Re-inventing the Wheel:
Differential Operators on the Sphere

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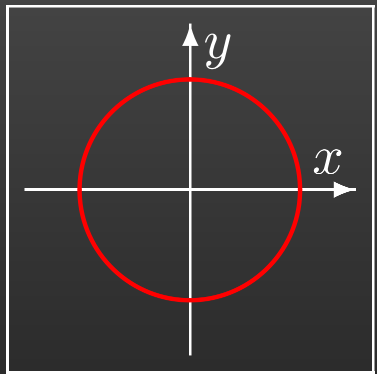
The round wheel

The round circle: $\{x^2 + y^2 = 1\} = \{(\cos\theta, \sin\theta)\}$



with symmetry group $\mathrm{SO}(2)$

$$\wr$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

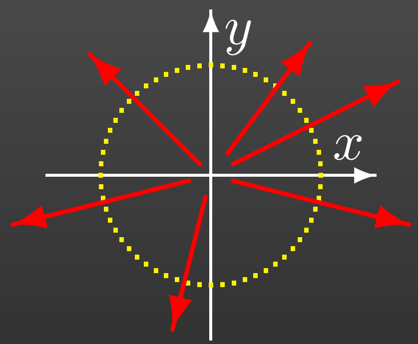
Invariant differential operators:–

$$f \mapsto 2\frac{d^3 f}{d\theta^3} + 7\frac{d^2 f}{d\theta^2} - \pi^{42}\frac{df}{d\theta} - f.$$

Any polynomial in $d/d\theta$ will do.

The projective wheel

The projective circle: $\{(x,y) \sim \lambda(x,y) \text{ for } \lambda > 0\}$



with symmetry group $\mathrm{SL}(2,\mathbb{R})$.

Homogeneous space: $\mathrm{SL}(2,\mathbb{R}) / \left\{ \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \right\}.$

Function on circle $\Leftrightarrow f(\lambda x, \lambda y) = f(x,y)$

$$\Leftrightarrow \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \quad \Leftrightarrow \quad \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (yg, -xg)$$

where $g(\lambda x, \lambda y) = \lambda^{-2} g(x,y)$.

An invariant operator

Claim: $f \mapsto g \equiv \frac{1}{y}\frac{\partial f}{\partial x} = -\frac{1}{x}\frac{\partial f}{\partial y}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant.

Proof: Fix skew tensor ϵ_{ij} (volume form). Then

$$\nabla_i f = \epsilon_{ij} x^j g \quad \text{defines} \quad g. \qquad \square$$

Write $g = \nabla f$ for

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E}(-2) = \Lambda^1.$$

exterior derivative

Homogeneous line bundles on $\mathrm{SL}(2,\mathbb{R})/P$

More invariant operators

Claim: there are $\mathrm{SL}(2,\mathbb{R})$-invariant operators

$$\nabla^{(\ell+1)} : \mathcal{E}(\ell) \longrightarrow \mathcal{E}(-\ell-2) \quad \forall\, \ell \geq 0.$$

Proof: $f_{ij\cdots k} \equiv \nabla_i \nabla_j \cdots \nabla_k f$ ($\ell + 1$ derivatives)

- $f_{ij\cdots k} = f_{(ij\cdots k)}$ and $x^i f_{ij\cdots k} = 0$
- exact sequence

$$0 \to \mathcal{E}(-\ell-2) \to \mathcal{E}_{(ij\cdots k)}(-1) \xrightarrow{x^i} \mathcal{E}_{(j\cdots k)} \to 0$$

$$\begin{array}{ccc} \Cup & & \Cup \\ g & \mapsto & \epsilon_{ip}\epsilon_{jq}\cdots\epsilon_{kr}x^p x^q \cdots x^r g. \end{array}$$

$\square$

Classification!

