

Complexes of differential operators

Michael Eastwood

[based on joint work with Hubert Goldschmidt,
and with Rod Gover and Katharina Neusser]

Australian National University

De Rham complex

in \mathbb{R}^3

$$f \xrightarrow{\text{grad}} \nabla_i f \quad \omega_i \xrightarrow{\text{curl}} \epsilon_i^{jk} \nabla_j \omega_k \quad \phi_i \xrightarrow{\text{div}} \nabla^i \phi_i$$

on a smooth manifold

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n \rightarrow 0$$

Locally exact

$$\Gamma(U, \Lambda^{p-1}) \xrightarrow{d} \Gamma(U, \Lambda^p) \xrightarrow{d} \Gamma(U, \Lambda^{p+1}) \quad \text{is exact} \quad p \geq 1$$
$$\ker : \Gamma(U, \Lambda^0) \xrightarrow{d} \Gamma(U, \Lambda^1) = \mathbb{R}$$

Rumin complex

in \mathbb{R}^3

X, Y vector fields. Suppose $X, Y, Z \equiv [X, Y]$ span.

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant. Let $H \equiv \text{span}\{X, Y\}$.

on a contact manifold

$$H \subset TM \quad \leftrightarrow \quad \Lambda^1 \rightarrow \Lambda^1_H$$

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathbb{R} & \rightarrow & \Lambda^0 & \rightarrow & \Lambda^1 \\
 & & & & & \searrow & \downarrow \\
 & & & & & & \Lambda^1_H
 \end{array}$$

defines $d_H : \Lambda^0 \rightarrow \Lambda^1_H$
s.t. $\mathbb{R} = \ker : \Lambda^0 \xrightarrow{d_H} \Lambda^1_H$

locally

Darboux $\leadsto [X, Z] = 0 = [Y, Z]$ wlg

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \Rightarrow \begin{cases} XYg - X^2h + Zg = 0 \\ YXh - Y^2g - Zh = 0 \end{cases} \quad \text{conversely? } \boxed{\text{yes!}}$$

Rumin complex cont'd

$$\begin{array}{ccccccccc}
 \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \xrightarrow{d} & \Lambda^4 & \xrightarrow{d} & \Lambda^5 \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 \Lambda^0 & & \Lambda^1_H & & \Lambda^2_H & & \Lambda^3_H & & \Lambda^4_H & & \\
 & & + & \nearrow \text{inje} & + & \nearrow \text{isom} & + & \nearrow \text{surj} & + & & \\
 & & L & & \Lambda^1_H \otimes L & & \Lambda^2_H \otimes L & & \Lambda^3_H \otimes L & & \Lambda^4_H \otimes L
 \end{array}$$

Diagram chase (spectral sequence) \rightsquigarrow

$$\begin{array}{ccccccccc}
 \Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H & \xrightarrow{d_H} & \Lambda^2_{H^\perp} & \xrightarrow{d_H^{(2)}} & \Lambda^2_{H^\perp} \otimes L & \xrightarrow{d_H} & \Lambda^3_H \otimes L & \xrightarrow{d_H} & \Lambda^5 \\
 \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
 \begin{array}{c} 0 \quad 0 \quad 0 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} & & \begin{array}{c} -2 \quad 1 \quad 0 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} & & \begin{array}{c} -3 \quad 0 \quad 1 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} & & \begin{array}{c} -5 \quad 0 \quad 1 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} & & \begin{array}{c} -6 \quad 1 \quad 0 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} & & \begin{array}{c} -6 \quad 0 \quad 0 \\ \times \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{---} \end{array} \\
 \dim = 1 & & \dim = 4 & & \dim = 5 & & \dim = 5 & & \dim = 4 & & \dim = 1
 \end{array}$$

Engel complex

in \mathbb{R}^4 X, Y vector fields. Suppose

- $X, Y, Z \equiv [X, Y], W \equiv [Y, Z]$ span,
- (Engel \leadsto wlg) all other commutators vanish.

