Homogeneous hypersurfaces

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[based on joint work with Vladimir Ezhov]

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A classification problem

$$\Sigma \hookrightarrow G/H$$

where $G/H =$ your favourite homogeneous space

Examples of G/H

$\mathbb{R}^n = (\mathrm{SO}(n) \ltimes \mathbb{R}^n)/\mathrm{SO}(n) =$ Euclidean space

$\mathbb{R}^n = (\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n)/\mathrm{GL}(n, \mathbb{R}) =$ affine space

$\mathbb{RP}_n = \mathrm{SL}(n+1, \mathbb{R})/P =$ real projective space

$S^n = \mathrm{SO}(n+1, 1)/P =$ conformal space

$\mathbb{C}^n = (\mathrm{GL}(n, \mathbb{C}) \ltimes \mathbb{C}^n)/\mathrm{GL}(n, \mathbb{C}) =$ complex affine space

Today: real or complex (equi-)affine space

Curves in the real affine plane

- straight line
- circle (or ellipse)
- parabola
- $Y = X^\alpha$
- $r = e^{\alpha\theta}$ (spira mirabilis (Jacob Bernoulli))
- $Y = e^X$
- $Y = X \log X$

Normal forms $\quad y = 0 \quad y = x^2 \quad y = x^2 \pm x^4 + bx^5 + \cdots$

$Y = X \log X \rightsquigarrow \quad y = x^2 + x^4 + \frac{8}{5}x^5 + \cdots$

$b^2 =$ invariant

$Y = e^X \rightsquigarrow \quad y = x^2 - x^4 + \frac{2\sqrt{2}}{5}x^5 + \cdots$

Non-degenerate equi-affine surfaces in $\mathbb{C}^3$

- $Z = XY$
- $Z^2 = XY + 1$
- $XYZ = 1$
- $X^2(Z + Y^2)^3 = 1$
- $Z = XY + X^3$ (Cayley surface)

} Guggenheimer 1963

- $Z = XY + \log X$ Nomizu & Sasaki 1991

Equi-affine symmetries of the Cayley surface

$$\begin{aligned} X &\mapsto X & \quad X &\mapsto X + s \\ Y &\mapsto Y + t & \quad Y &\mapsto Y - 3sX - \tfrac{3}{2}s^2 \\ Z &\mapsto Z + tX & \quad Z &\mapsto Z + sY - \tfrac{3}{2}s^2X - \tfrac{1}{2}s^3 \end{aligned}$$

(Abelian)

Infinitesimal symmetries

Cayley surface $z = xy + x^3$

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y + t \\ z &\mapsto z + tx \end{aligned} \qquad x\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x &\mapsto x + s \\ y &\mapsto y - 3sx - \tfrac{3}{2}s^2 \\ z &\mapsto z + sy - \tfrac{3}{2}s^2x - \tfrac{1}{2}s^3 \end{aligned} \qquad -3x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \qquad \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{z = xy + \cdots = F(x,y)}$$

$$\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -1\right]\left[M\begin{bmatrix} x \\ y \\ F(x,y) \end{bmatrix} + \beta\right] = 0$$

symmetry algebra $\subset \mathfrak{g}$

Normal forms

Non-degenerate

$$z = x^2 + y^2 + \cdots \qquad \text{or} \qquad z = xy + \cdots$$

Blaschke normal (Leichtweiß1989)

$$z = g_{ij}x^i x^j + a_{ijk}x^i x^j x^k + \cdots$$

$$x^i \mapsto x^i + r^i z \quad \rightsquigarrow \quad a^j{}_{jk} \mapsto a^j{}_{jk} + \tfrac{2(n+2)}{3} r_k \quad \rightsquigarrow \quad \boxed{a_{ijk} \text{ trace-free}}$$

