H-projective geometry - part 8

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[following the work of others]

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Almost complex structures

$\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n,\mathbb{R})$ by

$$\mathrm{GL}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(2n,\mathbb{R}) \text{ s.t. } M\mathbb{J} = \mathbb{J}M\},$$

where $\mathbb{J}$ is the $2n \times 2n$ matrix

$$\left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right).$$

Complexify

$$\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \cong \mathrm{GL}(n,\mathbb{C})^{\mathbb{C}} \hookrightarrow \mathrm{GL}(2n,\mathbb{C})$$

$$\mathbb{C}^{2n} \leftrightarrow \mathbb{C}T = T^{1,0} \oplus T^{0,1} = \{v \text{ s.t. } Jv = iv\} \oplus \{v \text{ s.t. } Jv = -iv\}.$$

Covering

 subgroup of $\mathrm{SL}(n+1,\mathbb{C})$

$$\left(\begin{array}{c|c} \lambda & 0\cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

Homomorphism  $\to \mathrm{GL}(n,\mathbb{C})$ given by

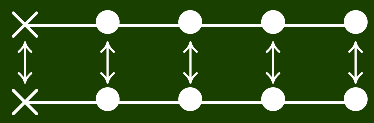
$$\left(\begin{array}{c|c} \lambda & 0\cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) \longmapsto \lambda^{-1}B.$$

It is an $(n+1)$-fold covering.

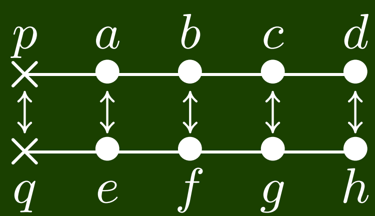
Representations

$$\mathrm{GL}(n,\mathbb{C})^{\mathbb{C}} \cong \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$$

Lie algebra


,

and decorate



$$5p + 4a + 3b + 2c + d \equiv 5q + 4e + 3f + 2g + h \bmod 6.$$

External tensor product

but keeps track of real structure.

Conjugate

$$\text{Conjugate of}\quad \begin{array}{ccccc} p & a & b & c & d \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ q & e & f & g & h \end{array} \quad\text{is}\quad \begin{array}{ccccc} q & e & f & g & h \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ p & a & b & c & d \end{array}.$$

Prototype

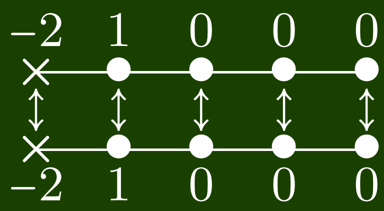
$$\mathbb{C}T = T^{1,0} \oplus T^{0,1} = \begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{array}.$$

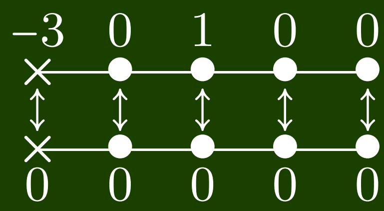
Dual

$$\Lambda^1 = \Lambda^{0,1} \oplus \Lambda^{1,0} = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ -2 & 1 & 0 & 0 & 0 \end{array} \oplus \begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

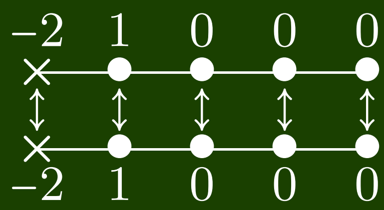
2-tensors

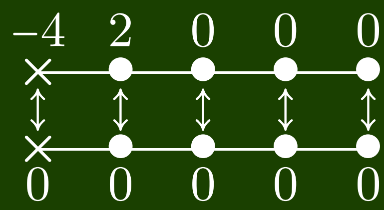
$$\Lambda^2 = \Lambda^{1,1} \oplus \left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right)$$

$\Lambda^2 =$ 

$\oplus \Big($ 

$\oplus$ Cc $\Big)$.