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant.

on an Engel manifold $H \subset TM \iff \Lambda^1 \twoheadrightarrow \Lambda^1_H$

$$\left. \begin{array}{l} Xf = g \\ Yf = h \end{array} \right\} \iff \left\{ \begin{array}{l} XYg - X^2h + Zg = 0 \\ Y^2Xh - Y^3g - YZh - Wh = 0 \end{array} \right. \quad \text{locally}$$

Complex $0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \rightarrow \Lambda^1_H \rightarrow \Delta^2 \rightarrow \Delta^3 \rightarrow \Lambda^4 \rightarrow 0$

ranks $1 \quad 2 \quad 2 \quad 2 \quad 1$

Five variables

in \mathbb{R}^5

X, Y vector fields. Suppose

- $\underbrace{X, Y, Z \equiv [X, Y], W \equiv [Y, Z], V \equiv [X, Z]}_{\text{span.}}$
- ~~(Engel) all other commutators vanish.~~

NB $Xf = 0, Yf = 0 \Rightarrow f$ constant.

$$H \subset TM \quad \Leftrightarrow \quad \Lambda^1 \rightarrow \Lambda^1_H \quad \rightsquigarrow \quad \Lambda^1 = \Lambda^1_H + \Lambda^2_H + \Lambda^1_H \otimes \Lambda^2_H$$

$$\begin{array}{cccccc}
 \Lambda^0 & \xrightarrow{d_H} & \Lambda^1_H & \xrightarrow{d_H^{(3)}} & \odot^2 \Lambda^1_H \otimes \Lambda^2_H & \xrightarrow{d_H^{(2)}} & \Delta^3 & \xrightarrow{d_H^{(3)}} & \Delta^4 & \xrightarrow{d_H} & \Lambda^5 \\
 \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
 \begin{array}{c} 0 \quad 0 \\ \text{---} \\ \bullet \end{array} & & \begin{array}{c} -2 \quad 1 \\ \text{---} \\ \bullet \end{array} & & \begin{array}{c} -5 \quad 2 \\ \text{---} \\ \bullet \end{array} & & \begin{array}{c} -6 \quad 2 \\ \text{---} \\ \bullet \end{array} & & \begin{array}{c} -6 \quad 1 \\ \text{---} \\ \bullet \end{array} & & \begin{array}{c} -6 \quad 0 \\ \text{---} \\ \bullet \end{array} \\
 \dim = 1 & & \dim = 2 & & \dim = 3 & & \dim = 3 & & \dim = 2 & & \dim = 1
 \end{array}$$

