

Integrability conditions for the Grushin operators

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Grushin 'distributions'

Standard coordinates (x, y) on \mathbb{R}^2 k^{th} Grushin fields

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x^k \frac{\partial}{\partial y} \quad \text{bracket generating}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

Cf $\frac{\partial}{\partial x} + ix \frac{\partial}{\partial y}$ Mizohata

Coordinates (x, y, t) on \mathbb{R}^3

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial y} \quad \text{Martinet (Cf Darboux)}$$

$$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + (ix - y) \frac{\partial}{\partial t} \quad \text{Lewy (Cauchy-Riemann-Heisenberg)}$$

Zeroth Grushin \rightsquigarrow de Rham complex

Standard coordinates (x, y) on \mathbb{R}^2

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \Rightarrow Yg - Xh = 0$$

locally the
converse is true

On any smooth 2-manifold

$$0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow 0,$$

the de Rham complex.

The Rumin complex

Coördinates (x, y, t) on \mathbb{R}^3

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv \frac{\partial}{\partial t} + x \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \Rightarrow \begin{cases} XYg - X^2h + Zg = 0 \\ YXh - Y^2g - Zh = 0 \end{cases}$$

locally the
converse is true

On any contact 3-manifold

$$0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{d_H^{(2)}} \Lambda^1_H \otimes L \xrightarrow{d_H} \Lambda^3 \rightarrow 0,$$

the Rumin complex.

Rumin cont'd

Contact 3-manifold: $H \subset TM$ such that $[H, H] = TM$.

bracket generating

$$0 \rightarrow L \rightarrow \Lambda^1 \rightarrow \Lambda^1_H \rightarrow 0$$

$$0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \rightarrow 0$$

$$\parallel$$

$$\Lambda^0$$

$$\parallel$$

$$\Lambda^1_H$$

$$\parallel$$

$$\Lambda^2_H$$

$$\parallel$$

+

L

\simeq Levi \rightarrow

$$\Lambda^1_H \otimes L$$

$$\Lambda^2_H \otimes L$$

$$\begin{matrix} 0 & 0 \\ \times & \bullet \end{matrix}$$

\rightarrow

$$\begin{matrix} -2 & 1 \\ \times & \bullet \end{matrix}$$

\rightarrow

$$\begin{matrix} -4 & 1 \\ \times & \bullet \end{matrix}$$

\rightarrow

$$\begin{matrix} -4 & 0 \\ \times & \bullet \end{matrix}$$

second order!

(2,3,5)-geometries

Traditional coordinates (x, y, p, q, z) on \mathbb{R}^5

$$X \equiv \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + q^2 \frac{\partial}{\partial z} \quad Y \equiv \frac{\partial}{\partial q}$$

generate a 2-plane distribution H in \mathbb{R}^5 and then

$$Z \equiv [X, Y] = -\frac{\partial}{\partial p} - 2q \frac{\partial}{\partial z}$$

and then

$$W \equiv [X, Z] = \frac{\partial}{\partial y} \quad U \equiv [Y, Z] = -2 \frac{\partial}{\partial z}$$

(and then everything else commutes).

(2,3,5) cont'd

$$H \subset TM$$

$$\underbrace{H \subseteq [H, H] \subseteq [H, [H, H]] = TM}_{\text{rank 2} \quad \text{rank 3} \quad \text{rank 5}}$$

bracket generating
and more besides

NB Frame $X, Y \in \Gamma(H)$. $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \rightarrow \Lambda^1_H \rightarrow ???$$


For any (2, 3, 5)-geometry

$$\begin{array}{ccccccc} 0 & 0 & \xrightarrow{1^{\text{st}}} & -2 & 1 & \xrightarrow{3^{\text{rd}}} & -5 & 2 & \xrightarrow{2^{\text{nd}}} & -6 & 2 & \xrightarrow{3^{\text{rd}}} & -6 & 1 & \xrightarrow{1^{\text{st}}} & -5 & 0 \\ \text{X} \text{---} \text{---} \text{---} \bullet & & & \text{X} \text{---} \text{---} \text{---} \bullet & & & \text{X} \text{---} \text{---} \text{---} \bullet & & & \text{X} \text{---} \text{---} \text{---} \bullet & & & \text{X} \text{---} \text{---} \text{---} \bullet & & & \text{X} \text{---} \text{---} \text{---} \bullet \end{array}$$

BGG ... R Bryant, ME, R Gover, K Neusser

First Grushin

Standard coordinates (x, y) on \mathbb{R}^2

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

NB $Xf = 0, Yf = 0 \Rightarrow f$ locally constant.