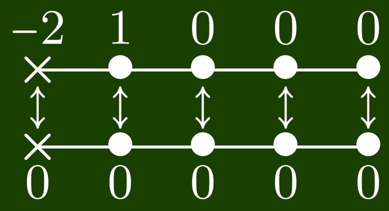
$\bigodot^2 \Lambda^1 =$ 

$\oplus \Big($ 

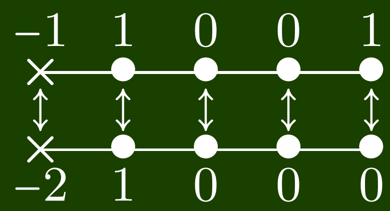
$\oplus$ Cc $\Big)$.

Torsion

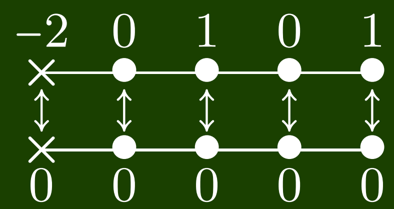
$\Lambda^2 \otimes T$ decomposes into five $\mathbb{R}$-irreducibles

$2 \times \Big($ 

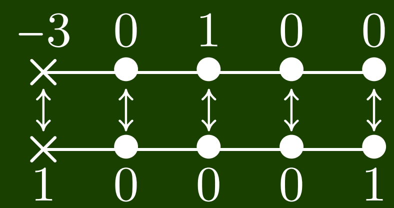
$\oplus$ Cc $\Big) \oplus$

$\Big($ 

$\oplus$ Cc $\Big) \oplus$

$\Big($ 

$\oplus$ Cc $\Big) \oplus$

$\Big($ 

$\oplus$ Cc $\Big)$

Complex connections

$\Lambda^1 \otimes \mathfrak{gl}(n,\mathbb{C})$ decomposes into six $\mathbb{R}$-irreducibles

$$3 \times \left(\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \text{Cc} \right) \oplus$$

$$\left(\begin{array}{ccccc} -1 & 1 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ -2 & 1 & 0 & 0 & 0 \end{array} \oplus \text{Cc} \right) \oplus$$

$$\left(\begin{array}{ccccc} -2 & 0 & 1 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \text{Cc} \right) \oplus$$

$$\left(\begin{array}{ccccc} -3 & 2 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \text{Cc} \right)$$

Compare

$$0 \to \left(\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathbb{C}c \right) \oplus \left(\begin{array}{ccccc} -3 & 2 & 0 & 0 & 1 \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathbb{C}c \right) \to$$

$$\Lambda^1 \otimes \mathfrak{gl}(n, \mathbb{C}) \xrightarrow{\partial} \Lambda^2 \otimes TM \to$$

$$\left(\begin{array}{ccccc} -3 & 0 & 1 & 0 & 0 \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{array} \oplus \mathbb{C}c \right) \to 0,$$

where $\Gamma_{ab}{}^c \overset{\partial}{\longmapsto} \Gamma_{[ab]}{}^c$.

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet\!\!-\!\!- & \bullet \\ -3 & 0 & 1 & 0 & 0 \end{array} \oplus \mathbb{C}c \right) \ni \text{ the Nijenhuis tensor}$$

H-projective geometry

$$0 \to \left(\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathrm{Cc} \right) \oplus \left(\begin{array}{ccccc} -3 & 2 & 0 & 0 & 1 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathrm{Cc} \right) \to$$

