature (n, n)-conformal spin structure to a projective structure of dimension n. For $n \geq 3$ the induced conformal Cartan connection is shown to be normal if and only if it is flat. The main technical work consists in showing that in the non-flat case the normalised conformal Cartan connection still allows a parallel (pure) spin-tractor and thus a corresponding (pure) twistor spinor on the conformal space. The talk is based on joint work with Katja Sagerschnig.

Kengo Hirachi Decomposition of critical GJMS operators on CR manifolds

On conformal manifolds, it is well-known that the critical GJMS operator has no constant term and can be decomposed into exterior derivative, an operator from 1-form to 1-form, and divergence. In this talk I will give an decomposition of (super) critical GJMS operators on CR manifolds by using (higher order) invariant differential operators which arise in BGG complex and discuss about the kernels of these invariant operators.

Alan Huckleberry Random matrix models of disordered bosons

Unlike the fermionic side where the underlying Lie groups of the classical ensembles are compact, the symmetric groups for ensembles of disordered bosons are typically noncompact. In that case the basic random matrix models consist of matrices in the Lie algebra $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{R})$. Assuming dynamical stability, their eigenvalues are required to be purely imaginary. In the lecture we will sketch our recent work with K. Schaffert (J. Phys. A: Math. Theor. **44**(2011) 335207) where a method is proposed for constructing ensembles $(\mathcal{E}, P)$ of G-invariant sets $\mathcal{E}$ of such matrices with probability measures P. These arise as moment map direct images from phase spaces X which play an important role in complex geometry and representation theory. In the toy-model case of $n = 1$, where X is the complex bidisk and P is the direct image of the uniform measure, an explicit description of the spectral measure is given.

Ernie Kalnins Classical superintegrability for the Turbiner, Winternitz and Tremblay system

Recently Turbiner, Winternitz and Tremblay introduced a superintegrable system that was characterised by symmetries that are of order greater than 2 in the canonical momenta. It was conjectured that if a parameter k appearing in the potentials they studied was rational then superintegrability at the classical level would follow. This talk will indicate how this is proven and give some hints as to what follows from this. We also study the classical Poisson algebras that result and also discuss what they might imply.

Jonathan Kress Superintegrability in a non-conformally-flat space

Until recently all known examples of superintegrable Hamiltonians were for systems on constant curvature spaces, or by Stäckel transform, on conformally flat spaces. Methods developed to investigate the superintegrability of the Tremblay-Turbiner-Winternitz system can be extended to a non-conformally flat superintegrable system in 4 dimensions possessing irreducible higher rank Killing tensors. The classical and quantum versions of this system will be discussed.

Wojciech Krynski Geometry of corank 2 distributions

We will present results on classification of distributions of corank 2. The problem of equivalence is solved completely for distributions of odd rank with so-called maximal first Kronecker index. For a given distribution we construct a canonical frame and we describe the most symmetric models. The talk is based on joint work with I. Zelenko and on a joint paper with B. Jakubczyk and F. Pelletier.

Thomas Leistner Conformal structures with explicit ambient metrics

One of the main results in conformal geometry is the ambient metric construction by C. Fefferman and R. Graham, which is fundamental for the construction of invariants for a conformal structure. Its computation is straightforward for conformal classes with an Einstein metric, but more involved for general conformal classes. In the talk, we will present two examples of conformal structures for which we have an explicit expression for the ambient metric. For the first class of examples the conformal structure is defined by a so-called (Lorentzian) pp-wave metrics, which are not necessarily conformally Einstein. We compute explicitly their ambient metric and the obstruction tensor in even dimensions. The other class of examples is given by exceptional conformal metrics in signature (2,3) that were found by P. Nurowski. For an 8-parameter family of these conformal metrics we show that the holonomy of their ambient metric is equal to the non-compact exceptional group G_2. This is joint work with P. Nurowski, University of Warsaw.