Theorem We've found all $\mathrm{SL}(2,\mathbb{R})$-invariant linear differential operators on the circle: if $D:\mathcal{E}(\ell)\to\mathcal{E}(m)$ is a non-constant invariant linear differential operator, then

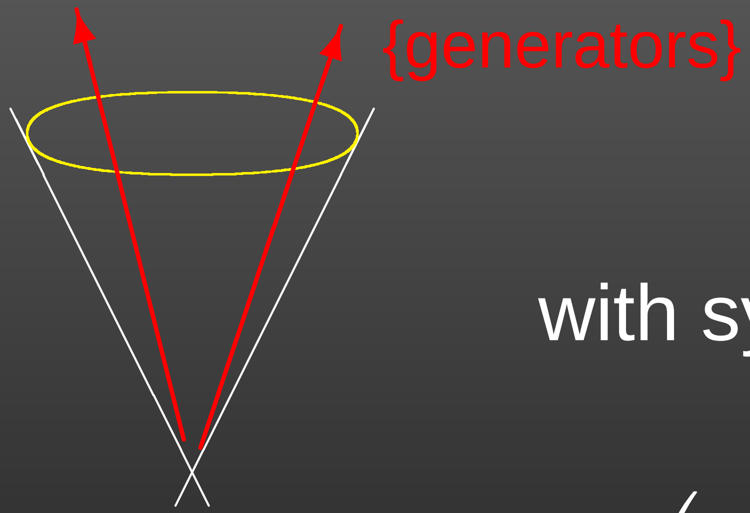
- $\ell \geq 0$ and $m = -\ell - 2$
- $D =$ constant $\times\ \nabla^{(\ell+1)}$.

Best proof $\leftrightarrow$ representation theory of $\mathfrak{sl}(2,\mathbb{R})$.

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \cdots$$

$[x,y] = h \quad [h,x] = 2x \quad [h,y] = -2y$ &c.

Another wheel



with symmetry group $\mathrm{SO}(2,1)$.

But, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$ induces

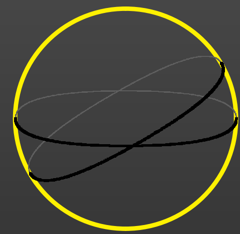
$$\mathrm{SL}(2,\mathbb{R}) \xrightarrow{\simeq\, 2:1} \mathrm{SO}(2,1)$$

$\rightsquigarrow$ $(\mathrm{SL}(2,\mathbb{R})/P)/\pm\,\mathrm{Id}$ (not much different).

The 2-sphere

Various symmetry groups:–

- The round sphere $= \mathrm{SO}(3)/\mathrm{SO}(2)$
- The projective sphere $= \mathrm{SL}(3,\mathbb{R})/P$
- The conformal sphere $= \mathrm{SO}(3,1)/P$
- The Riemann sphere $= \mathrm{SL}(2,\mathbb{C})/P$



Projective $\neq$ conformal $\left\{ \begin{array}{l} \dim_{\mathbb{R}} \mathrm{SL}(3,\mathbb{R}) = 8 \\ \dim_{\mathbb{R}} \mathrm{SO}(3,1) = 6 \end{array} \right.$

Conformal $=$ Riemann

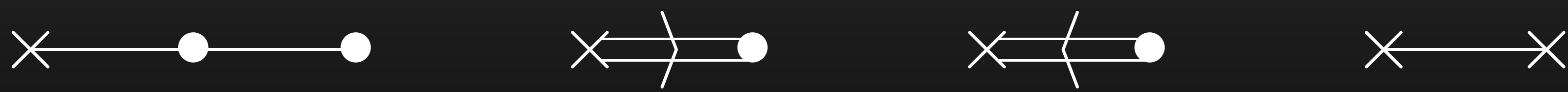
$$\mathrm{SL}(2,\mathbb{C}) \xrightarrow{\simeq\ 2:1} \mathrm{SO}(3,1).$$

The 3-sphere

Many symmetry groups, including

- The projective 3-sphere $= \mathrm{SL}(4, \mathbb{R})/P$
- The conformal 3-sphere $= \mathrm{SO}(4, 1)/P$
- The contact projective 3-sphere $= \mathrm{Sp}(4, \mathbb{R})/P$
- The CR 3-sphere $= \mathrm{SU}(2, 1)/P$

They are all different:–



Also as complex homogeneous spaces.

$S^3 = \mathrm{SU}(2) \Rightarrow$ many more possibilities. . .

$\mathrm{SL}(4,\mathbb{R})$-invariant operators

- $f \in \Gamma(S^3, \mathcal{E}(0)) \Rightarrow x^i \nabla_i f = 0$. But...

$$0 \to \Lambda^1 \to \mathcal{E}_i(-1) \xrightarrow{x^i} \mathcal{E} \to 0$$ Euler sequence

bundle of 1-forms on S^3

and $\nabla : \Lambda^0 \to \Lambda^1$ is invariant (exterior derivative).

