The range of the Killing operator on projective space

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Killing operator in flat space

Killing operator: $\boxed{X_a \mapsto \nabla_{(a}X_{b)}}$

Solve in flat space: $K_{ab} \equiv \nabla_a X_b$ is skew.

Claim: $\nabla_a K_{bc} = 0$.
$$\begin{aligned} \nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a \\ &= 0, \text{ as required.} \end{aligned}$$

Hence, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{aligned} \nabla_a X_b &= K_{ab} \\ \nabla_a K_{bc} &= 0 \end{aligned}}$ closed!

Conclusion: $\boxed{X_a = m_{ab}x^b + r_a}$ where $m_{ab} = -m_{ba}$.

rotations

translations

Curved prolongation

$$\nabla_{(a}X_{b)} = 0 \iff \nabla_a X_b = K_{ab} \text{ for } K_{ab} = K_{[ab]}.$$

But then, $\nabla_a K_{bc} = \nabla_c K_{ba} - \nabla_b K_{ca}$

$$= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a$$

$$= R_{bc}{}^d{}_a X_d.$$

Therefore, $\nabla_{(a}X_{b)} = 0 \iff \boxed{\begin{array}{rcl} \nabla_a X_b & = & K_{ab} \\ \nabla_a K_{bc} & = & R_{bc}{}^d{}_a X_d \end{array}}$

Hence, Killing fields ↔ covariant constant sections of $V \equiv \Lambda^1 \oplus \Lambda^2$ with connection

$$\left[\begin{array}{c} X_b \\ K_{bc} \end{array}\right] \xmapsto{\nabla_a} \left[\begin{array}{c} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} - R_{bc}{}^d{}_a X_d \end{array}\right].$$

Curvature

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix} X_c \\ K_{cd} \end{bmatrix} = \begin{bmatrix} 0 \leftsquigarrow \text{by design} \\ \underbrace{R \bowtie K}_{\boxed{2R_{ab}{}^{e}{}_{[c}K_{d]e} + 2R_{cd}{}^{e}{}_{[a}K_{b]e}}} + \underbrace{(\nabla R) \bowtie X}_{\boxed{(\nabla_b R_{cd}{}^{e}{}_{a})X_e - (\nabla_a R_{cd}{}^{e}{}_{b})X_e}} \end{bmatrix}$$

Flat $\iff$ $R_{abcd} = \lambda(g_{ac}g_{bd} - g_{bc}g_{ad})$

$\iff$ constant curvature.

