A non-normal Fefferman-type construction

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Plan

The Fefferman-construction

The original Fefferman construction [Fefferman,'76] canonically associated a conformal structure on a circle bundle over a CR-structure. It was shown by Sparling and discussed by [Graham, '87] that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits a light-like conformal Killing field which also satisfies additional (conformally invariant) properties.

The characterising property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [Čap-Gover, '10] that a conformal structure $(M, \mathcal{C})$ is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies $\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

Generalisation: Fefferman-type constructions

A generalisation of the original Fefferman-construction was described in [Čap, '05], and in recent years a number of constructions have been discussed in that framework:

- The original construction was treated by [Čap-Gover, '10]
- A construction of [Biquard, '00] of conformal structures from quaternionic contact structures was treated by [Alt, '10]
- Nurowski's conformal structures that are associated to generic rank 2 distributions on 5-manifolds and Bryant's [Bryant, '06] conformal structures associated to generic rank 3 distributions on 6-manifolds were discussed in [H.-Sagerschnig, '10, '11]

In all cited cases the Fefferman-type construction is *normal*, which immediately implies that the holonomy of the conformal structure reduces and makes it possible to derive a holonomy-based characterisation of the induced structures.

A non-normal construction

Here we discuss a *non-normal Fefferman-type construction*. We associate a split signature (n, n) conformal spin structure to a projective structure of dimension n.

The original motivation for this Fefferman-type construction was work by [Dunajski-Tod, '10]:

Extending a construction due to [Walker, '54], which associates a pseudo-Riemannian split signature (n, n)-metric to an affine torsion-free connection on an n-manifold, they associate a conformal split signature (n, n)-metric to a projective class of torsion-free affine connections on an n-manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. For $n = 2$ this construction was also observed in work by [Nurowski-Sparling, '03]. The precise relation between the cited works and the construction here has been shown recently by [Šilhan-Žádník, '11].

Parabolic geometries

Parabolic geometries are Cartan geometries of type (G, P), with P a parabolic subgroup of a Lie group G: A parabolic geometry of the given type an a manifold M is described by a principal P-bundle $\mathcal{G} \to M$ that is endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{p})$.

Parabolic geometries allow uniform regularity and normality conditions, and if these conditions are satisfied, the parabolic structure is an equivalently description of an underlying geometric structure, like projective, conformal or CR-structures.

The curvature of ω can be regarded as a function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and the normalisation condition employs the Kostant co-differential $\partial^* : \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.: The Cartan connection ω is normal if and only if κ lies in the kernel of ∂^*.

Fefferman-type constructions

A Fefferman-type construction [Čap, '05] is a natural procedure that starts with a parabolic geometry of a type (G, P) on a manifold M and associates a parabolic geometry of another type $(\tilde{G}, \tilde{P})$ on a (possibly larger) manifold $\tilde{M}$.

The algebraic input for this is an inclusion of Lie groups $i : G \hookrightarrow \tilde{G}$ with the property that $Q := G \cap \tilde{P} \subset P$. The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is then possible if G acts locally transitive on the homogeneous space $\tilde{G}/\tilde{P}$.

The first step is to form the correspondence space $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. $\mathcal{G} \to \tilde{M}$ is then a Q-principal bundle endowed with the Cartan connection form ω of type (G, Q).

The second step is to form the extended Cartan bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ and canonically extend ω to a Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. Then $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

Normality and holonomy

The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is called *normal* if normality of ω automatically implies normality of $\tilde{\omega}$. This has immediate strong consequences:

To form the *holonomy* $\mathrm{Hol}(\omega)$ of a parabolic geometry $(\mathcal{G}, \omega)$, one extends $\mathcal{G}$ to a principal G-bundle $\hat{G} = \mathcal{G} \times_P G$ and canonically extends ω to the a principal connection form $\hat{\omega}$ on $\hat{G}$. Then $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.

It immediately follows from the Fefferman-type construction that $\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega)$, and if the construction is normal, $\mathrm{Hol}(\tilde{\omega})$ is the well-defined holonomy of the parabolic geometry on $\tilde{M}$.

In particular, this implies that if ω is non-flat, $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a non-flat parabolic geometry on $\tilde{M}$ with holonomy contained in $G \subset \tilde{G}$.

