Prolongation 101: DG seminar 7/5/13

1) <u>An example</u>

trace-free part $(\nabla_a \sigma^b) = 0$

Prolong: $\nabla_a \sigma^b = \delta_a{}^b \mu$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma^c = R_{ab}{}^c{}_d \sigma^d$$

$$\circledast \quad \| \qquad\qquad \Rightarrow (1-n)\nabla_b \mu = R_{ab}{}^a{}_d \sigma^d$$

$$\nabla_a \delta_b{}^c \mu - \nabla_b \delta_a{}^c \mu \qquad \Leftrightarrow \boxed{\nabla_a \mu + \frac{1}{n-1} R_{ab} \sigma^c = 0} \quad \| \; R_{bd}\sigma^d$$

So we conclude that

$$\nabla_a \sigma^b - \delta_a{}^b \mu = 0$$

$$\nabla_a \mu + P_{ab}\sigma^c = 0$$

$$\underbrace{\nabla_a \begin{pmatrix} \sigma^b \\ \mu \end{pmatrix} \equiv \begin{pmatrix} \nabla_a \sigma^b - \delta_a{}^b \mu \\ \nabla_a \mu + P_{ab}\sigma^c \end{pmatrix}}_{\text{projective tractors from thin air!}}$$

$$\begin{pmatrix} T \\ \oplus \\ \mathbb{R} \end{pmatrix}$$

(Overly) Optimistic rave.

Compute:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\begin{pmatrix} \sigma^c \\ \mu \end{pmatrix} = \begin{pmatrix} W_{ab}{}^c{}_d \sigma^d \\ (\nabla_a P_{bc} - \nabla_b P_{ac})\sigma^c \end{pmatrix} \quad \text{tractor curvature}$$

<u>but</u> $\qquad R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \text{trace pieces}$

we could have deduced this at $\circledast$ (but we chose not so to do).

(Optimism begins to fade.)

For simplicity let's suppose (w.l.g. in some sense) that the Ricci curvature is symmetric

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c P_{bd} - \delta_b{}^c P_{ad}$$

$P_{ab} =$ { Schouten tensor
projective

2) <u>Another example</u>

$$\nabla_{(a}\sigma_{b)} = 0.$$

$$\nabla_a \sigma_b = \underset{\text{skew}}{\mu_{ab}} \Rightarrow \nabla_a \mu_{bc} = \nabla_c \mu_{ba} - \nabla_b \mu_{ca}$$

$$= (\nabla_c \nabla_b - \nabla_b \nabla_c)\sigma_a$$

$$= R_{bc}{}^{d}{}_{a}\sigma_d$$

$$= (W_{bc}{}^{d}{}_{a} + \delta_b{}^{d} P_{ca} - \delta_c{}^{d} P_{ba})\sigma_d$$

$$\Rightarrow \nabla_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} \equiv \begin{pmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^{d}{}_{a}\sigma_d \end{pmatrix} = 0$$

$$\| $$

$$\begin{pmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} + P_{ab}\sigma_c - P_{ac}\sigma_b \end{pmatrix} - \begin{pmatrix} 0 \\ W_{bc}{}^{d}{}_{a}\sigma_d \end{pmatrix}$$

The tractor connection

a projectively invariant correction term

$$\Lambda^2 \begin{pmatrix} T \\ \oplus \\ \mathbb{R} \end{pmatrix}^* = \Lambda^2 \begin{pmatrix} \mathbb{R} \\ \oplus \\ \Lambda^1 \end{pmatrix} = \begin{matrix} \Lambda^1 \\ \oplus \\ \Lambda^2 \end{matrix}$$

(Optimism is on the way out.)

Many more examples show that this is the norm rather than the exception (Dennis says that Robert says that Cartan does not construct a Cartan connection in his 5 variables paper).

