



Conformally Fedosov Manifolds

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[joint work with Jan Slovák]

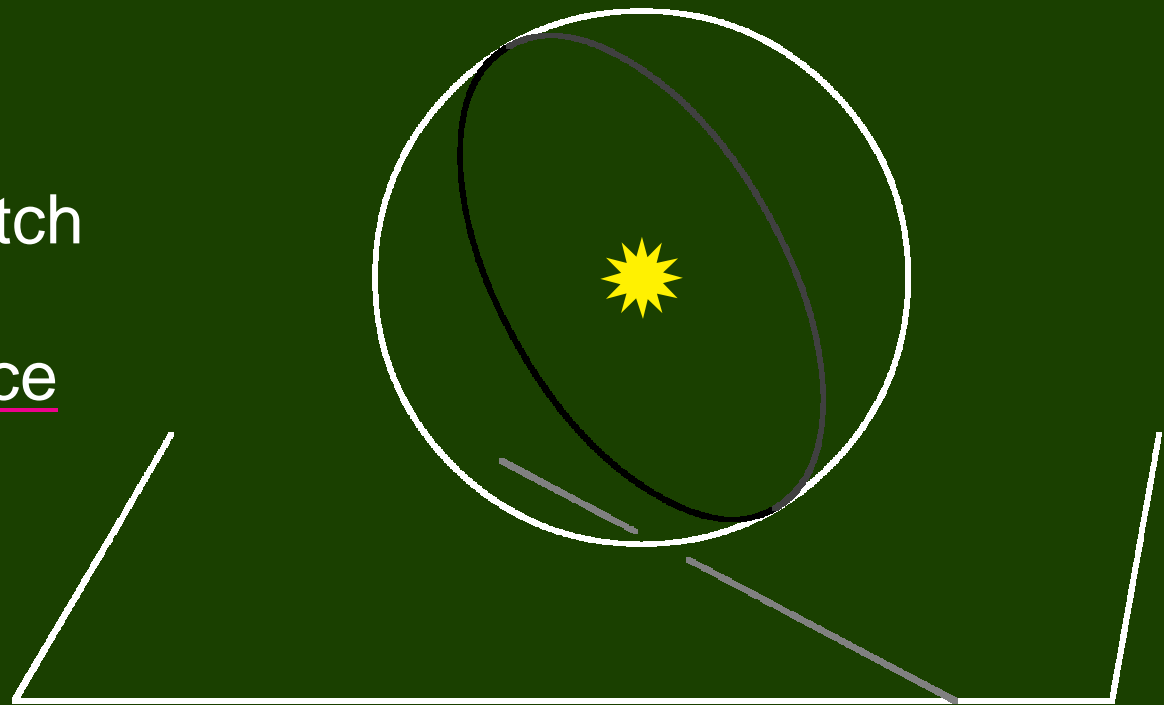
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Projective differential geometry

Defⁿ $\hat{\nabla} \sim \nabla \iff$ same geodesics (unparameterised)

EG (Thales 600 BC) The round sphere is projectively flat

Affine coordinate patch
 $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a
projective equivalence



Operational Defⁿ

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

Conformally symplectic geometry

symplectic geometry: $\left\{ \begin{array}{l} \bullet \text{ non-degenerate 2-form } J \\ \bullet J \text{ is closed: } dJ = 0 \end{array} \right.$

conformally " " " ": $\left\{ \begin{array}{l} \bullet \text{ non-degenerate 2-form } J \\ \bullet J \text{ is conformally closed: } dJ = 2\alpha \wedge J \\ \text{(where } \alpha \text{ is a closed 1-form)} \end{array} \right.$

$$J \mapsto \hat{J} = \Omega^2 J \quad \rightsquigarrow \quad \alpha \mapsto \hat{\alpha} = \alpha + \Upsilon \quad \text{where } \Upsilon = d \log \Omega$$

Example

$$J \equiv (1/\|x\|)^2 (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \dots) \quad \text{on } \mathbb{R}^{2n} \setminus \{0\}$$

is invariant under dilation $x \mapsto \lambda x$

so descends to $S^1 \times S^{2n-1}$

Combine!