- $f \in \Gamma(S^3, \mathcal{E}(1)) \Rightarrow x^i \nabla_i \nabla_j f = 0$. But...

$$0 \to \bigodot^2 \Lambda^1(1) \to \mathcal{E}_{(ij)}(-1) \xrightarrow{x^i} \mathcal{E}_j \to 0$$ exact

and so $\nabla^{(2)} : \Lambda^0(1) \to \bigodot^2 \Lambda^1(1)$ is invariant.

More. . .

- $f \in \Gamma(S^3, \Lambda^1(2))$ and twisted Euler

$$0 \to \Lambda^1(2) \to \mathcal{E}_i(1) \xrightarrow{x^i} \mathcal{E}(2) \to 0$$

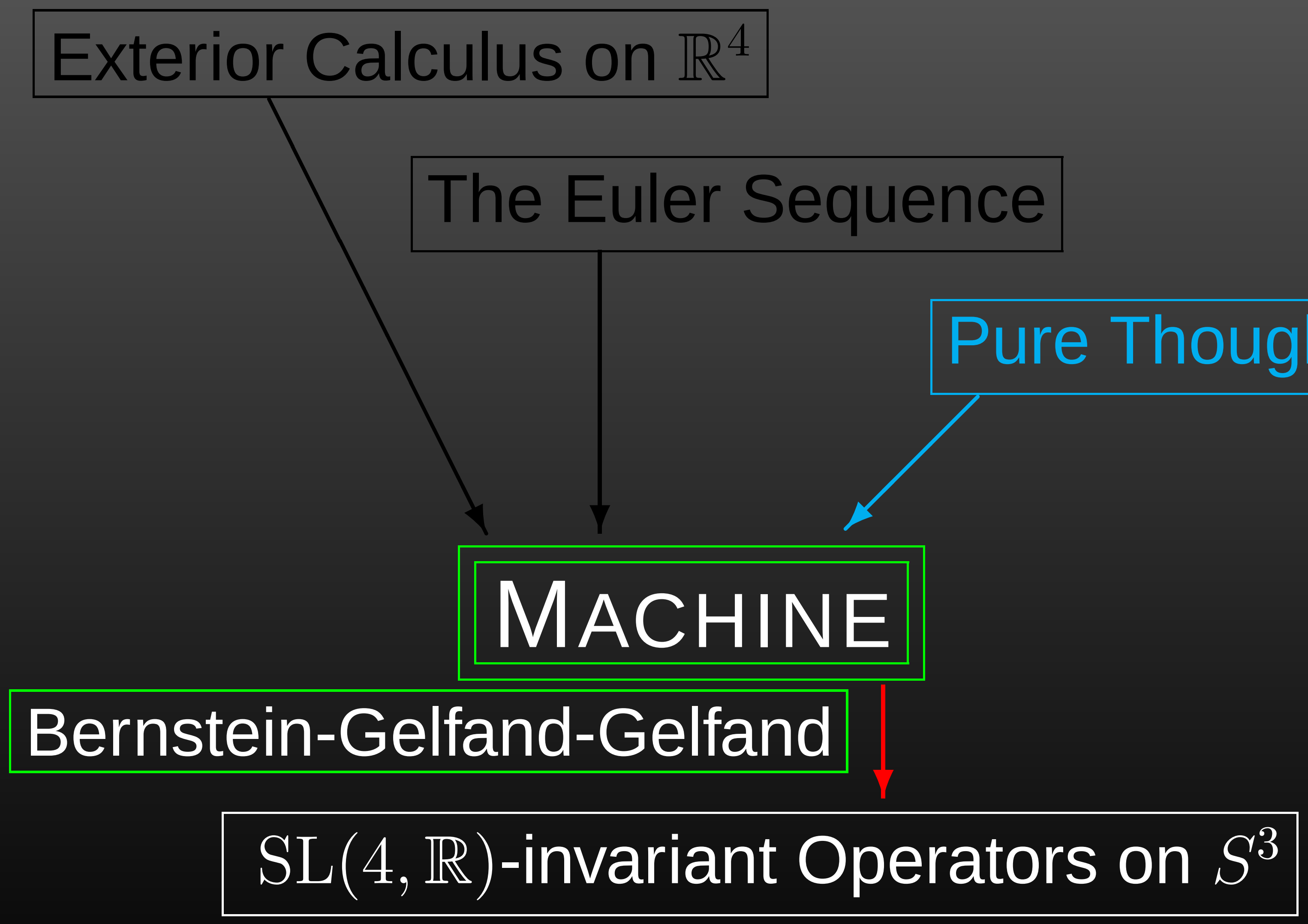
$\Rightarrow x^i \nabla_i f_j = f_j$ and $x^i f_i = 0$. Therefore,

$$\begin{aligned} x^i \nabla_{(i} f_{j)} &= \tfrac{1}{2}(x^i \nabla_i f_j + \nabla_j (x^i f_i) - (\nabla_j x^i) f_i) \\ &= \tfrac{1}{2}(f_j + 0 - f_j) \;=\; 0. \end{aligned}$$