Gaven Martin Quasiregular mappings, curvature & dynamics

We survey recent developments in the area of geometric function theory and nonlinear analysis and in particular those that pertain to recent developments linking these areas to dynamics and rigidity theory in dimension $n \geq 3$. A self mapping (endomorphism) of an n-manifold is rational or uniformly quasiregular if it preserves some bounded measurable conformal structure. Because of Rickman's version of Montel's theorem there is a close analogy between the dynamics of rational endomorphisms of closed manifolds and the classical Fatou-Julia theory of iteration of rational mappings of the complex plane. The theory is particularly interesting on the Riemann n-sphere where many classical results find their analogue, some of which we discuss here. We present the most recent results toward a solution of the Lichnerowicz problem of classifying those manifolds admitting rational conformal endomorphisms. As a by product we discover interesting new rigidity theorems for open self maps of closed n-manifolds whose fundamental group is word hyperbolic.

Lionel Mason The Penrose transform and gerbes in 6-dimensions

The Penrose transform establishes a correspondence between solutions to conformally invariant zero rest mass equations in arbitrary dimensions and cohomology classes on twistor spaces. This work is motivated by recent studies of scattering amplitudes in higher dimensions and their relationship to the Penrose transform and we focus on dimension 6. We obtain canonical distributional representatives for momentum eigenstates. These give an easy direct proof of the Penrose transform and an identification of the twistor transform between different twistor representatives for the same fields in terms obstructions to a formal neighbourhood extension problem. We show that the conformally invariant three-point amplitude for the wave equation has a canonical expression on twistor space and give a formulation of this ϕ^3 theory on twistor space. These ideas also apply in modified form in split signature where we use the split signature analogues of the momentum eigenstates to construct a 6-dimensional half Fourier transform between momentum space representations of massless fields and their twistor space representations. We use this to give an explanation and extension of Sparling's 'χ-transform' to a correspondence between the kernels and cokernels of the GJMS operators on real twistor space and solutions to solutions to the ZRM equations.

Vladimir Matveev A linear algebra trick that solved two problems: first problem of Lie and superintegrable systems with one linear and one cubic integral

I show a simple trick from linear algebra that could be, in my opinion, helpful in the geometric theory of PDE. I will demonstrate its usefulness by presenting solutions of two classical problems:

The first problem came from the theory of superintegrable systems, and is to describe all 2D metrics admitting one integral linear in momenta and one integral cubic in momenta. The results of this part of my talk are joint with Shevchishin. I will describe all such metrics locally, in a neighborhood of almost every point, and construct examples on closed surfaces. The same trick can be used in studying other superintegrable systems; I will conclude this part of my talk with a list of problems.

The second problem was explicitly formulated by Sophus Lie 1882, and is to describe all 2D metrics admitting nontrivial projective vector field(s). A part of the result related to the existence of two projective vector fields is joint with R. Bryant and G. Manno. I will give such a description in a neighborhood of almost every point, and, in the case when the metric actually admits two projective vector fields, in a neighborhood of every point.

Tohru Morimoto Differential equations on filtered manifolds - invitation to nilpotent analysis

A *filtered manifold* is a differentiable manifold M equipped with a filtration of the tangent bundle $\{T^pM\}_{p\in Z}$ such that i) T^pM is a subbundle of the tangent bundle TM of M, ii) $T^pM \subset T^qM$ for $p > q$, $T^0M = 0$, and $T^{-\mu}M = TM$ for a positive integer μ, therefore

$$0 = T^0M \subset T^{-1}M \subset \cdots \subset T^{-\mu}M = TM,$$

and iii) $[\underline{T^pM}, \underline{T^qM}] \subset \underline{T^{p+q}M}$ for any $p, q \in Z$, where $\underline{T^pM}$ denotes the sheaf of sections of T^pM. *The symbol algebra of a filtered manifold M at a point $x \in M$* is the nilpotent graded Lie algebra $\mathrm{gr}T_xM = \sum_i \mathrm{gr}_i T_xM$, where $\mathrm{gr}_iT_xM = T^i_xM/T^{i+1}_xM$. Note that the graded vector space $\mathrm{g}T_xM$ is naturally equipped with a structure of a Lie algebra satisfying