General theory $\mathcal{D}: E \to F$ (1st order for simplicity)

$$\sigma(\mathcal{D}): \Lambda^1 \otimes E \to F \quad \text{suppose surjective}$$

$$K \equiv \ker \sigma(\mathcal{D}).$$

[In our examples: $T \to \Lambda^1 \otimes_0 T$

$$K = \Lambda^0 \xrightarrow{\delta} \Lambda^1 \otimes T \twoheadrightarrow \Lambda^1 \otimes_0 T$$

$$\Lambda^1 \to \odot^2 \Lambda^1$$

$$K = \Lambda^2 \to \Lambda^1 \otimes \Lambda^1 \twoheadrightarrow \odot^2 \Lambda^1 \,]$$

Generally,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & E \arrow[r] \arrow[l, bend left] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda^1 \otimes E \arrow[r] \arrow[d] & J^1 E \arrow[r] \arrow[d, "\mathcal{D}"] & E \arrow[r] \arrow[l, bend left] & 0 \\
 & F \arrow[r, equal] \arrow[d] & F \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

choose a splitting

induced splitting

Then $K' = \begin{matrix} E \\ \oplus \\ E' \cong K \end{matrix} \xrightarrow{\tilde{\mathcal{D}}} \begin{matrix} \Lambda^1 \otimes E \\ \oplus \\ \Lambda^2 \otimes E \end{matrix}$

But symbol is no longer surjective:-

$$\begin{tikzcd}
\Lambda^1 \otimes E \arrow[r, "\mathbb{1}"] & \Lambda^1 \otimes E \\
\Lambda^1 \otimes K \arrow[r, "\partial"] \arrow[d] & \Lambda^2 \otimes E \\
\Lambda^1 \otimes \Lambda^1 \otimes E \arrow[ur, "\wedge \otimes \mathbb{1}"'] &
\end{tikzcd}$$

$$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \nabla\sigma - \mu \\ \nabla\mu - \kappa\sigma \end{pmatrix}$$

in fact $\tilde{\mathcal{D}}$ is canonically constructed from $\mathcal{D}$.

So force the issue by writing F' for $\begin{matrix} \Lambda^1 \otimes E \\ \oplus \\ \partial(\Lambda^1 \otimes K) \end{matrix} \subset \begin{matrix} \Lambda^1 \otimes E \\ \oplus \\ \Lambda^2 \otimes E \end{matrix}$

and we've constructed $\mathcal{D}': E' \to F'$

Then $K' = \ker \sigma(\mathcal{D}') = (\Lambda^1 \otimes K) \cap \odot^2 \Lambda^1 \otimes E$

and now iterate.

In our examples.

1) $K = \Lambda^0 \overset{\delta}{\hookrightarrow} \Lambda^1 \otimes T$

$$\Lambda^1 \otimes K \hookrightarrow \Lambda^1 \otimes \Lambda^1 \otimes E \to \Lambda^2 \otimes E$$

$$\Lambda^1 \hookrightarrow \Lambda^1 \otimes \Lambda^1 \otimes E \to \Lambda^2 \otimes T$$

$$\mu_a \longmapsto \mu_a \delta_b{}^c \mapsto \mu_{[a}\delta_{b]}{}^c$$ injective so $K' = 0$

and there's a canonical splitting

$$\Lambda^2 \otimes T = \Lambda^1 \oplus \Lambda^2 \otimes_0 T$$

so everything is fine: prolongation gives a unique connection.

2) $K = \Lambda^2 \hookrightarrow \Lambda^1 \otimes \overset{E}{\overset{\shortparallel}{\Lambda^1}}$

$$\underbrace{\Lambda^1 \otimes \Lambda^2 \hookrightarrow \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1 \to \Lambda^2 \otimes \Lambda^1}_{\partial}$$

is an isomorphism. No choice entailed in splitting and get a connection.

More general theory

$$K \subset \Lambda^1 \otimes E$$
$$K' = (\Lambda^1 \otimes K) \cap \odot^2 \Lambda^1 \otimes E$$
$$K'' = (\odot^2 \Lambda \otimes K) \cap \odot^3 \Lambda^1 \otimes E \quad \text{etc}$$

$K^{(l)} = 0$ for $l \gg 0 \Rightarrow$ finite type and

then at least, prolongation terminates and we obtain a connection on

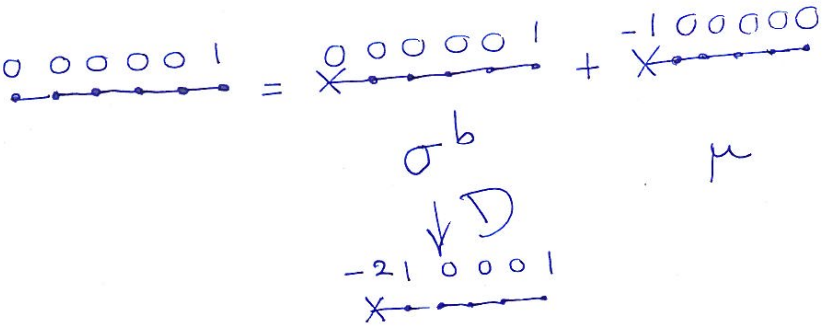
$$\underbrace{E + K + K' + \cdots + K^{(l)}}_{} + \underbrace{\cdots}_{\text{terminates}}$$