Pick invariant

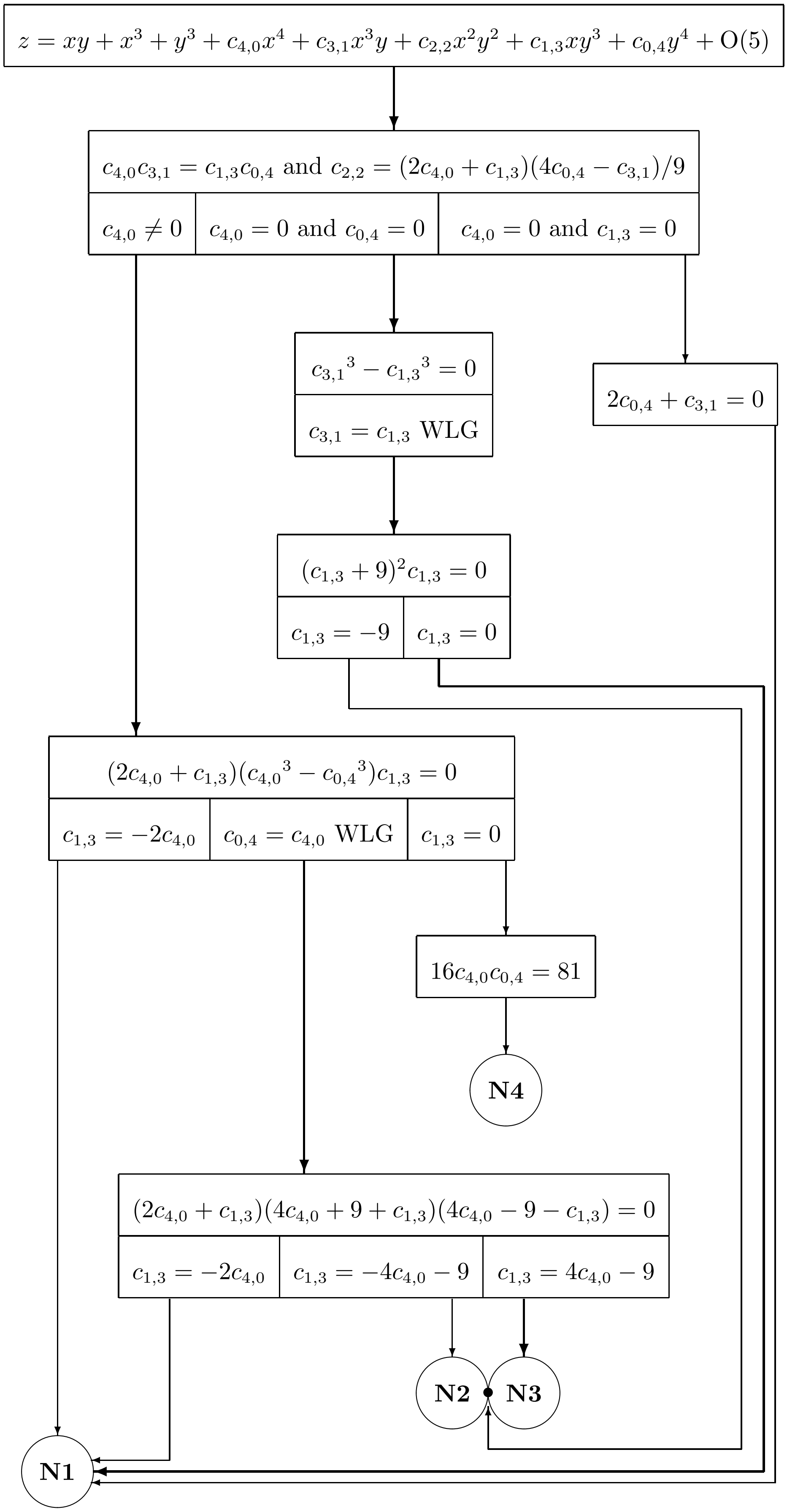
$$a_{ijk}a^{ijk} \qquad \text{e.g.} \quad z = xy + ax^3 + by^3 + \cdots \quad \rightsquigarrow \quad ab$$

$$\therefore \quad z = xy + x^3 + y^3 + \cdots \qquad \text{or} \qquad z = xy + x^3 + \cdots$$