But $0 \to \bigodot^2 \Lambda^1(2) \to \mathcal{E}_{(ij)} \xrightarrow{x^i} \mathcal{E}_j(1) \to 0$ is exact

and so $\boxed{\nabla : \Lambda^1(2) \to \bigodot^2 \Lambda^1(2)}$ is invariant.

Moral



Exterior Calculus on $\mathbb{R}^4$
The Euler Sequence
Pure Thought
MACHINE
Bernstein-Gelfand-Gelfand
$\mathrm{SL}(4,\mathbb{R})$-invariant Operators on S^3

Classification!

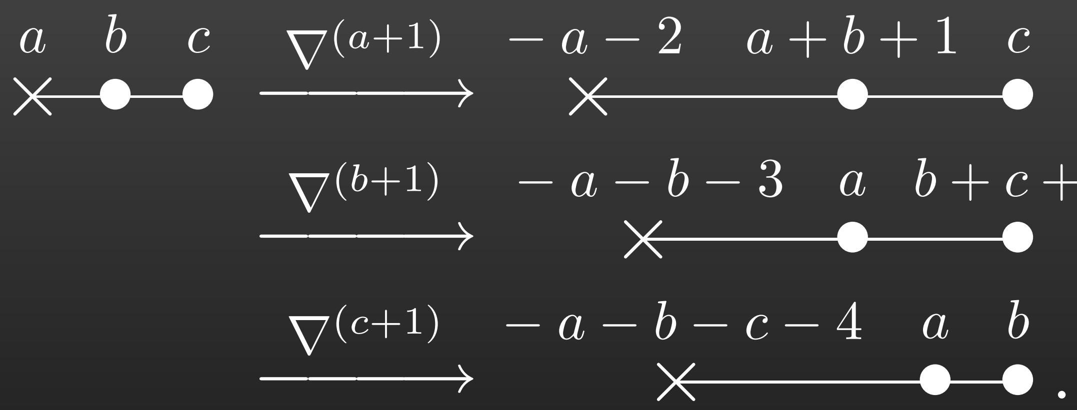
Theorem We've found all $\mathrm{SL}(4,\mathbb{R})$-invariant linear differential operators on projective S^3: if $D : E \to F$ is a non-constant invariant linear differential operator between irreducible homogeneous bundles, then

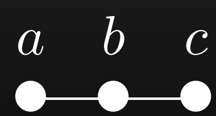
- D may be constructed by our machine
- there is an explicit list of such D.

Best proof $\leftrightarrow$ representation theory of $\mathfrak{sl}(4,\mathbb{R})$.

The list

For non-negative integers a, b, c, there are invariant operators



- All invariant operators are captured (uniquely)
- the kernel of the first one is 

- it's locally exact! (BGG resolution)

Examples

$a = 0,\ b = 0,\ c = 0 \rightsquigarrow$ de Rham

$$\begin{array}{ccccccccccc}
0\longrightarrow & \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} & \longrightarrow & \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow 0 \\
& \| & & \| & & \| & & \| & & \| & \\
0\longrightarrow & \mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \longrightarrow 0
\end{array}$$

$a = 0,\ b = 1,\ c = 0 \rightsquigarrow$ linear elasticity!

$$\begin{array}{ccccccccccc}
0\longrightarrow & \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet} & \longrightarrow & \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet} & \longrightarrow & \overset{-5}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} & \longrightarrow 0 \\
& \| & & \| & & \| & & & & & \\
0\longrightarrow & \Lambda^2\mathbb{R}^4 & \longrightarrow & \Lambda^1(2) & \xrightarrow{\nabla} & \bigodot^2\Lambda^1(2) & \xrightarrow{\nabla^{(2)}} & \cdots & & &
\end{array}$$

What's happening on $\mathbb{R}^3 \hookrightarrow S^3$?