Hard algebra – art form more than science. Phillip Griffiths

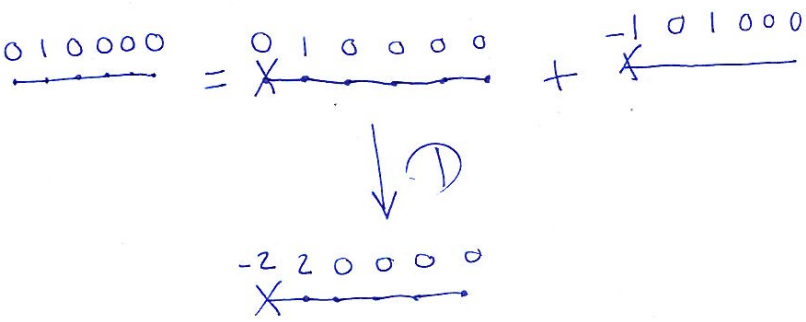
<u>Good news</u> In some circumstances Kostant's BBW theorem computes $K^{(\ell)}$ (and much more besides).

(beware of the order we're playing the moves)

Example 1)



Example 2)



<u>Bad news</u> Even in these parabolic-invariant cases, the follow-your-nose construction of the connection can fail

<u>EG</u> <u>conformal</u> Killing equation

$$\nabla_{(a}\sigma_{b)}^{\circ} = 0$$

$$\nabla_a \sigma_b = \mu_{ab} + \nu g_{ab}$$

Following your nose leads more down the projective path that the conformal

— see Examples 0.0.1 ← awkward

0.0.2 ← <u>good</u>

This goodness is generalised by Hammerl, Somberg, Souček, Šilhan ← Dennis next week

Here's the same example but done in two ways, one of which is better!

0.0.1. *Example.* Consider the partial differential equation

$$\text{the trace-free part of } \nabla_{(a}\sigma_{b)} = 0. \tag{1}$$

Even in this simple case, prolongation is already fairly involved. The details can be omitted on first reading and the main features are described in §0.0.3 below. We can rewrite (1) as

$$\nabla_a\sigma_b = \mu_{ab} + \nu g_{ab} \quad \text{where } \mu_{ab} \text{ is skew.} \tag{2}$$

Then $\nabla_{[a}\mu_{bc]} = 0$, so

$$\begin{aligned} \nabla_a\mu_{bc} = \nabla_c\mu_{ba} - \nabla_b\mu_{ca} &= \nabla_c(\nabla_b\sigma_a - \nu g_{ba}) - \nabla_b(\nabla_c\sigma_a - \nu g_{ca}) \\ &= R_{bc}{}^d{}_a\sigma_d - g_{ab}\nabla_c\nu + g_{ac}\nabla_b\nu. \end{aligned} \tag{3}$$

Tracing over a and b gives

$$\nabla^b\mu_{bc} = -R_c{}^d\sigma_d - (n-1)\nabla_c\nu.$$

Let us introduce $\rho_c = \frac{1}{n-1}\nabla^b\mu_{bc}$ and rearrange this last equation as

$$\nabla_a\nu = -\rho_a - \tfrac{1}{n-1}R_a{}^b\sigma_b. \tag{4}$$

It may be used to eliminate $\nabla_c\nu$ from (3) to obtain

$$\nabla_a\mu_{bc} = g_{ab}\rho_c - g_{ac}\rho_b + K_{abc}, \tag{5}$$

where

$$K_{abc} = R_{bc}{}^d{}_a\sigma_d + \tfrac{1}{n-1}g_{ab}R_c{}^d\sigma_d - \tfrac{1}{n-1}g_{ac}R_b{}^d\sigma_d. \tag{6}$$

Notice that K_{abc} is totally trace-free. Now apply ∇_d to (5) and skew over a and d to obtain

$$R_{da}{}^e{}_b\mu_{ce} - R_{da}{}^e{}_c\mu_{be} = \nabla_d K_{abc} - \nabla_a K_{dbc} + g_{ab}\nabla_d\rho_c - g_{db}\nabla_a\rho_c - g_{ac}\nabla_d\rho_b + g_{dc}\nabla_a\rho_b.$$