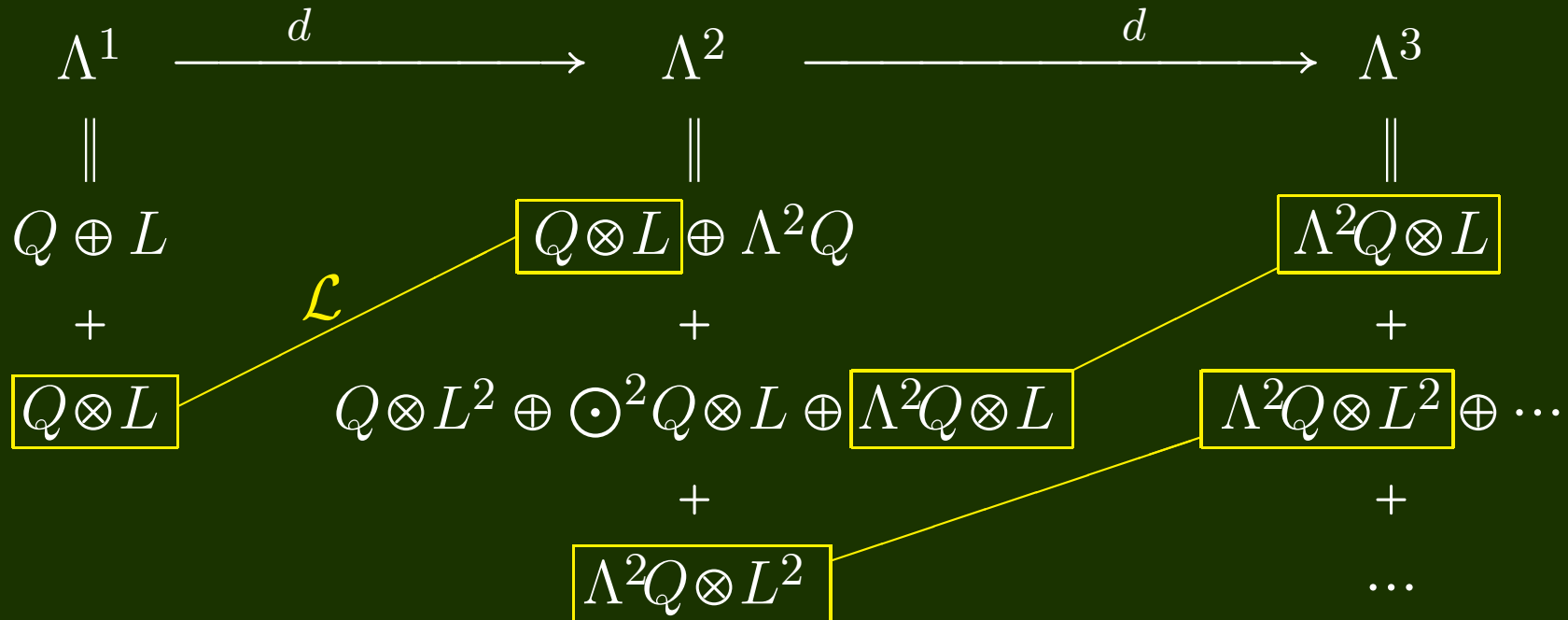


$\dim M = 5, TM \supset H, \text{rank } H = 3.$ Suppose $[H, H] = TM.$

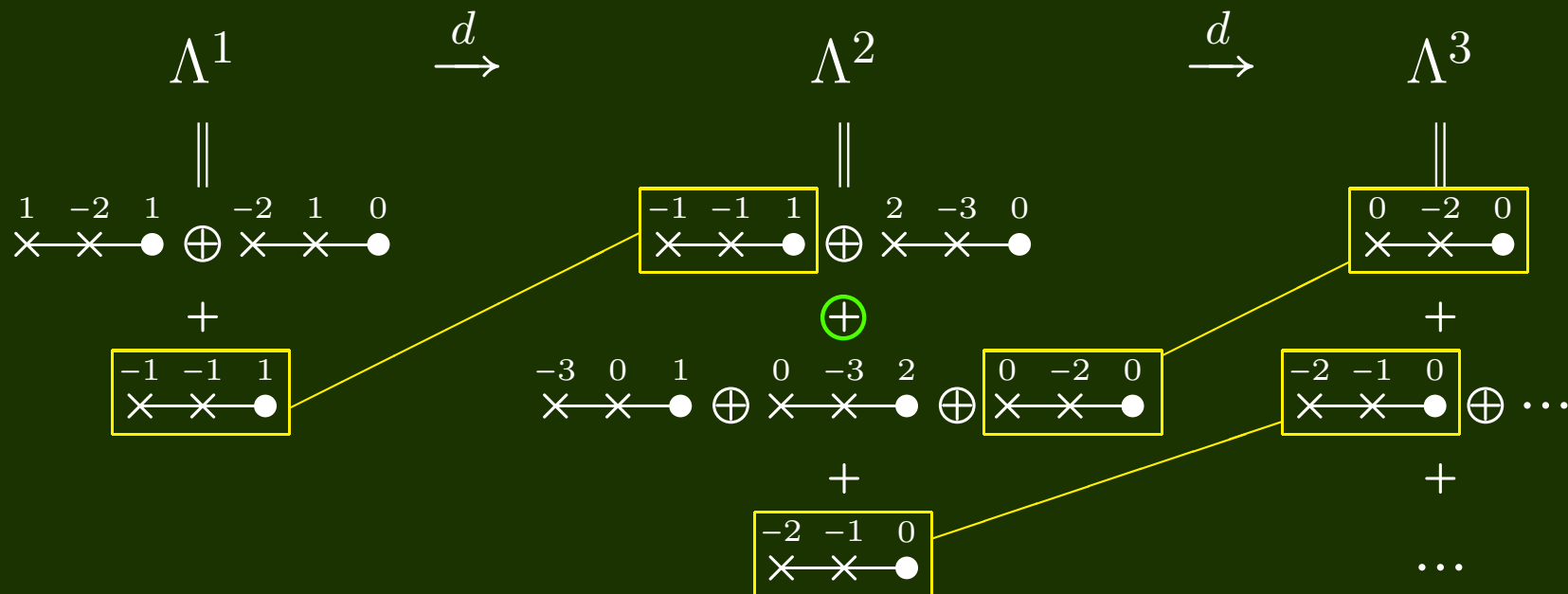
$$K \equiv \ker : \Lambda^1 \rightarrow \Lambda^1_H \quad K \rightarrow \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \Lambda^2_H \quad \mathcal{L}(K) \subset \Lambda^2_H$$