$$[\mathrm{gr}_pT_xM, \mathrm{gr}_qT_xM] \subset \mathrm{gr}_{p+q}T_xM.$$

Since the grading is concentrated in negative degree, the Lie algebra $\mathrm{g}T_xM$ is nilpotent. Note that if the filtration is trivial, that is $TM = T^{-1}M$, then the filtered manifold is nothing but a differentiable manifold and the symbol algebra $\mathrm{gr}T_xM = \sum_i \mathrm{gr}_iT_xM$ is just the tangent vector space T_xM, i.e., an abelian Lie algebra. Thus the generalization from the manifolds to the filtered manifolds leads to a generalization from the abelians to the nilpotents. Of fundamental importance is the notion of weighted order for differential operators on a filtered manifold M. We say that the weighted order of X is less than or equal to k ($\text{w-ord}\, X \leq k$) for a vector field X on M if X is a section of $T^{-k}M$, and this definition immediately extends to any differential operator on M. Under the slogan of nilpotent analysis we have been studying differential equations on filtered manifolds by letting these symbol algebras and weighted orders play key rôles. In this talk I will give a short survey on nilpotent analysis, discussing: weighted jet bundles, formal theory of differential equations, finite type and infinite type equations, weightedly involutive equations, generalization of Cartan-Kähler theorem, differential equations associated with a representation of a graded Lie algebra, linear differential equations and extrinsic geometries, their invariants, and related problems.

Willard Miller Recurrence relations and higher order superintegrability in classical and quantum mechanics

We present a method to prove quantum superintegrability of an integrable Schrdinger operator eigenvalue equation, based on recurrence relations obeyed by the eigenfunctions of the system with respect to separable coordinates. We show that the method provides rigorous proofs of superintegrability and explicit constructions of higher order generators for the symmetry algebra. We apply it to several families of systems, each depending on rational parameters, including the Tremblay, Turbiner and Winternitz system and extended Kepler-Coulomb systems. We show that the explicit information supplied by the special function recurrence relations allows us to prove that the symmetry algebra generated by the lowest order generators closes and to determine the associated structure equations. Also we present the analogs of these constructions for the associated classical Hamiltonian systems. (Joint work with E. G. Kalnins and J. M. Kress.)

Jean-Philippe Nicolas A conformal approach to scattering theory

Time dependent scattering theory is a set of techniques aiming to analyze the asymptotic properties of solutions to evolution equations usually of hyperbolic or parabolic type. The typical framework is a fixed Riemannian manifold with asymptotic ends on which fields evolve with a time parameter. A complete scattering theory in its basic form states that in the asymptotic regions the field approaches solutions of simplified equations and moreover, the data for these solutions encode the full information of the evolution of the field. In the framework of general relativity, scattering theory deals with covariant equations, i.e. hyperbolic, and the constructions obtained so far rely heavily on spectral analysis. Although these techniques are powerful, they suffer from a major limitation: they require a time independent background. This limitation is purely attached to the method used and not to the problem studied; the relevant information for scattering is the asymptotics of the metric, not its time dependence or even its local structure. In a work in collaboration with Lionel Mason, we have proposed an alternative approach to scattering in relativity based on conformal compactifications. The idea was originally proposed by Friedlander and pushed further by Baez, Segal and Zhou, but only for static or flat situations. We work on asymptotically simple spacetimes with generic time-dependence. A Penrose compactification allows to re-interpret a complete scattering theory as the well-posedness of a Goursat problem on Scri. Using methods adapted from Hörmander to solve the Goursat problem allows us to recover wave operators with a geometrical choice of comparison dynamics. The talk will present the essential ideas of scattering theory, of the conformal approach, some recent progress and the programme of conformal scattering.

Gerd Schmalz Free CR manifolds

I will show how the main invariants of CR manifolds of CR dimension n and codimension n^2 can be derived using the framework of parabolic geometry. This is joint work with Jan Slovák (Masaryk University Brno, Czech Republic).