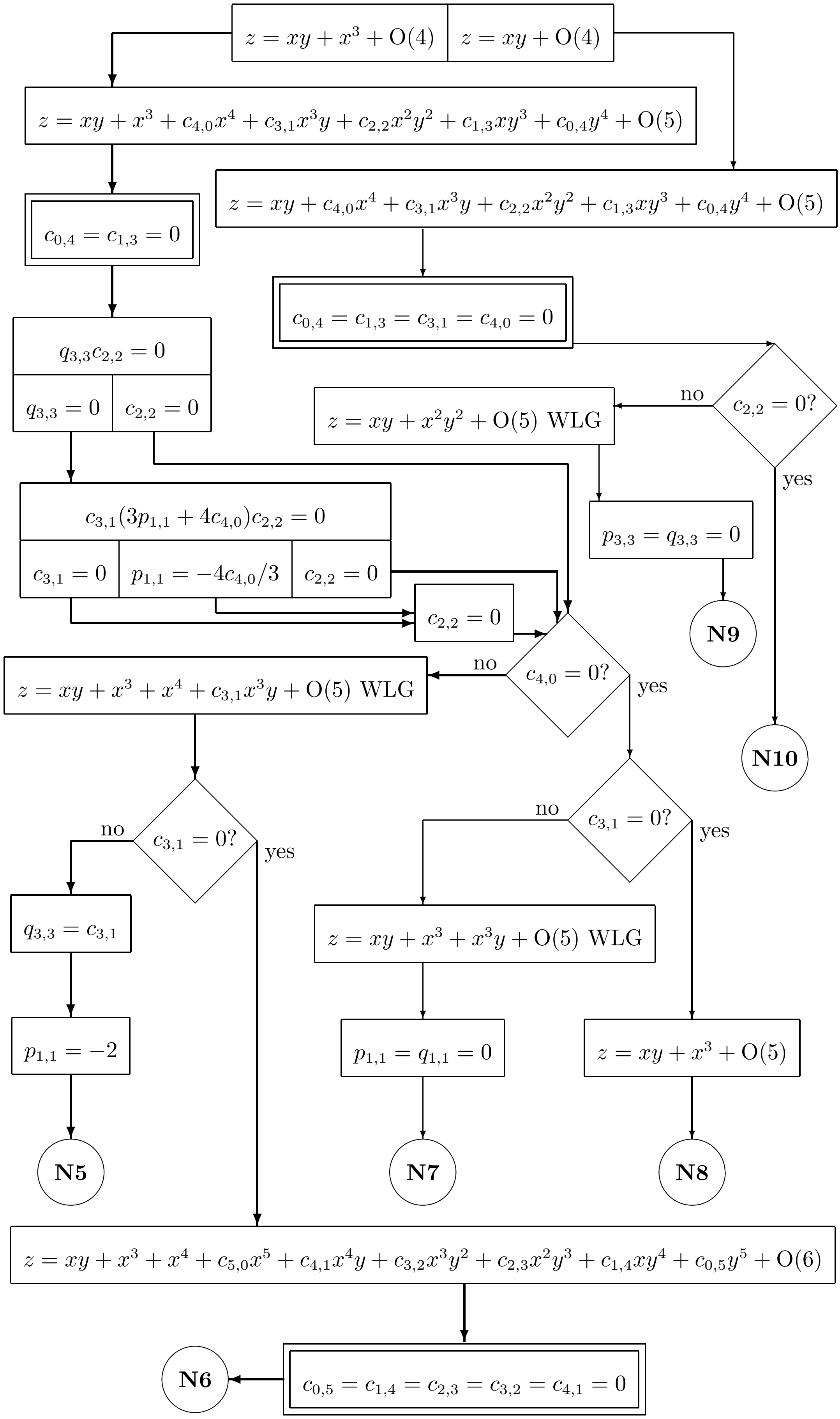
Affine homogeneous surfaces in $\mathbb{C}^3$