Recall: $\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$

NB: $\hat{\nabla}_{(a} J_{b)c} = \nabla_{(a} J_{b)c} - 3\Upsilon_{(a} J_{b)c}$

Decree (for conformally Fedosov)

- $[J_{ab}]$ conformally symplectic
- $[\nabla_a]$ projective structure
- $\nabla_{[a} J_{bc]} = 2\alpha_{[a} J_{bc]} \quad (\nabla_{[a} \alpha_{b]} = 0)$
- $\nabla_{(a} J_{b)c} = \beta_{(a} J_{b)c}$

Normalise: $\alpha_a = \beta_b$. Then $\nabla_a J_{bc} = 2J_{a[b} \alpha_{c]}$ quite strong

Remaining conformal change

$$\hat{J}_{ab} = \Omega^2 J_{ab} \quad \hat{\alpha}_a = \alpha_a + \Upsilon_a \quad \Upsilon_a = \nabla_a \log \Omega$$

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a$$

Fedosov gauge

Locally: choose $\alpha_a = 0$ ($\iff \nabla_a J_{bc} = 0$)

Remaining freedom

$\hat{J}_{ab} = \Omega^2 J_{ab}$ where Ω is constant (local)

$\hat{\nabla}_a = \nabla_a$ canonically defined global connection

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c \equiv R_{ab}{}^c{}_d X^d$$

$$R_{abcd} \equiv R_{ab}{}^e{}_d J_{ec} \implies$$

$$R_{abcd} = R_{[ab](cd)} \quad R_{[abc]d} = 0$$

\implies

$$R_{abcd} = \underset{\substack{\uparrow \\ J\text{-trace-free}}}{V_{abcd}} + J_{ac} \underset{\substack{\uparrow \\ \text{symmetric}}}{\Phi_{bd}} - J_{bc} \Phi_{ad} + J_{ad} \Phi_{bc} - J_{bd} \Phi_{ac} + 2J_{ab} \Phi_{cd}$$

cf. Weyl + Schouten

Complex projective space

With Fubini-Study metric \rightsquigarrow

$$R_{ab}{}^c{}_d = \delta_a^c g_{bd} - \delta_b^c g_{ad} + J_{ad} J^c{}_b - J_{bd} J^c{}_a + 2J_{ab} J^c{}_d$$

so

$$R_{abcd} = J_{ac} g_{bd} - J_{bc} g_{ad} + J_{ad} g_{bc} - J_{bd} g_{ac} + 2J_{ab} g_{cd}$$

\implies

$$V_{abcd} = 0 \quad \Phi_{ab} = g_{ab}$$

Projective viewpoint

(Conformal viewpoint = exercise)

$$R_{ab}{}^c{}_d = J_{ad} J^c{}_b - J_{bd} J^c{}_a + 2J_{ab} J^c{}_d - \frac{3}{2n-1} (\delta_a^c g_{bd} - \delta_b^c g_{ad}) \\ + 2 \frac{n+1}{2n-1} (\delta_a^c g_{bd} - \delta_b^c g_{ad})$$

Projective Weyl tensor !!

Conformal differential geometry

Densities of weight w $g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab} \implies \hat{\phi} = \Omega^w \phi$

Tractors $\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$ transform according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b - \frac{1}{2} \Upsilon^b \Upsilon_b \sigma \end{bmatrix}, \quad \text{where} \quad \begin{aligned} \Upsilon_b &\equiv \nabla_b \log \Omega \\ \Upsilon^b &= g^{bc} \Upsilon_c \end{aligned}$$

Tractor connection (É. Cartan / T.Y. Thomas)

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_a^b \mu_b \end{bmatrix}$$

flat \iff Weyl = 0
 [Cotton-York = 0]
 [when dim = 3]

Structure group = $SO(n+1, 1)$

Conformally Fedosov tractors

Densities of weight w $J_{ab} \mapsto \hat{J}_{ab} = \Omega^2 J_{ab} \implies \hat{\phi} = \Omega^w \phi$

Tractors $\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$ transform according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b + \Upsilon^b \alpha_b \sigma \end{bmatrix}, \quad \text{where} \quad \begin{aligned} \Upsilon_b &\equiv \nabla_b \log \Omega \\ \Upsilon^b &\equiv J^{bc} \Upsilon_c \end{aligned}$$

Tractor connection in Fedosov gauge

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b - J_{ab} \rho + \Phi_{ab} \sigma \\ \nabla_a \rho - \Phi_a^b \mu_b - S_a \sigma \end{bmatrix} \quad \begin{aligned} \Phi_a^b &= J^{bc} \Phi_{ac} \\ S_a &= \frac{1}{2n+1} \nabla^b \Phi_{ab} \end{aligned}$$

