Theoretical tomography past and present

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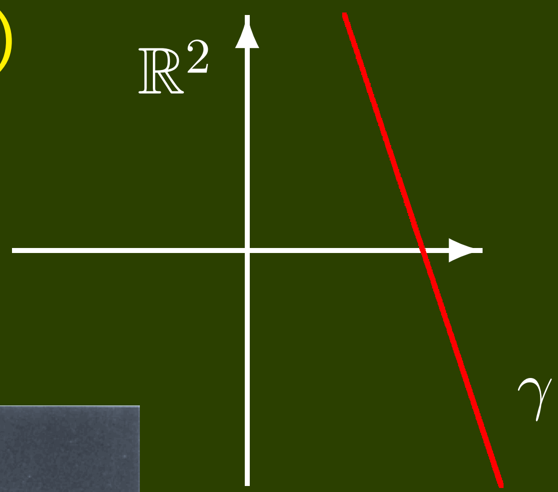
Radon transform

Johann Radon (1917)



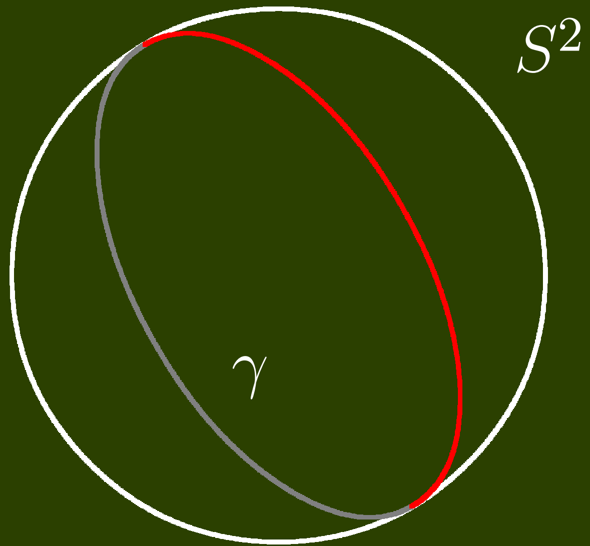
$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$

$$\phi(\gamma) = \int_\gamma f$$

$$\phi(a, b) = \int_{-\infty}^{\infty} f(a + bt, t)dt$$

Funk transform

Paul Funk (1913)



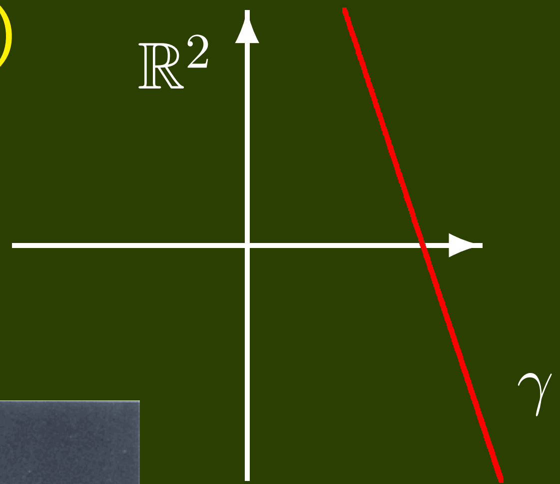
$$f \in \Gamma(S^2, \mathcal{E}) \qquad \phi(\gamma) = \oint_\gamma f$$

NB	$\mathcal{F} : \Gamma(S^2, \mathcal{E}) \to \Gamma(S^2, \mathcal{E})$
NB!	$\mathcal{F} : \Gamma_{\mathrm{even}}(S^2, \mathcal{E}) \to \Gamma_{\mathrm{even}}(S^2, \mathcal{E})$
NB!!	$\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}) \to \Gamma(\mathbb{RP}_2, \mathcal{E})$
NB!!!	$\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}) \to \Gamma(\mathbb{RP}_2^*, \mathcal{E})$

isomorphism!