N1	$z = xy + x^3 + y^3 + ax^4 - 2bx^3y - 2axy^3 + by^4 + O(5)$
N2	$z = xy + x^3 + y^3$ $+bx^4 - (4b+9)x^3y - \frac{1}{9}(2b+9)(8b+9)x^2y^2 - (4b+9)xy^3 + by^4 + O(5)$
N3	$z = xy + x^3 + y^3$ $+bx^4 + (4b-9)x^3y + (6b-9)x^2y^2 + (4b-9)xy^3 + by^4 + O(5)$
N4	$z = xy + x^3 + y^3 + \frac{9}{4}bx^4 + \frac{9}{2}x^2y^2 + \frac{9}{4}(1/b)y^4 + O(5)$ where $b \neq 0$
N5	$z = xy + x^3 + x^4 + bx^3y + O(5)$ where $b \neq 0$
N6	$z = xy + x^3 + x^4 + bx^5 + O(6)$
N7	$z = xy + x^3 + x^3y + O(5)$ (the Archimedean screw)
N8	$z = xy + x^3 + O(5) = xy + x^3$ (the Cayley surface)
N9	$z = xy + x^2y^2 + O(5) = (1 - \sqrt{1-4xy})/2$ (hyperboloid)
N10	$z = xy + O(5) = xy$
D1	$z = x^2 + x^2y + x^3y + x^2y^2 + x^5 + 3x^3y^2 + x^2y^3 + O(6)$ ($\leftarrow$ twisted cubic)
D2	$z = x^2 + x^4 + ax^5 + O(6)$ = a function of x alone (cylinder)
D3	$z = x^2 + x^2y + x^2y^2 + x^5 + x^2y^3 + 4x^5y + x^2y^4 + ax^7 + 10x^5y^2 + x^2y^5 + O(8)$ (cone)
D4	$z = x^2 + x^2y + x^2y^2 + x^2y^3 + O(6) = x^2/(1-y)$ (cone)
D5	$z = x^2 + O(5) = x^2$ (cylinder)
F	$z = O(3) = 0$

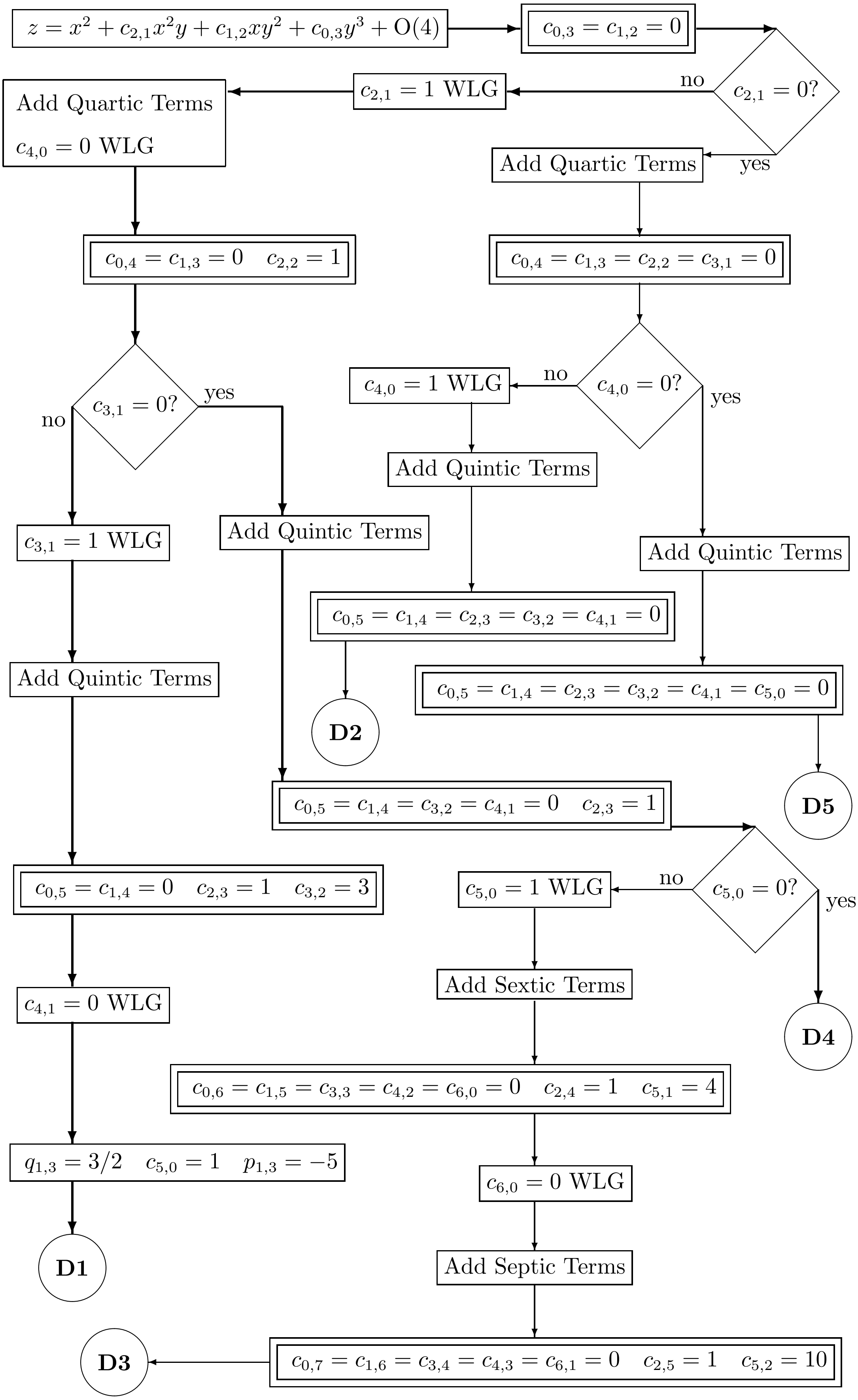
Non-degenerate with non-vanishing Pick invariant