de Rham

$$f \mapsto \nabla_a f \qquad f_a \mapsto \nabla_{[a} f_{b]} \qquad f_{ab} \mapsto \nabla_{[a} f_{bc]}$$

$$f \xrightarrow{\text{grad}} \nabla_a f \quad f_a \xrightarrow{\text{curl}} \epsilon^{abc} \nabla_b f_c \quad f^a \xrightarrow{\text{div}} \nabla_a f^a$$

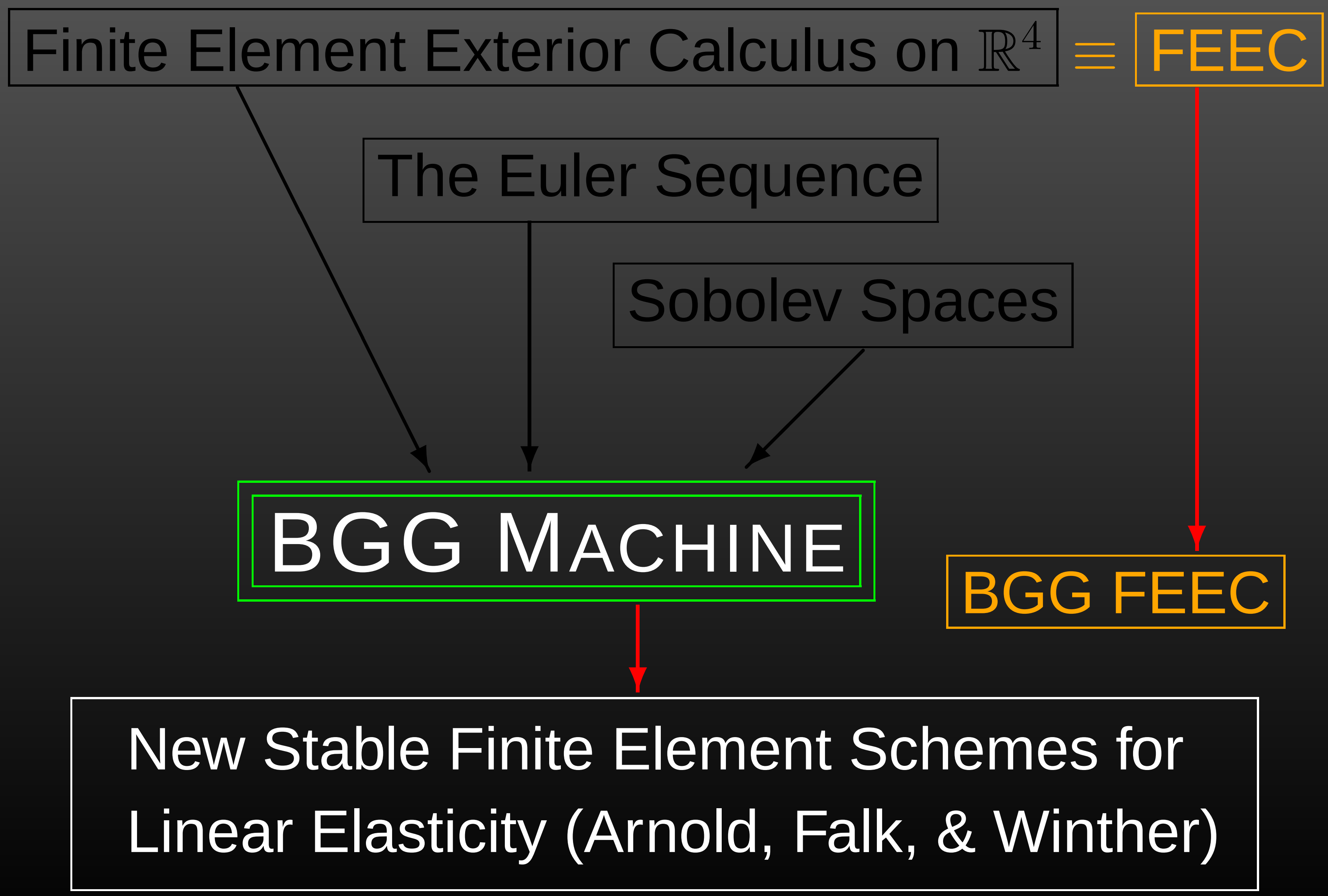
linear elasticity $=$ Riemannian deformation