$$D \subset H \quad \text{defined by } H = \Lambda^3 H \otimes \Lambda^2_H$$

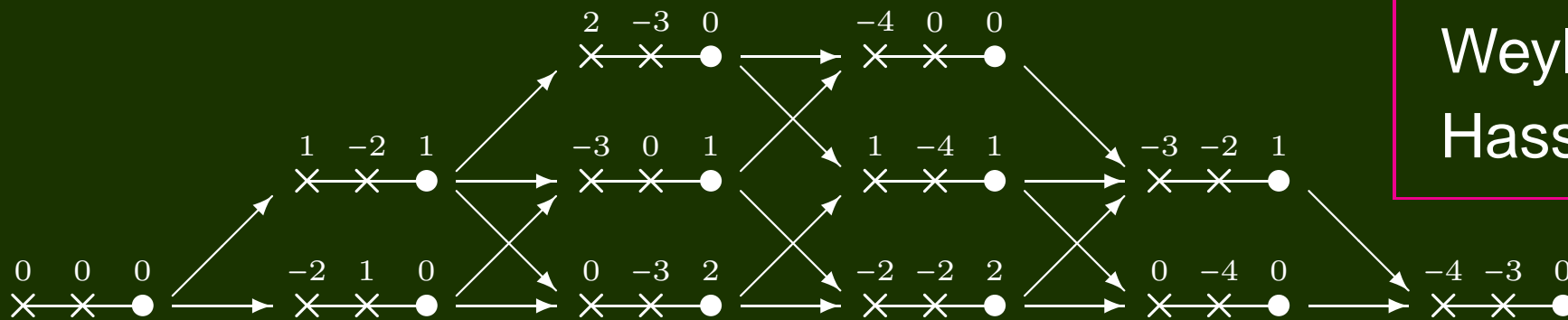
Choose complementary line bundle $H = \xi \oplus D$ ($\Leftrightarrow \Lambda^1_H = Q \oplus L$)



××• cont'd



Bernstein-Gelfand-Gelfand complex



Affine action of Weyl group on Hasse diagram

Parabolic geometry

Geometries modelled on homogeneous spaces

$$\boxed{G/P} \quad \begin{cases} G \text{ simple Lie group} \\ P \text{ parabolic subgroup} \end{cases}$$