Non-degenerate with vanishing Pick invariant



Degenerate



$z = x^2 + c_{2,1}x^2y + c_{1,2}xy^2 + c_{0,3}y^3 + \mathrm{O}(4)$
$c_{0,3} = c_{1,2} = 0$
$c_{2,1} = 0?$
no
yes
$c_{2,1} = 1$ WLG
Add Quartic Terms
$c_{4,0} = 0$ WLG
Add Quartic Terms
$c_{0,4} = c_{1,3} = 0 \quad c_{2,2} = 1$
$c_{0,4} = c_{1,3} = c_{2,2} = c_{3,1} = 0$
$c_{3,1} = 0?$
no
yes
$c_{4,0} = 0?$
no
yes
$c_{4,0} = 1$ WLG
Add Quintic Terms
$c_{3,1} = 1$ WLG
Add Quintic Terms
Add Quintic Terms
$c_{0,5} = c_{1,4} = c_{2,3} = c_{3,2} = c_{4,1} = 0$
Add Quintic Terms
$c_{0,5} = c_{1,4} = c_{2,3} = c_{3,2} = c_{4,1} = c_{5,0} = 0$
D2
D5
$c_{0,5} = c_{1,4} = c_{3,2} = c_{4,1} = 0 \quad c_{2,3} = 1$
$c_{0,5} = c_{1,4} = 0 \quad c_{2,3} = 1 \quad c_{3,2} = 3$
$c_{5,0} = 0?$
no
yes
$c_{5,0} = 1$ WLG
Add Sextic Terms
D4
$c_{4,1} = 0$ WLG
$c_{0,6} = c_{1,5} = c_{3,3} = c_{4,2} = c_{6,0} = 0 \quad c_{2,4} = 1 \quad c_{5,1} = 4$
$q_{1,3} = 3/2 \quad c_{5,0} = 1 \quad p_{1,3} = -5$
$c_{6,0} = 0$ WLG
D1
Add Septic Terms
D3
$c_{0,7} = c_{1,6} = c_{3,4} = c_{4,3} = c_{6,1} = 0 \quad c_{2,5} = 1 \quad c_{5,2} = 10$