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} XYg - X^2h + Zg = 0 \\ YXh - Y^2g - Zh = 0 \end{array} \right. \quad \boxed{\text{conversely?}}$$

Locally (on $U^{\text{open}} \subseteq \mathbb{R}^2$ contractible)

$$\left. \begin{array}{l} Xf = g \\ Yf = h + C \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} XYg - X^2h + Zg = 0 \\ YXh - Y^2g - Zh = 0 \end{array} \right.$$

\uparrow
 $\boxed{\text{constant}}$

First Grushin versus contact

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = \frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2$$

versus

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv \frac{\partial}{\partial t} + x \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = \frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^3$$

NB $0 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \xrightarrow{\partial/\partial t} \mathcal{E}_3 \rightarrow 0$ $\partial/\partial t$ is a symmetry.

Symmetry reduction

$$\begin{array}{ccccccccc}
 0 & \rightarrow & \begin{array}{c} 0 \quad 0 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -2 \quad 1 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -4 \quad 1 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -4 \quad 0 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & 0 \\
 & & \uparrow \frac{\partial}{\partial t} & & \uparrow \frac{\partial}{\partial t} & & \uparrow \frac{\partial}{\partial t} & & \uparrow \frac{\partial}{\partial t} & & \\
 0 & \rightarrow & \begin{array}{c} 0 \quad 0 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -2 \quad 1 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -4 \quad 1 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & \begin{array}{c} -4 \quad 0 \\ \times \leftarrow \bullet \end{array} & \longrightarrow & 0
 \end{array}$$

Diagram chasing \rightsquigarrow

First Grushin complex

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = \frac{\partial}{\partial y}$$

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \begin{array}{c} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_2 \end{array} \longrightarrow \begin{array}{c} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_2 \end{array} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

$$f \longmapsto \begin{bmatrix} Xf \\ Yf \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} \longmapsto \begin{bmatrix} X^2h - (XY + Z)g \\ Y^2g - (YX - Z)h \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \longmapsto Xq + Yp$$

here

and here

else exact

Local cohomology is \mathbb{R}

O Calin, D-C Chang, ME

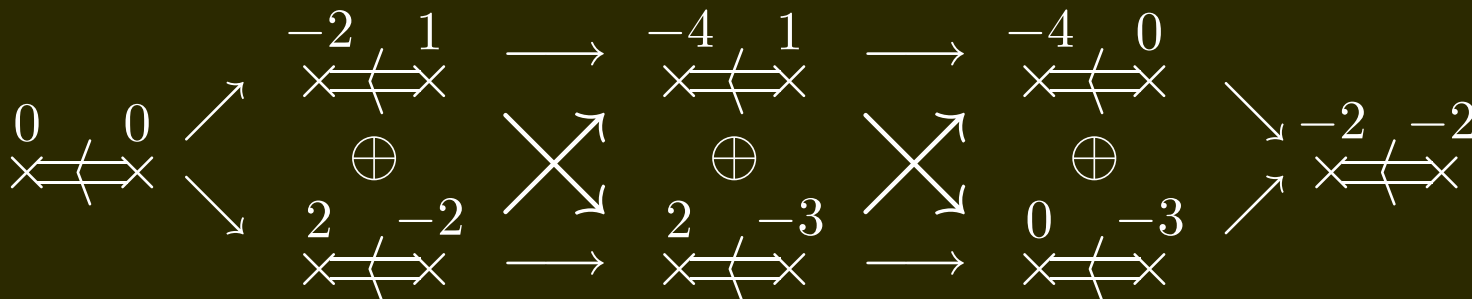
Second Grushin

$$X \equiv \frac{\partial}{\partial x} \quad Y \equiv x^2 \frac{\partial}{\partial y} \quad Z \equiv [X, Y] = 2x \frac{\partial}{\partial y} \quad W \equiv [X, Z] = 2 \frac{\partial}{\partial y}$$