Structure group = $\text{Sp}(2n + 2, \mathbb{R})$

Tractor curvature

In Fedosov gauge

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ V_{abcd} \mu^d + Y_{abc} \sigma \\ Y_{abc} \mu^c - \frac{1}{2n} (\nabla^c Y_{abc} - V_{abce} \Phi^{ce}) \sigma \end{bmatrix} - 2J_{ab} \begin{bmatrix} \rho \\ S_c \sigma - \Phi_{cd} \mu^d \\ S_c \mu^c - \frac{1}{2n} (\Phi_{de} \Phi^{de} + \nabla^c S_c) \sigma \end{bmatrix}$$

where (cf. Cotton-York)

$$Y_{abc} \equiv \nabla_a \Phi_{bc} - \nabla_b \Phi_{ac} + J_{ac} S_b - J_{bc} S_a + 2J_{ab} S_c$$

Bianchi $\implies \nabla^d V_{abcd} + (2n + 1) Y_{abc} = 0$ so $V_{abcd} = 0$ \implies

this part vanishes

Consequences of $V_{abcd} \equiv 0$

$$V_{abcd} = 0 \iff (\nabla_a \nabla_b - \nabla_b \nabla_a) \Sigma = 2J_{ab} \Theta \Sigma$$

Bianchi $\implies \nabla_{[a} (J_{bc]} \Theta) = 0$

\implies in Fedosov gauge $J_{[bc} \nabla_a] \Theta = 0 \implies \nabla_a \Theta = 0$

$$\Theta \in \Gamma(\text{End}(\mathbb{T})) \cong \Gamma(\otimes^2 \mathbb{T}) \rightsquigarrow \Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Phi_{bc} & S_b \\ 0 & S_c & -\frac{1}{2n}(\Phi^{de} \Phi_{de} + \nabla^c S_c) \end{bmatrix}$$

$$\nabla_a \Theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\nabla_a \Phi_{bc} - J_{ab} S_c - J_{ac} S_b & * \\ 0 & * & * \end{bmatrix}$$

Therefore $V_{abcd} = 0 \implies \nabla_a \Phi_{bc} + J_{ab} S_c + J_{ac} S_b = 0$

Further consequences of $V_{abcd} \equiv 0$

In Fedosov gauge

$$V_{abcd} = 0 \Rightarrow \nabla_a \Theta = 0 \iff \nabla_a \Phi_{bc} + J_{ab} S_c + J_{ac} S_b = 0$$

$$\iff \nabla_a \Phi^{bc} + \delta_a^b S^c + \delta_b^c S^a = 0$$

$$\iff \text{trace-free-part}(\nabla_a \Phi^{bc} = 0)$$

mobility equations

$$\iff \begin{aligned} \nabla_a \Phi^{bc} + \delta_a^b S^c + \delta_b^c S^a &= 0 \\ \nabla_a S^b + \delta_a^b X - \Phi_{ac} \Phi^{bc} &= 0 \\ \nabla_a X - 2\Phi_{ab} S^b &= 0 \end{aligned}$$

extremely strong!!

Examples with $V_{abcd} \equiv 0$

- $\mathbb{C}\mathbb{P}_n$ recall $V_{abcd} = 0$ and $\Phi_{ab} = g_{ab} \rightsquigarrow$

$$\Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Phi_{bc} & S_b \\ 0 & S_c & -\frac{1}{2n}(\Phi^{de}\Phi_{de} + \nabla^c S_c) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -g_{bc} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- $S^1 \times S^{2n-1}$ with $J_{ab} = (1/\|x\|)^2(dx^1 \wedge dx^2 + \dots)$
 $\nabla_a =$ the flat connection \rightsquigarrow

locally

$$\begin{aligned} V_{abcd} &= 0 \\ \Phi_{ab} &= 0 \end{aligned}$$

$$\Theta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Any more?

(Potential) applications

- Integral geometry on $\mathbb{C}\mathbb{P}_n$ (with H. Goldschmidt)

Elliptic complex on any (conformally) symplectic manifold

$$\begin{array}{ccccccccccccccc}
 0 & \rightarrow & & \xrightarrow{d} & & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^2 & \xrightarrow{d_{\perp}} & \dots & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^{n-1} & \xrightarrow{d_{\perp}} & \Lambda_{\perp}^n & & \\
 & & & & & & & & & & & & & & \downarrow d_{\perp}^{(2)} \\
 0 & \leftarrow & \Lambda^0 & \xleftarrow{d_{\perp}} & \Lambda^1 & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^2 & \xleftarrow{d_{\perp}} & \dots & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^{n-1} & \xleftarrow{d_{\perp}} & \Lambda_{\perp}^n & &
 \end{array}$$

$V_{abcd} = 0 \implies$
OK to couple with \mathbb{T}

Smith (1976, $n = 2$)
Rumin-Seshadri
Tseng-Yau

- Characterisation of $\mathbb{C}\mathbb{P}_n$ (with L. Stolovitch)

Blaschke conjecture ...



THE END

THANK YOU