Non-degenerate equi-affine hypersurfaces in $\mathbb{C}^4$

	Equation	Basepoint
I1	$W = XY + Z^2$	$(0,0,0,0)$
I2	$W^2 = XY + Z^2 + 1$	$(1,0,0,0)$
I3	$W = XY + Z^2 + X^3$	$(0,0,0,0)$
I4	$W = XY + Z^2 + \log X$	$(0,1,0,0)$
I5	$W(XY + Z)^2 = 1$	$(1,0,0,1)$
I6	$W^2(XY + Z^2)^3 = 1$	$(1,0,0,1)$
A1	$WXYZ = 1$	$(1,1,1,1)$
A2	$WZ + WY^2 + X^2Z + X^2Y^2 = 1$	$(1,0,0,1)$
A3	$6WXYZ - 4X^3Z - 3X^2Y^2 + W^2Z^2 + 4WY^3 = 1$	$(1,0,0,1)$
A4	$W = XY + Z^2 + X^2Z + \alpha X^4$	$(0,0,0,0)$

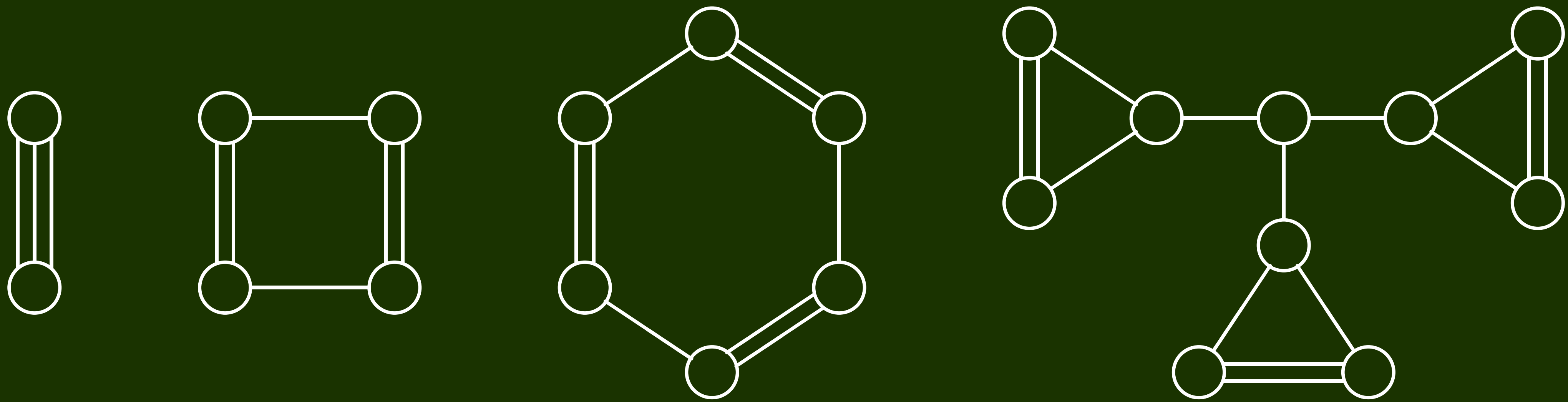
Non-degenerate equi-affine hypersurfaces in $\mathbb{C}^4$

	Equation	Basepoint
B1	$WZ^2 = Z + X^3 + XYZ$	$(1,0,0,1)$
B2	$(W + XY + X^3)^2 Z = 1$	$(1,0,0,1)$
B3	$(WZ + Y^2 + X^2 Z)^4 = Z$	$(1,0,0,1)$
B4	$(W^2 + X^2)^2 (Z + Y^2)^3 = 1$	$(1,0,0,1)$
B5	$(W + X^2 Z + XY)^2 Z = 1$	$(1,0,0,1)$
B6	$(W + XY)^2 (Z + X^2) = 1$	$(1,0,0,1)$
B7	$WZ^2 = Z + X^2 + XYZ$	$(1,0,0,1)$
B8	$(W + YZ + X^2 Z)^5 = Z$	$(1,0,0,1)$
B9	$(WZ + X^2 + YZ^2)^5 = Z^4$	$(1,0,0,1)$
B10	$W = XY + Z^2 + XZ^2$	$(0,0,0,0)$
B11	$W^2 = XY + X^2 Y + X^2 Z$	$(1,1,0,1)$

Invariants

Pick invariant $= a_{ijk}a^{ijk} \qquad a_{ijk}a^{ijl}a_{mnl}a^{mnk} \qquad \cdots$

$$\|$$



THE END

THANK YOU