Examples

• conformal geometry $SO(n+1, 1)/P$

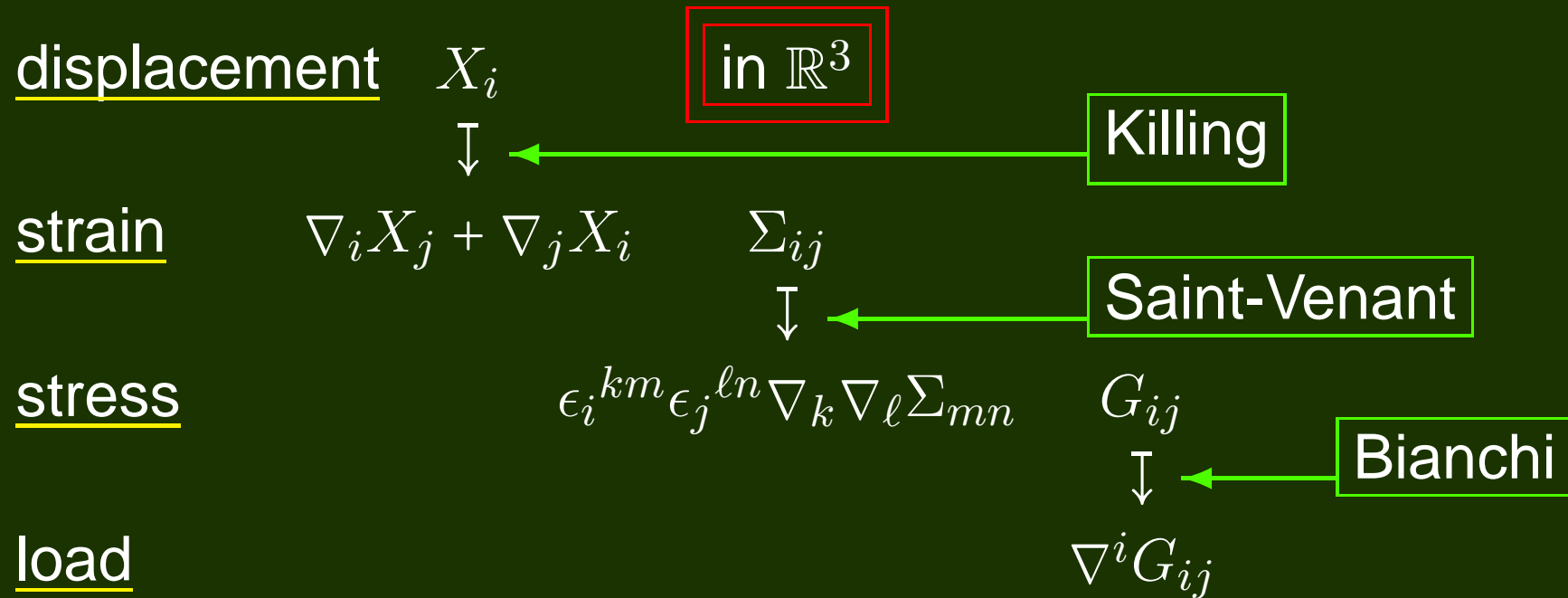
• CR geometry $SU(n+1, 1)/P$

• projective geometry $SL(n+1, \mathbf{R})/$

$$\left\{ \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & * & \cdots & * \end{bmatrix} \right\}$$

I wish to say that I believe that projective differential geometry will be of increasing importance
Shiing-Shen Chern, 1988

Application: linear elasticity



BGG $\begin{array}{c} 0 & 1 & 0 \\ \times & \bullet & \bullet \end{array} \xrightarrow{\nabla} \begin{array}{c} -2 & 2 & 0 \\ \times & \bullet & \bullet \end{array} \xrightarrow{\nabla^{(2)}} \begin{array}{c} -4 & 0 & 2 \\ \times & \bullet & \bullet \end{array} \xrightarrow{\nabla} \begin{array}{c} -5 & 0 & 1 \\ \times & \bullet & \bullet \end{array}$ on \mathbb{RP}_3

\leadsto (Arnold-Falk-Winther) new stable finite element schemes

Application: analysis on $\mathbb{C}\mathbb{P}_n$

on a symplectic manifold

Rumin-Seshadri complex

$$\begin{array}{cccccccccccc}
 \boxed{\Lambda^0} & \xrightarrow{d} & \boxed{\Lambda^1} & \xrightarrow{d_\perp} & \Lambda^2_\perp & \xrightarrow{d_\perp} & \Lambda^3_\perp & \xrightarrow{d_\perp} & \dots & \xrightarrow{d_\perp} & \Lambda^n_\perp \\
 & & & & & & & & & & \downarrow d_\perp^{(2)} \\
 \Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda^2_\perp & \xleftarrow{d_\perp} & \Lambda^3_\perp & \xleftarrow{d_\perp} & \dots & \xleftarrow{d_\perp} & \Lambda^n_\perp
 \end{array}$$

\square local cohomology = \mathbb{R} (cf. Rumin on $S^{2n+1} \rightarrow \mathbb{C}\mathbb{P}_n$)

- $\Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^2_\perp)$ exact
- $\Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1) \xrightarrow{\nabla} \Gamma(\mathbb{C}\mathbb{P}_n, \odot^2 \Lambda^1) \xrightarrow{\nabla^{(2)}} \Gamma(\mathbb{C}\mathbb{P}_n, \boxplus_\perp \Lambda^1)$ exact



THANK YOU