Sphere or $\mathbb{RP}_n$ has symmetries of maximal dimension

$$\dim \Lambda^1 + \dim \Lambda^2 = n + n(n-1)/2 = \mathrm{n(n+1)/2}$$

$$= \dim \mathrm{SO}(n+1) = \dim \mathfrak{so}(n+1).$$

Coupled de Rham sequence

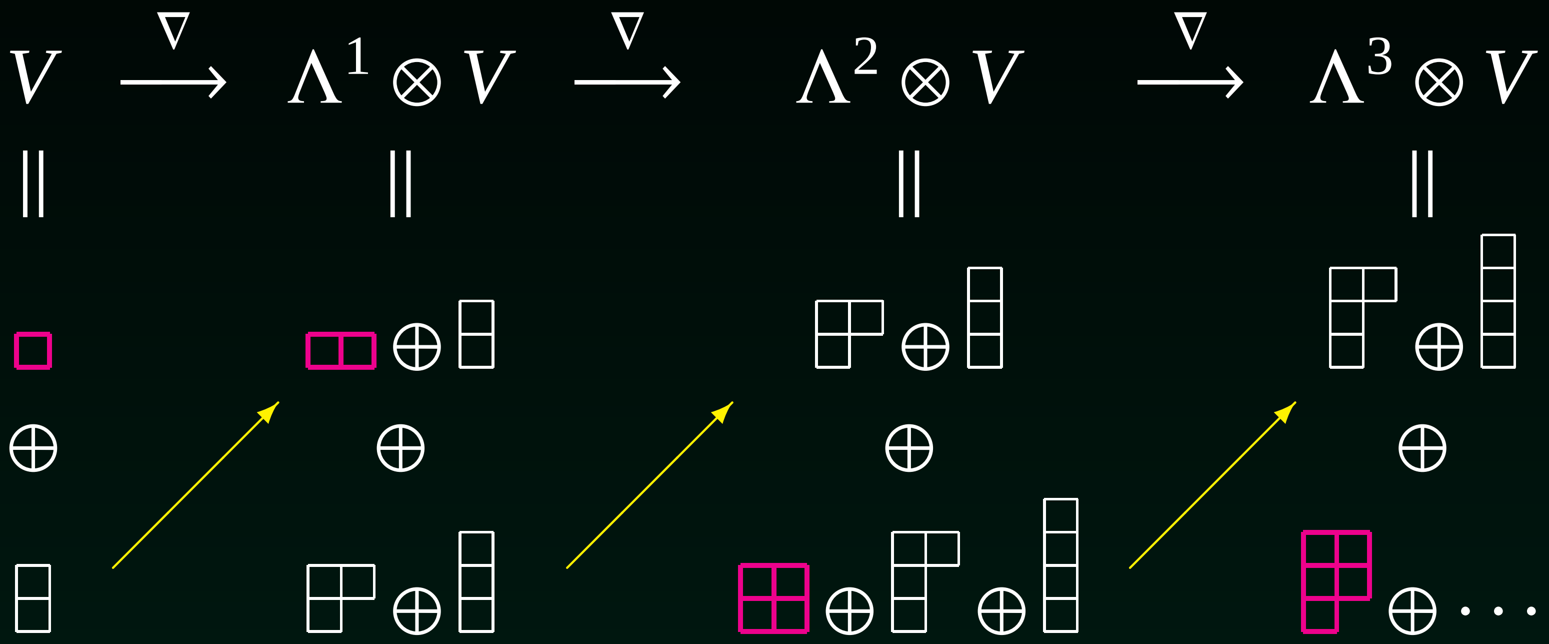
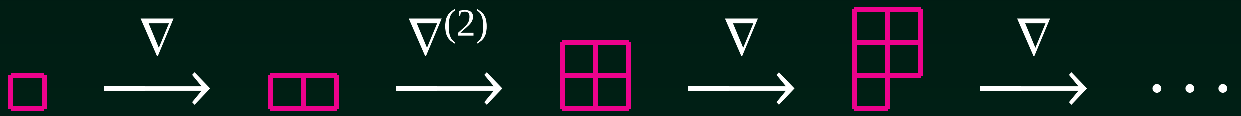


Diagram chasing in the constant curvature case $\rightsquigarrow$



a locally exact complex!

BGG resolutions

The locally exact complexes

Riemannian deformation

$$\ydiagram{1} \xrightarrow{\nabla} \ydiagram{2} \xrightarrow{\nabla^{(2)}} \ydiagram{2,2} \xrightarrow{\nabla} \ydiagram{2,2,1} \xrightarrow{\nabla} \cdots$$

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots$$

are Bernstein-Gelfand-Gelfand resolutions.

de Rham

On the sphere or real projective space $\ydiagram{2} \xrightarrow{\nabla^{(2)}} \ydiagram{2,2}$ is

$\omega_{ab} \mapsto \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd})$, where

$$\pi(\Phi_{acbd} = \Phi_{[ab][cd]}) \qquad \ydiagram{2} \otimes \ydiagram{2} \longrightarrow \ydiagram{2,2}$$

Range?

Range of the Killing operator? Given $\omega_{ab} = \omega_{(ab)}$,

$$\omega_{ab} = \nabla_{(a}X_{b)} \quad \Leftrightarrow \quad \begin{bmatrix} \omega_{ab} \\ 2\nabla_{[b}\omega_{c]a} \end{bmatrix} = \nabla_a \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix}.$$

On the sphere or real projective space

$$\nabla_a \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} = \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab}X_c - g_{ac}X_b \end{bmatrix}$$

NB: $H^1(S^n, \mathbb{R}) = 0$ and $H^1(\mathbb{RP}_n, \mathbb{R}) = 0$.

$$\omega_{ab} = \nabla_{(a}X_{b)} \Leftrightarrow \boxed{\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0}$$

Killing operator on $\mathbf{CP}_{\mathbf{n}}$

Connection on $\Lambda^1 \oplus \Lambda^2$:–

$$\left[\begin{array}{l} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b - J_{ab} J_c{}^d X_d + J_{ac} J_b{}^d X_d + 2 J_{bc} J_a{}^d X_d \end{array}\right]$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \end{array}\right] = 4 \left[\begin{array}{c} 0 \\ J_{c[a} \tilde{K}_{b]d} - J_{d[a} \tilde{K}_{b]c} - J_{ab} \tilde{K}_{cd} - J_{cd} \tilde{K}_{ab} \end{array}\right]$$

where $\tilde{K}_{ab} \equiv J_{[a}{}^e K_{b]e}$.