Tracing over a and b gives

$$R_d{}^e\mu_{ce} - R_d{}^{be}{}_c\mu_{be} = -\nabla^b K_{dbc} + (n-2)\nabla_d\rho_c + g_{dc}\nabla^b\rho_b$$

but tracing again, over c and d, gives $0 = 2(n-1)\nabla^b\rho_b$. Therefore,

$$\nabla_a\rho_b = \tfrac{1}{n-2}\left(R_a{}^c\mu_{bc} - R_a{}^{cd}{}_b\mu_{cd} - \nabla^c K_{abc}\right). \tag{7}$$

At this point it is clear that the system has closed: it comprises (2), (4), (5), and in (7) one has to expand $\nabla^c K_{abc}$ using (6) and (2).

0.0.2. *Example.* Consider the partial differential equation

$$\text{the trace-free part of } \nabla_{(a}\sigma_{b)} = 0. \tag{8}$$

Even in this simple case, prolongation is already fairly involved. The details can be omitted on first reading and the main features are described in §0.0.3 below. We can rewrite (8) as

$$\nabla_a\sigma_b = \mu_{ab} + \nu g_{ab} \quad \text{where } \mu_{ab} \text{ is skew.} \tag{9}$$

Then $\nabla_{[a}\mu_{bc]} = 0$, so

$$\begin{aligned} \nabla_a\mu_{bc} = \nabla_c\mu_{ba} - \nabla_b\mu_{ca} &= \nabla_c(\nabla_b\sigma_a - \nu g_{ba}) - \nabla_b(\nabla_c\sigma_a - \nu g_{ca}) \\ &= R_{bc}{}^d{}_a\sigma_d - g_{ab}\nabla_c\nu + g_{ac}\nabla_b\nu. \end{aligned} \tag{10}$$

If we introduce the totally trace-free *Weyl curvature* tensor C_{abcd} and symmetric *Schouten* tensor P_{ab} by

$$R_{abcd} = C_{abcd} + \mathrm{P}_{ac}g_{bd} - \mathrm{P}_{bc}g_{ad} + \mathrm{P}_{bd}g_{ac} - \mathrm{P}_{ad}g_{bc}, \tag{11}$$

then (10) may be conveniently rewritten as

$$\nabla_a\mu_{bc} = g_{ab}\rho_c - g_{ac}\rho_b - \mathrm{P}_{ab}\sigma_c + \mathrm{P}_{ac}\sigma_b + C_{bc}{}^d{}_a\sigma_d, \tag{12}$$

where

$$\nabla_a\nu = -\rho_a - \mathrm{P}_a{}^b\sigma_b. \tag{13}$$

Now apply ∇_d to (12) and skew over a and d to obtain

$$\begin{aligned} R_{da}{}^e{}_b\mu_{ce} - R_{da}{}^e{}_c\mu_{be} &= g_{ab}\nabla_d\rho_c - g_{db}\nabla_a\rho_c - g_{ac}\nabla_d\rho_b + g_{dc}\nabla_a\rho_b \\ &\quad - \mathrm{P}_{ab}\nabla_d\sigma_c + \mathrm{P}_{db}\nabla_a\sigma_c + \mathrm{P}_{ac}\nabla_d\sigma_b - \mathrm{P}_{dc}\nabla_a\sigma_b \\ &\quad - Y_{dab}\sigma_c + Y_{dac}\sigma_b \\ &\quad + (\nabla_d C_{bc}{}^e{}_a)\sigma_e - (\nabla_a C_{bc}{}^e{}_d)\sigma_e + C_{bc}{}^e{}_a\nabla_d\sigma_e - C_{bc}{}^e{}_d\nabla_a\sigma_e, \end{aligned}$$

where

$$Y_{abc} = \nabla_a\mathrm{P}_{bc} - \nabla_b\mathrm{P}_{ac}$$

is the *Cotton-York* tensor. If we trace this equation over a and b, substituting from (11) for R_{abcd} and using the Bianchi identity $\nabla^b C_{bcde} = (n-3)Y_{dec}$, we find that

$$(n-2)\nabla_d\rho_c + g_{cd}\nabla^a\rho_a = (n-2)\mathrm{P}_{cd}\nu + g_{cd}\mathrm{P}\nu - (n-2)\mathrm{P}_d{}^b\mu_{bc} + (n-2)Y^b{}_{dc}\sigma_b,$$

where $\mathrm{P} = \mathrm{P}^a{}_a$. If we trace this equation over a and b and substitute back in, then we discover that

$$\nabla_a\rho_b = \mathrm{P}_a{}^c\mu_{bc} + \mathrm{P}_{ab}\nu + Y^c{}_{ab}\sigma_c \tag{14}$$

and the system has closed: it comprises (9), (12), (13), and (14).