Radon transform revisited

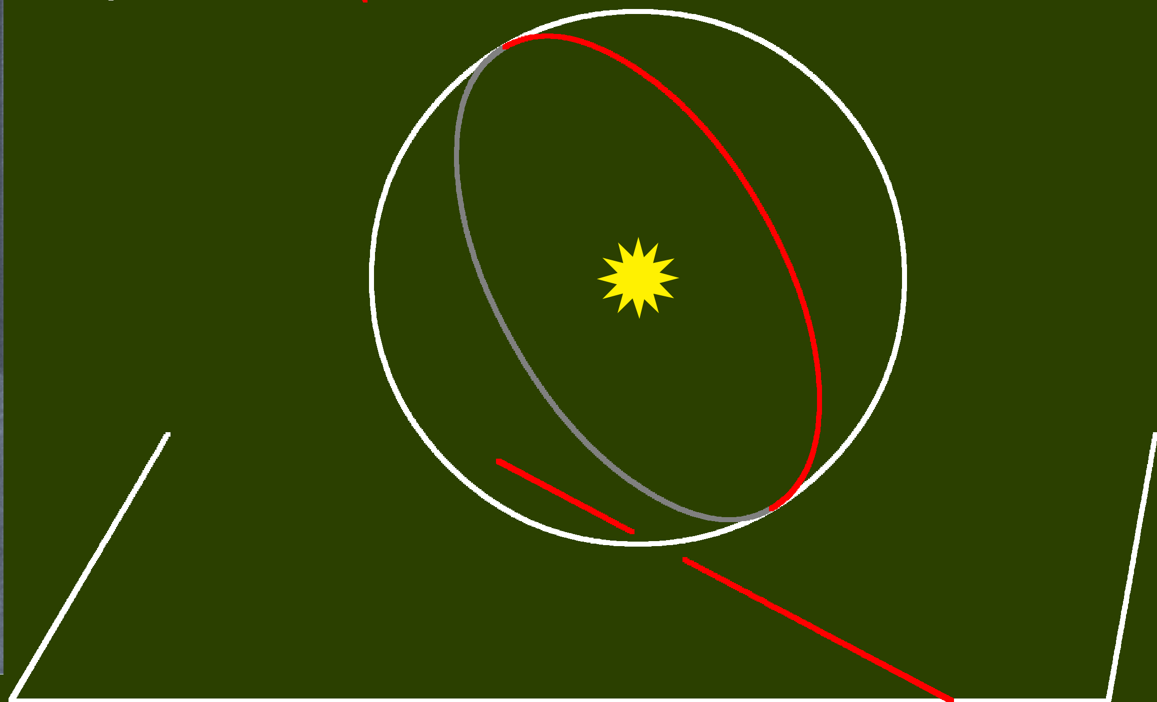
Johann Radon (1917)



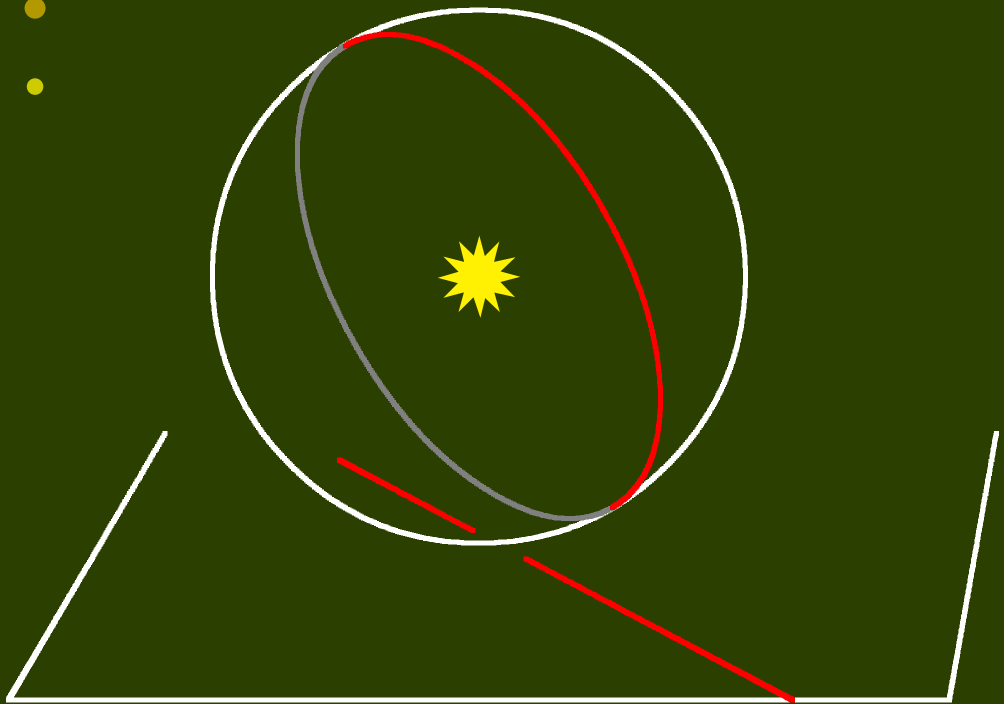
$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$