Better connection

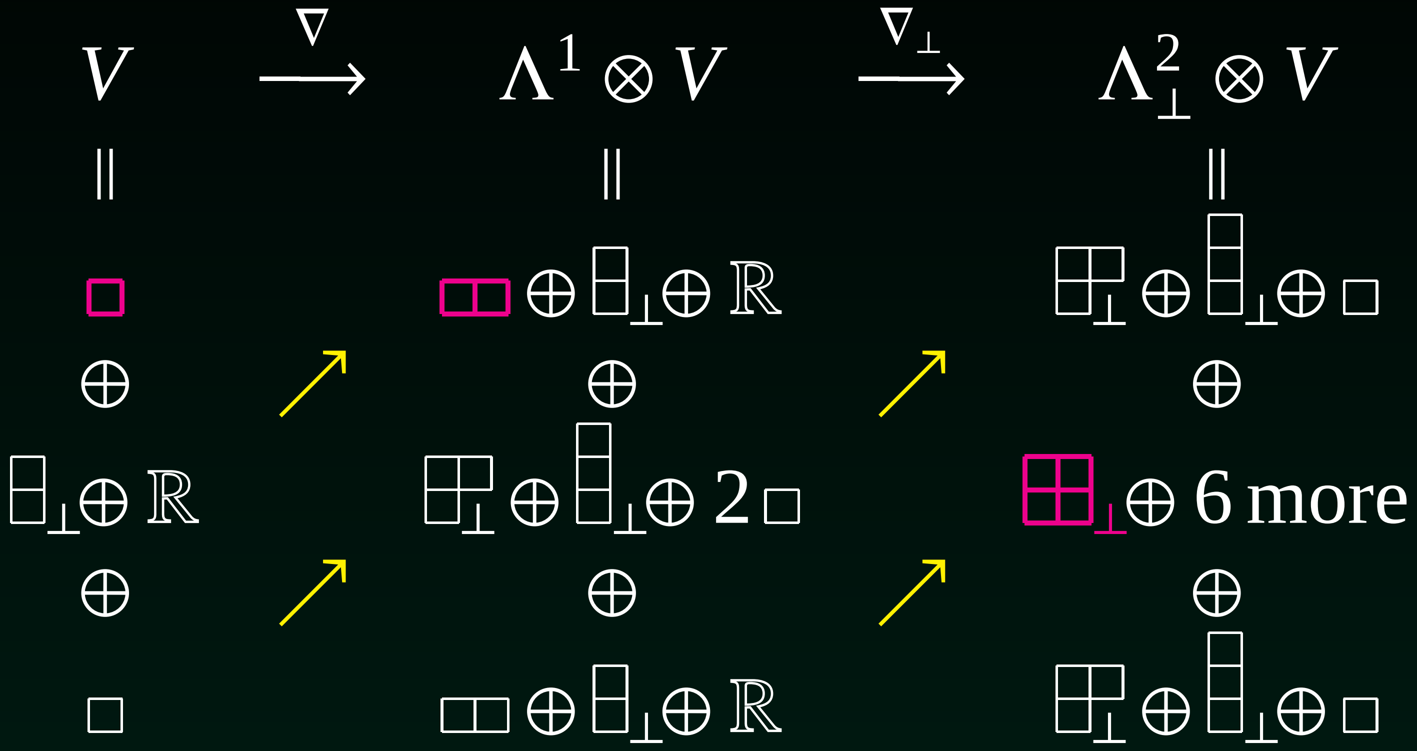
Better connection on $\Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$:–
(from joint work with Hubert Goldschmidt)

$$\left[\begin{array}{c} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} + g_{ab} X_c - g_{ac} X_b + J_{ab} L_c - J_{ac} L_b - J_{bc} L_a + J_{bc} J_a{}^d X_d \\ \nabla_a L_b + J_a{}^d K_{bd} \end{array}\right]$$

Curvature

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \left[\begin{array}{c} X_c \\ K_{cd} \\ L_c \end{array}\right] = 2J_{ab} \left[\begin{array}{c} L_c + J_c{}^e X_e \\ -J_c{}^e K_{de} + J_d{}^e K_{ce} \\ -X_c + J_c{}^e L_e \end{array}\right]$$

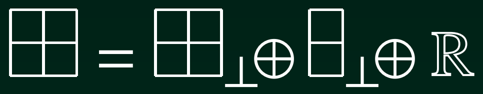
Better coupling



(from Lie algebra cohomology (Heisenberg algebra))

$$\omega_{ab} = \nabla_{(a}X_{b)} \Leftrightarrow \boxed{\pi_\perp(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0}$$

$\perp$= trace free part w.r.t. J_{ab}, e.g.



THANK YOU

THE END