Jan Slovák

Fefferman circle bundle on free CR-distributions

Following the lecture by Gerd Schmalz, I shall discuss an analog of Fefferman's circle bundle construction for the free CR distributions of dimension $n > 1$. Indeed, there is a straightforward construction of such a circle bundle whose geometry is defined by identifying the tangent bundle with skew-Hermitian automorphisms of an auxiliary $(n + 1)$–dimensional complex vector bundle. This is a joint work with Gerd Schmalz.

Petr Somberg Universal splitting operators

I will discuss various constructions, properties and a few geometrical applications of universal splitting operators. I will present several explicit examples.

Vladimír Souček Branching laws for Verma modules and applications in parabolic geometries

As is well known in parabolic geometry, duality between the space of infinite jets in origin and the corresponding Verma modules induces a bijective correspondence between homogeneous differential operators between sections of homogeneous vector bundles and homomorphisms of appropriate Verma modules. The same duality can be used also for a study of families of invariant different operators of Juhl type. This leads directly to question of branching laws for generalized Verma modules.

Not too much is known on such branching laws and the branching can often be quite wild. We shall formulate the branching problem in a broader category of modules - the BGG category $\mathcal{O}^{\mathfrak{p}}$, and we shall describe framework leading to discretely decomposable branching laws. A general result will be described in terms of the Grothendieck group of the category $\mathcal{O}^{\mathfrak{p}}$.

In one-graded case, there is an efficient method to describe more precise result on branching laws. There is an explicit method how to describe singular vectors describing the branching using appropriate ODE of second order. It leads immediately to a explicit formulae for singular vectors in terms of suitable orthogonal polynomials (of hypergeometric type). The result will be illustrated in a few special cases in conformal, Grassmannian and other geometries. In some cases, the method leads to a description of bilinear invariant differential operators.

In some cases, the results obtained can be generalized to curved parabolic geometries. We shall show examples of this using the language of semi-holonomic Verma modules.

Arman Taghavi-Chabert A higher-dimensional generalisation of the Goldberg-Sachs theorem

In four dimensions, the Goldberg-Sachs theorem gives necessary and sufficient conditions on the Weyl tensor and Cotton-York tensor for the existence of a locally integrable distribution of complex null 2-planes on a real or complex (pseudo-) Riemannian manifold. We show how the theorem can be generalised to higher dimensions in the holomorphic category, and time permitting, we discuss its real versions.

Paul Tod Three-dimensional Einstein-Weyl geometry

I shall review various features of three-dimensional Einstein-Weyl geometry, including the connection to 3^{rd}-order ODEs, and some of its recent occurrences in some special 4- and 5-dimensional Lorentzian geometries.

Peter Vassiliou Systems of Lie type and the Cauchy problem for Darboux integrable partial differential equations

Systems of *Lie type* are ordinary differential equations (ODE) which possess "superposition" formulas. All linear ODE are of Lie type but there are nonlinear examples beginning with the Riccati equation. A hyperbolic partial differential equation (PDE) is *Darboux integrable* if its characteristic systems satisfy a certain constraint. Well known examples include the Liouville equation and the partial differential equation giving rise to the flat Cartan connection associated with generic rank 2 distributions over 5-manifolds. In this talk I show how to formulate the Cauchy problem for such PDE so that the solution corresponds to the flow of Cauchy data by a system of Lie type.

Keizo Yamaguchi Second reduction theorem in contact geometry of second order

Classical theory for systems of first order partial differential equations for a scalar function can be rephrased as the submanifold theory of contact manifolds (geometric first order jet spaces). In the same spirit, we have developed the geometric theory of systems of partial differential equations of second order for a scalar function as the *Contact Geometry of Second Order*, following E. Cartan. We will formulate the submanifold theory of second order jet spaces as the geometry of PD manifolds $(R; D^1, D^2)$ of second order. Main topic in this talk will be the Second Reduction Theorem for PD manifold of second order. By utilizing Parabolic Geometry, we will give, combined with reduction theorems, several classes of systems of partial differential equations of second order, for which the model equation of each class admits the Lie algebra of infinitesimal contact transformations, which is finite dimensional and simple.