versus Engel on \mathbb{R}^4

$$\begin{aligned} X &= \frac{\partial}{\partial x} & Y &= \frac{\partial}{\partial s} + x \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial y} \\ Z \equiv [X, Y] &= \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial y} & W \equiv [X, Z] &= 2 \frac{\partial}{\partial y} \end{aligned}$$

BGG ... R Bryant, ME, R Gover, K Neusser (B Doubrov)



Second Grushin complex

On \mathbb{R}^4

$$\begin{array}{ccccccccc}
 \Lambda^0 & \rightarrow & \Lambda^1 & \rightarrow & \Lambda^2 & \rightarrow & \Lambda^3 & \rightarrow & \Lambda^4 \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 1 \boxed{1} & & 2 \boxed{2} & & 1 & & 1 & & 1 \\
 & & + & \nearrow & + & & & & \\
 & & 1 & & 2 \boxed{1} & & 1 & & \\
 & & + & \nearrow & + & \nearrow & + & & \\
 & & 1 & & 2 \boxed{1} & & 1 & & \\
 & & & & + & \nearrow & + & & \\
 & & & & 1 & & 2 \boxed{2} & & 1 \boxed{1}
 \end{array}$$

$0 \rightarrow \boxed{1} \xrightarrow{1^{\text{st}}} \boxed{2} \begin{array}{l} \nearrow 2^{\text{nd}} \boxed{1} \\ \searrow 3^{\text{rd}} \boxed{1} \end{array} \oplus \begin{array}{l} \boxed{1} \\ \boxed{1} \end{array} \begin{array}{l} \nearrow 3^{\text{rd}} \boxed{2} \\ \searrow 2^{\text{nd}} \boxed{2} \end{array} \xrightarrow{1^{\text{st}}} \boxed{1} \rightarrow 0$