Induced solutions of BGG-equations

In many cases the inclusion $G \hookrightarrow \tilde{G}$ is realized as the stabilizer of an element in a $\tilde{G}$-representation V. It is well known that the *tractor bundle* $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ carries the *tractor connection* ∇ that is naturally induced from the Cartan connection form $\tilde{\omega}$. Then $\mathrm{Hol}(\tilde{\omega}) \subset G$ is equivalent to the existence of a parallel section $s \in \Gamma(\mathcal{V})$ of a suitable type.

By the general theory of BGG-operators on parabolic geometries as developed by [Čap-Slovak-Souček, '01], such a parallel section s is equivalent to a normal solution of the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ associated to $\mathcal{V}$.

This $1:1$-correspondence is realized by a natural (tensorial) projection $\Pi_0 : \mathcal{V} \to \mathcal{H}_0$ and the *first BGG-splitting operator* $L_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ of $\mathcal{V}$.

Examples of normal Fefferman-type constructions of conformal structures

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$
  signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  $+$ lightlike conformal Killing field (with additional properties)
- $\mathrm{Sp}(n+1, 1) \hookrightarrow \mathrm{SO}(4n+4, 4)$:
  quaternionic contact structure $\rightsquigarrow$
  signature $(4n+3, 3)$ conformal structure
  $+$ 2 orthogonal lightlike conformal Killing fields
- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2, 3)$-conformal spin structure $+$ generic twistor spinor
- $\mathrm{Spin}(3, 4) \hookrightarrow \mathrm{Spin}(4, 4)$:
  generic rank 3-distribution on 6-manifold $\rightsquigarrow$
  signature $(3, 3)$-conformal spin structure $+$ generic twistor spinor

$\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1, n+1)$

The Fefferman-type construction is based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1, n+1)$:

Denote by $\Delta = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ the real 2^{n+1}-dimensional spin representation of $\tilde{G} = \mathsf{Spin}(n+1, n+1)$. Then we fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}, s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing - here s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or $\Delta_-^{n+1,n+1}$ if n is odd.

These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1,n+1}$ of s_E, s_F with respect to Clifford multiplication are complementary maximally isotropic subspaces.

Then $G := \{g \in \mathsf{Spin}(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong \mathrm{SL}(n+1)$, defines an embedding $G = \mathrm{SL}(n+1) \overset{i}{\hookrightarrow} \mathsf{Spin}(n+1, n+1)$.

Fefferman-space $\tilde{M}$ and induced structure

One computes $\tilde{M} = \mathcal{G} \times_Q P/Q \cong (T^*M \otimes \mathcal{E}[2])/\{0\}$. Here we use the notation $\mathcal{E}[w]$ for suitably weighted (projective) version of the density bundle.

The invariant spinors s_E and s_F give rise to pure spin tractors:

The spin tractor bundle of $(M, \mathcal{C})$ is $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_\pm^{n+1,n+1} = \mathcal{G} \times_Q \Delta_\pm^{n+1,n+1}$. Since $s_E \in \Delta_\pm^{n+1,n+1}$ and $s_F \in \Delta_-^{n+1,n+1}$ are Q-invariant, they induce canonical sections $\mathbf{s}_E \in \Gamma(\mathcal{S}_\pm)$ and $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ obtained via the Fefferman construction induces a tractor connection ∇ on each conformal tractor bundle; by construction the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are parallel with respect to the induced tractor connections on the respective spin tractor bundles. But these are not necessarily the normal conformal tractor connection!

Normality of induced Cartan connection

Proposition

For $n = 2$ the Fefferman-type construction $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3,3)$ is normal. For $n \geq 3$ The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only ω is flat, in which case also $\tilde{\omega}$ is flat.

An outline of the argument:

The normalization condition on a conformal structure automatically implies that it is also torsion-free, i.e., that $\tilde{\kappa} : \tilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{p}$ has values in $\tilde{p} \subset \tilde{\mathfrak{g}}$. If the Fefferman-type construction $\mathrm{SL}(n+1) \rightsquigarrow \mathrm{Spin}(n+1, n+1)$ is normal, this forces the curvature of the projective structure κ to have values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{q}$, since $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$.

Normality

In the case where $n = 2$ the projective curvature consists only of the projective Cotton tensor, and has values in $\mathfrak{p}_+ \subset \mathfrak{q}$. Then a detailed discussion of the normalization condition indeed implies normality of the Fefferman-type construction.