$$\phi_a \mapsto \nabla_{(a} \phi_{b)} \quad \phi_{ab} \mapsto \epsilon^{ace} \epsilon^{bdf} \nabla_c \nabla_d \phi_{ef} \quad \phi^{ab} \mapsto \nabla_b \phi^{ab}$$

displacement $\mapsto$ strain $\mapsto$ stress $\mapsto$ load

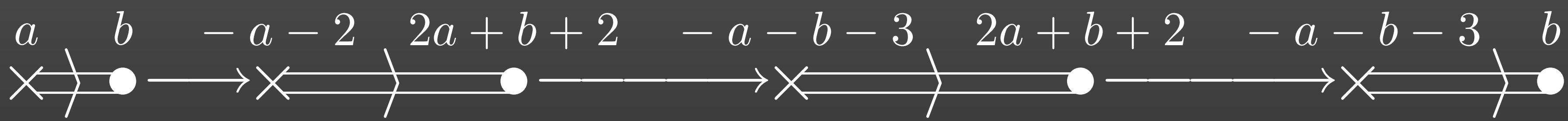
cöord change $\xrightarrow{\text{Killing}}$ metric $\mapsto$ curvature $\xrightarrow{\text{Bianchi}}$

Application to numerical analysis

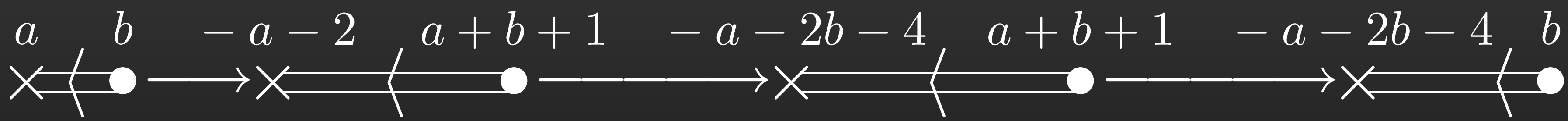


More BGG complexes on 3-sphere

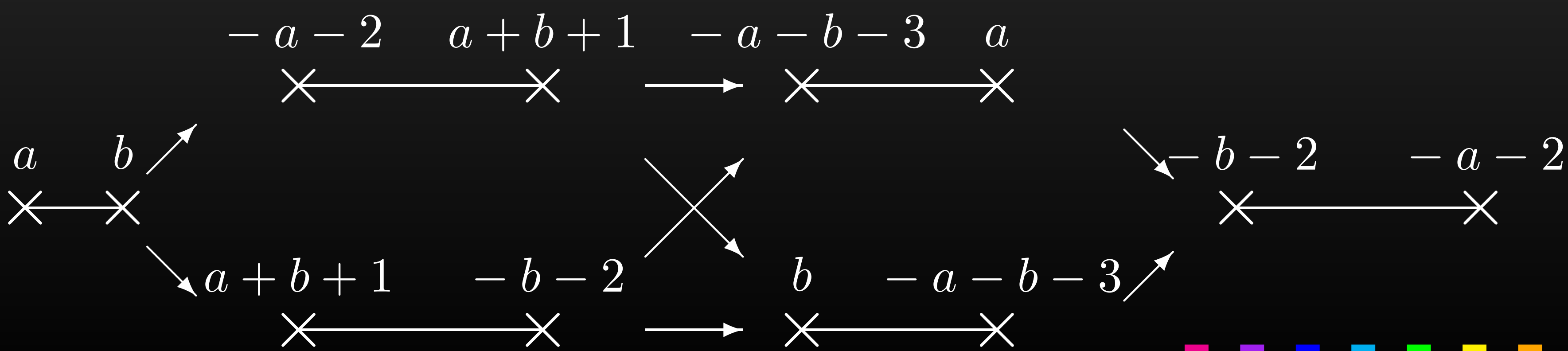
Conformal



Contact projective



CR



What's going on here?

- Representation theory of simple Lie groups.
- G is simple. P is parabolic.
- The (affine) action of the Weyl group of G.
- Hasse diagrams.
- Central character.
- The Jantzen-Zuckerman translation principle.
- Verma modules.
- Geometric interpretation.
- Parabolic geometry: Čap, Slovák, Souček,. . .

Further Reading

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- A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, Ann. Math. 154 (2001) 97–113.
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- A. Čap and J. Slovák, Parabolic Geometries 1, Surveys vol. 154, AMS 2009.

THANK YOU