reduce using $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$

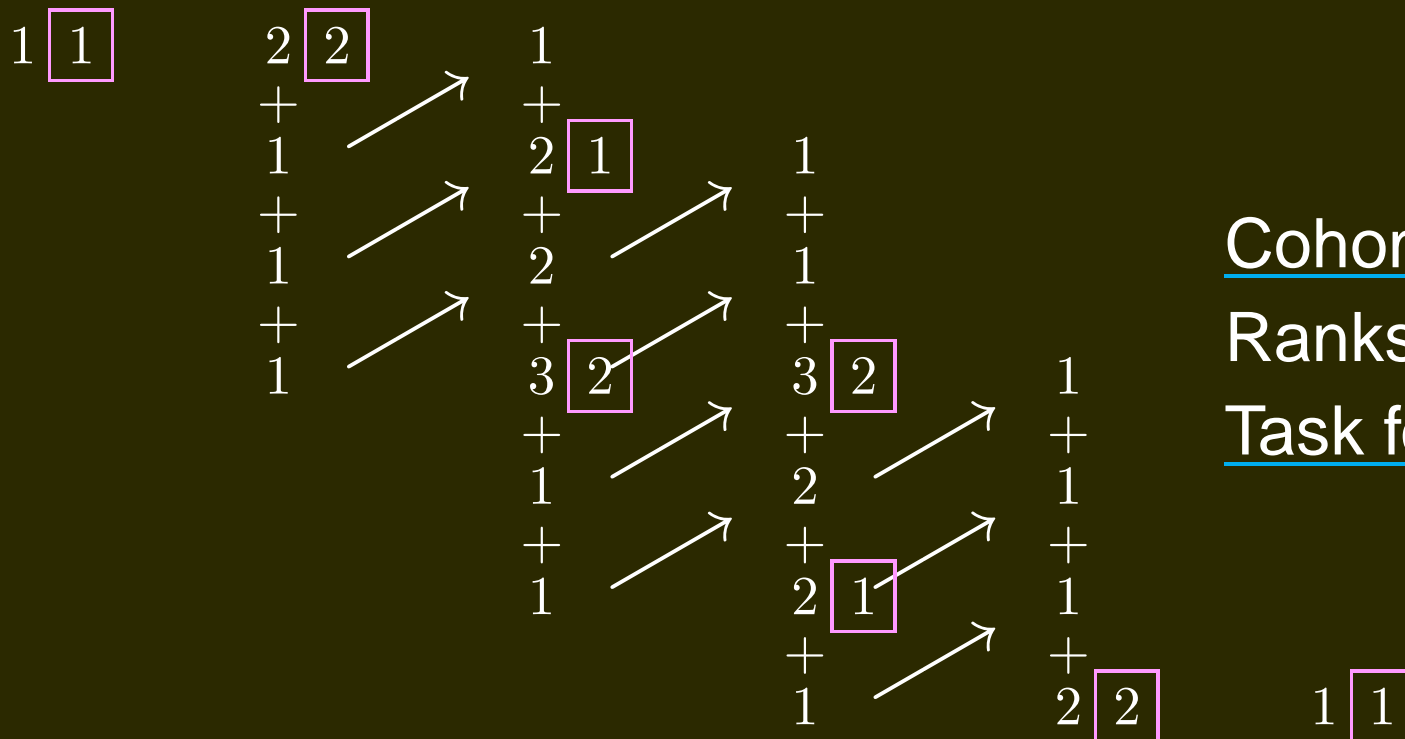
O Calin, D-C Chang, ME
arXiv:1207.5440

Third Grushin complex

$$X = \frac{\partial}{\partial x} \quad Y = x^3 \frac{\partial}{\partial y} \quad Z \equiv [X, Y] \quad W \equiv [X, Z] \quad U \equiv [X, W]$$

Look for symmetry reduction from \mathbb{R}^5

$$X = \frac{\partial}{\partial x} \quad Y = \frac{\partial}{\partial r} + x \frac{\partial}{\partial s} + x^2 \frac{\partial}{\partial t} + x^3 \frac{\partial}{\partial y} \quad \dots$$



Cohomology?

Ranks = 1, 2, 3, 3, 2, 1?

Task for babies?

Task for babies?

Yes! Everything works as expected \rightsquigarrow a complex

$$0 \rightarrow \boxed{1} \xrightarrow{1^{\text{st}}} \boxed{2} \begin{array}{l} \nearrow 2^{\text{nd}} \\ \searrow 4^{\text{th}} \end{array} \begin{array}{l} \boxed{1} \\ \oplus \\ \boxed{2} \end{array} \begin{array}{l} \rightarrow \\ \times \\ \rightarrow \end{array} \begin{array}{l} \boxed{2} \\ \oplus \\ \boxed{1} \end{array} \begin{array}{l} \searrow \\ \nearrow \end{array} \boxed{2} \rightarrow \boxed{1} \rightarrow 0$$

reduce using

$$\frac{\partial}{\partial r} \quad \frac{\partial}{\partial s} \quad \frac{\partial}{\partial t}$$

Integrability conditions

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \Rightarrow \begin{cases} Y^2g = YXh - Zh \\ X^4h = X^3Yg + X^2Zg + XWg + Ug \\ XYZg - 2YWg + Z^2g = X^2Zh - 3XWh + 2Uh \end{cases}$$

Remark On \mathbb{R}^2 also $2YZg = 2XZh - 5Wh$

Conundrum: who ordered that?

ME, J Slovák, V Souček

Fourth Grushin

Not a task for babies!

Slow growth distribution (2, 3, 4, 5, 6)

Not a parabolic geometry: $\frac{SL(2, \mathbb{R}) \ltimes \odot^4 \mathbb{R}^2}{\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}}$

Cf B Doubrov & A Medvedev

EG Lie algebra cohomology \Rightarrow ranks are

1, 2, 3, 4, 3, 2, 1

and then

1, 2, 4, 6, 6, 4, 2, 1

for slow growth (2, 3, 4, 5, 6, 7) et cetera



THE END

THANK YOU