However when $n \geq 3$, the projective curvature is uniquely determined by the projective Weyl tensor, and this has values in a P-module larger than $\mathfrak{q}$. But then, if the curvature κ doesn't vanish, it immediately follows from equivariancy-properties that κ has values outside of $\mathfrak{q} \subset \mathfrak{p}$.

The normal case $n = 2$:

Immediate consequence of normality:

Proposition

The split-signature conformal structures obtained from two-dimensional projective structures have the following properties:

1. *The conformal holonomy* $\mathrm{Hol}(\tilde{\omega})$ *is contained in* $\mathrm{SL}(3)$*.*
2. *The spin tractor bundle has two sections* $\mathbf{s}_E$ *and* $\mathbf{s}_F$ *with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e.* $\nabla^{\mathcal{S}_+,nor}\mathbf{s}_E = 0$ *and* $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$*. Thus they correspond to two pure twistor spinors* $\chi_E \in \Gamma(\mathbf{S}_+[\frac{1}{2}])$ *and* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$*.*

Here $\mathbf{S}[\frac{1}{2}] = \mathbf{S}_+[\frac{1}{2}] \oplus \mathbf{S}_-[\frac{1}{2}]$ is the weighted conformal spin bundle on $\tilde{M}$ which is associated to the real $4 = 2^2$-dimensional spin representation of $\mathrm{CSpin}(2,2) = \mathbb{R}_+ \times \mathrm{Spin}(2,2)$. The second part of the proposition is then a consequence of the $1:1$-correspondence between parallel spin tractors in $\mathcal{S}_\pm$ and twistor spinors $\chi \in \mathbf{S}_\pm[\frac{1}{2}]$, satisfying $D\chi + \frac{1}{4}\gamma \not{D}\chi = 0$.

The non-normal case: Normalization and preserved tractor spinor

Since for $n \geq 3$ the induced Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ is not already the normal conformal connection form, one needs a modification $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{p}})$ with the property that $\tilde{\omega}^{nor} = \tilde{\omega} + \Psi$ is normal, i.e., the curvature function $\tilde{\kappa}^{nor}$ of the modified Cartan connection form lies in the Kernel of the Kostant co-differential $\tilde{\partial}^* : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$.

While it is difficult to obtain an explicit formula for this modification Ψ, it is possible to show specific properties; in particular, the normalized connection still preserves one of the pure tractor spinors:

The non-normal case: Normalization and preserved tractor spinor

Theorem

$\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. In particular, the conformal spin structure $(M, \mathcal{C})$ carries a canonical (pure) twistor spinor $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

Corollary

The conformal holonomy $\mathrm{Hol}(\mathcal{C})$ is contained in the isotropy subgroup of $s_F \in \Delta_-^{n+1,n+1}$ in $\mathrm{Spin}(n+1, n+1)$; this is $SL(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subset \mathrm{Spin}(n+1, n+1)$.

An outline of the normalization:

1. The first step is to compute the failure $\tilde{\partial^*}\tilde{\kappa} : \tilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ of $\tilde{\omega}$ to be normal, which then yields a first modification Ψ' such that the new connection $\tilde{\omega}' = \tilde{\omega} + \Psi'$ is normal up to terms in the highest filtration component.
2. The main observation then is that Ψ' still annihilates the canonical spinor in $\tilde{\mathcal{S}}_-$,
$$\Psi' \mathbf{s}_F = 0,$$
and in particular also the modified tractor connection still preserves this spinor:
$$\tilde{\nabla}' \mathbf{s}_F = 0.$$
3. Now for a conformal structure, there is only one more normalization step that yields $\tilde{\omega}^{nor}$, and the necessary modification then has values in highest homogeneity. This yields that the tractor spinor $\mathbf{s}_F$ still projects to a twistor spinor, and a simple integrability condition then already forces $\mathbf{s}_F$ to be $\tilde{\nabla}^{nor}$-parallel.

Outlook

The following questions are currently treated in joint work with K. Sagerschnig, J. Šilhan and V. Žádník:

- In the normal case $n = 2$: [Dunajski-Tod, '10] showed that the original projective structure on M is metrizable if and only if the induced signature $(2, 2)$ conformal structure includes a Kähler or para-Kähler metric. Using the description of these geometric solutions as parallel sections of suitably modified tractor connections we show this 1:1-correspondence as a direct consequence of normality.
- For the non-normal case $n \geq 3$, it is an open problem to characterise the induced conformal structures. The twistor spinor described above will play a role in this, but additional characterising data will be necessary.

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