0.0.3. *Discussion.* The conclusion is that (1) is equivalent to $\widetilde{\nabla}\Sigma = 0$ where

$$\Sigma = \begin{pmatrix} & \sigma_b & \\ \mu_{bc} & & \nu \\ & \rho_b & \end{pmatrix} \quad \text{is a section of the bundle} \quad V = \begin{matrix} & \Lambda^1 & \\ \Lambda^2 & \bigoplus & \Lambda^0 \\ & \Lambda^1 & \end{matrix}$$

and $\widetilde{\nabla} : V \to \Lambda^1 \otimes V$ is an explicit connection of the form

$$\widetilde{\nabla} \begin{pmatrix} & \sigma & \\ \mu & & \nu \\ & \rho & \end{pmatrix} = \begin{pmatrix} & \nabla\sigma - \mu - \nu & \\ \nabla\mu - \rho - R \bowtie \sigma & & \nabla\nu - \rho - R \bowtie \sigma \\ & \nabla\rho - R \bowtie \mu - R \bowtie \nu - (\nabla R) \bowtie \sigma & \end{pmatrix}, \tag{15}$$

where each $\bowtie$ indicates an appropriate linear combination of contractions of its ingredients.

The equation (1) is well-known. It says that the vector field σ^a is a conformal Killing field—its flow preserves the metric up to scale. From this geometric interpretation it follows easily that the space of solutions is bounded by $\dim \mathfrak{so}(n+1,1)$ since $\mathfrak{so}(n+1,1)$ is the conformal algebra in the flat case. This bound is confirmed by the technique of prolongation:–

$$\operatorname{rank} V = 2 \operatorname{rank} \Lambda^1 + \operatorname{rank} \Lambda^2 + \operatorname{rank} \Lambda^0 = 2n + \frac{n(n-1)}{2} + 1 = \frac{(n+1)(n+2)}{2}.$$

In [1], Semmelmann uses this technique to establish similar bounds on the dimension of spaces of conformal Killing forms. Specifically, he finds an explicit connection (also having the form (15)) on the bundle

$$V = \begin{matrix} & \Lambda^p & \\ \Lambda^{p+1} & \bigoplus & \Lambda^{p-1} \\ & \Lambda^p & \end{matrix} \quad \text{with rank} \begin{pmatrix} n+2 \\ p+1 \end{pmatrix}$$

so that conformal Killing p-forms are equivalent to parallel sections of this bundle.

However! *Example 0.0.2* is better since it may be written as $\widetilde{\nabla}_a\Sigma = 0$ where $\widetilde{\nabla}_a$ is the connection on the bundle V given by

$$\widetilde{\nabla}_a \begin{pmatrix} & \sigma_b & \\ \mu_{bc} & & \nu \\ & \rho_b & \end{pmatrix} = D_a \begin{pmatrix} & \sigma_b & \\ \mu_{bc} & & \nu \\ & \rho_b & \end{pmatrix} - \begin{pmatrix} & 0 & \\ C_{bc}{}^d{}_a\sigma_d & & 0 \\ & Y^c{}_{ab}\sigma_c & \end{pmatrix},$$

where we recognise

$$D_a \begin{pmatrix} & \sigma_b & \\ \mu_{bc} & & \nu \\ & \rho_b & \end{pmatrix} \equiv \begin{pmatrix} & \nabla_a\sigma_b - \mu_{ab} - \nu g_{ab} & \\ \nabla_a\mu_{bc} - g_{ab}\rho_c + g_{ac}\rho_b + \mathrm{P}_{ab}\sigma_c - \mathrm{P}_{ac}\sigma_b & & \nabla_a\nu + \rho_a + \mathrm{P}_a{}^b\sigma_b \\ & \nabla_a\rho_b - \mathrm{P}_a{}^c\mu_{bc} - \mathrm{P}_{ab}\nu & \end{pmatrix}$$

as the invariant tractor connection and the other *curvature correction* term as being conformally invariant to boot. So this one is gloriously orange whilst *Example 0.0.1* is a sickly puce at best.

References

[1] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Zeit. **245** (2003) 503–527.