$$\phi(\gamma) = \int_\gamma f$$

Radon=Funk!



Radon=Funk



Fudge factor

f on S^2 versus f on $\mathbb{R}^2$

via F on $\mathbb{R}^3 \setminus \{0\}$ s.t.

$$F(\lambda x) = \lambda^{-2} F(x)$$

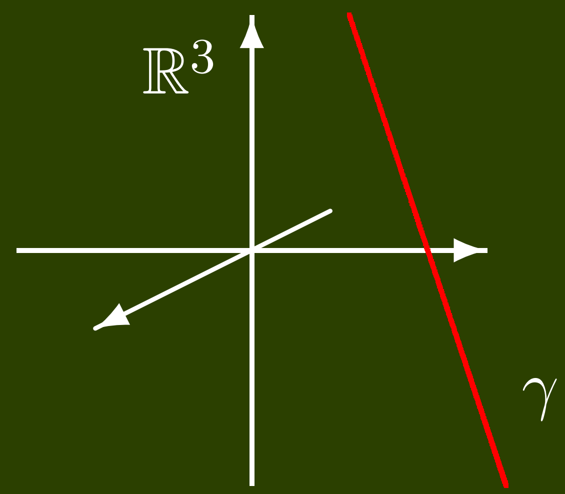
$$\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \to \Gamma(\mathbb{RP}_2^*, \tilde{\mathcal{E}}(-1))$$

symmetries

Radon	Euclidean symmetry: $\mathrm{SO}(2) \ltimes \mathbb{R}^2$	Fourier transform
Funk	spherical symmetry: $\mathrm{SO}(3)$	spherical harmonics
Funk-Radon	projective symmetry: $\mathrm{SL}(3, \mathbb{R})$	complex analysis!?

X-ray transform

Fritz John (1938)



$$f \in \Gamma_*(\mathbb{R}^3, \mathcal{E}) \qquad \phi(\gamma) = \int_\gamma f$$

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}(-1))$$

$$\boxed{\phi(w,x,y,z) = \int_{-\infty}^{\infty} f(w+xt, y+zt, t)\, dt}$$

$$\boxed{\frac{\partial^2 \phi}{\partial w \partial z} = \frac{\partial^2 \phi}{\partial x \partial y}}$$

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^4) & \\ \swarrow & & \searrow \\ \mathbb{RP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

ultrahyperbolic wave equation

Bateman's formula

Harry Bateman (1904)!!! (cf. Edmund Whittaker (1902))

$$\phi(w,x,y,z) = \oint_\gamma f\left((w+ix)+(iy+z)\zeta, (iy-z)+(w-ix)\zeta, \zeta\right) d\zeta$$

$$\boxed{\frac{\partial^2\phi}{\partial w^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0}$$

$$\tau : \mathbb{CP}_3 \longrightarrow S^4$$

Penrose transform ~(1976)!

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\simeq} \{\text{harmonic functions on } U\}$$

Present and future

- Double fibration transforms

$$\mathfrak{X}_D = \{(z, C) \in D \times \mathcal{M}_D \mid z \in C\}$$

$$\begin{array}{ccc} & \mathfrak{X}_D & \\ \swarrow & & \searrow \\ G/Q = Z \supseteq D & & \mathcal{M}_D \end{array}$$

- X-ray on other homogeneous spaces G/H, especially
  - Complex projective space
  - Quaternionic projective space
  - Octonionic plane

THE END

THANK YOU