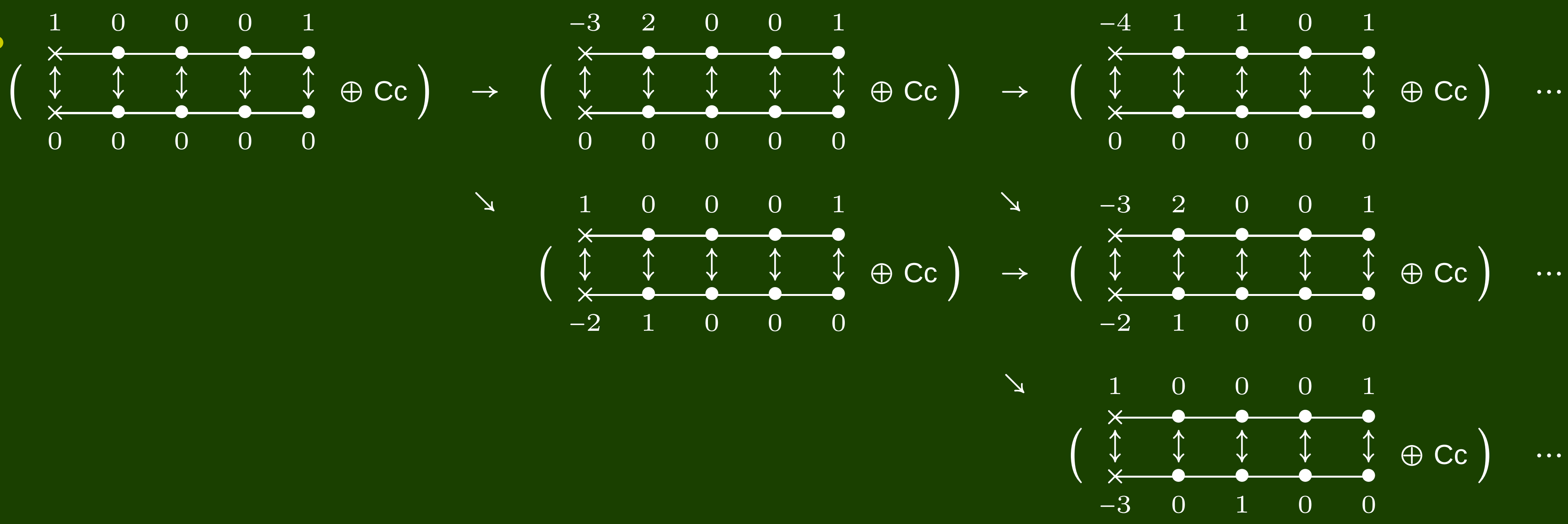
$$\Lambda^1 \otimes \mathfrak{gl}(n, \mathbb{C}) \xrightarrow{\partial} \Lambda^2 \otimes TM \to$$

$$\left(\begin{array}{ccccc} -3 & 0 & 1 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 1 & 0 & 0 & 0 & 1 \end{array} \oplus \mathrm{Cc} \right) \to 0,$$

where $\Gamma_{ab}{}^c \stackrel{\partial}{\longmapsto} \Gamma_{[ab]}{}^c$.

$$\left(\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ \times & \bullet & \bullet & \bullet & \bullet \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \times & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{array} \oplus \mathrm{Cc} \right) \ni \text{ H-projective freedom}$$

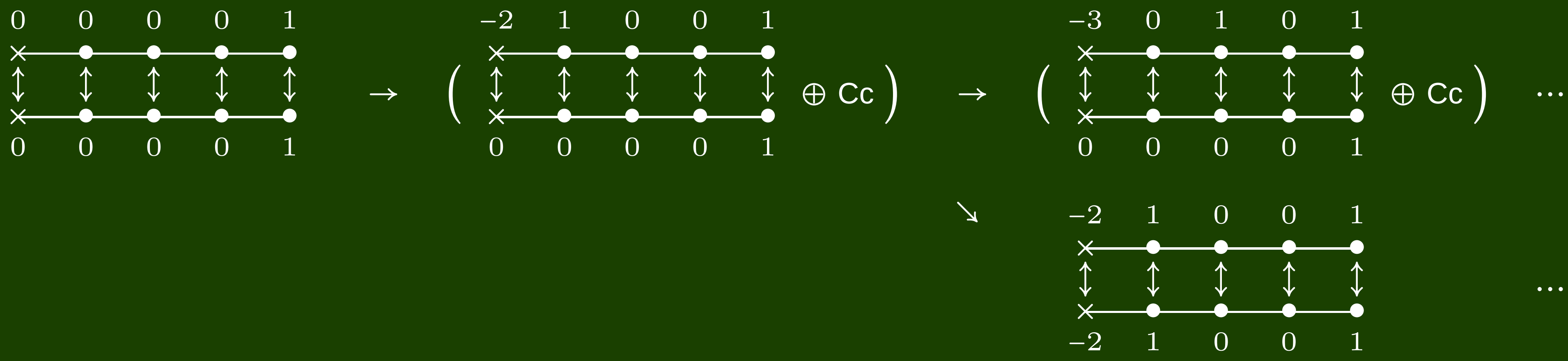
Deformation complex



$\updownarrow$ $\qquad$ $\updownarrow$ $\qquad$ $\updownarrow$

vector fields $\longrightarrow$ infinitesimal deformations $\longrightarrow$ harmonic curvature.

H-mobility complex



THE END

THANK YOU