Conformal and CR Geometry from the Parabolic Viewpoint

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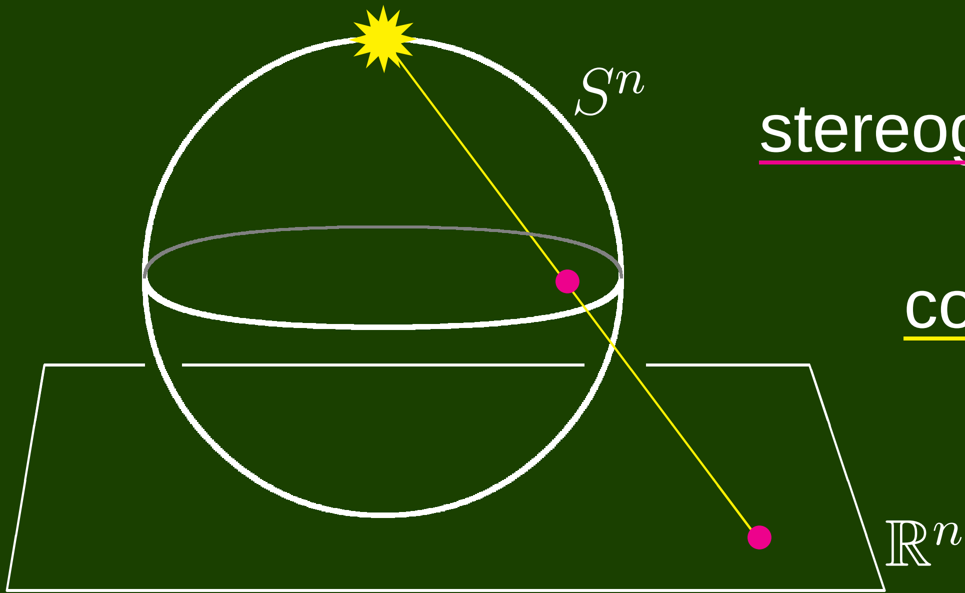
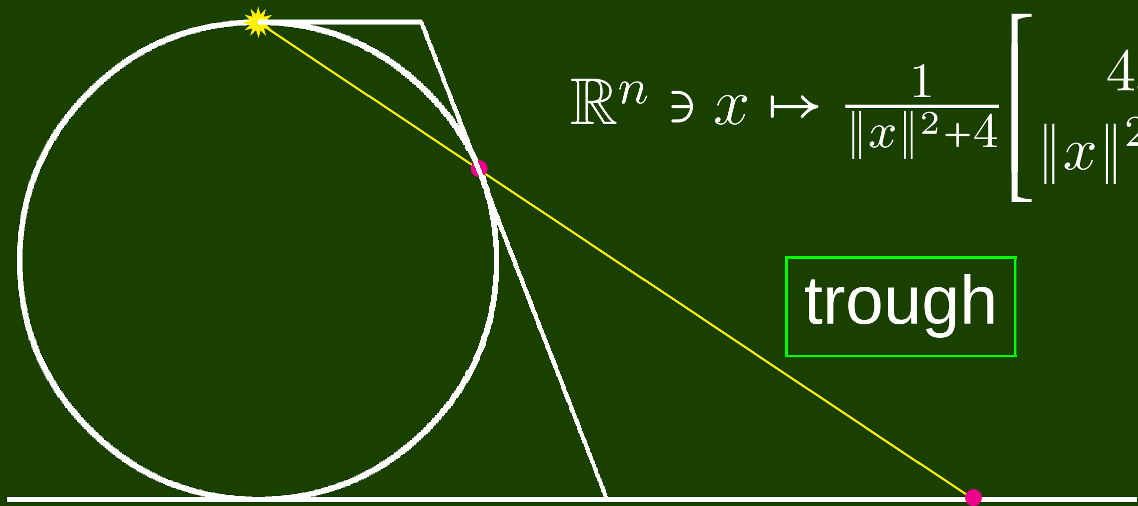
The flat model: conformal case



stereographic projection

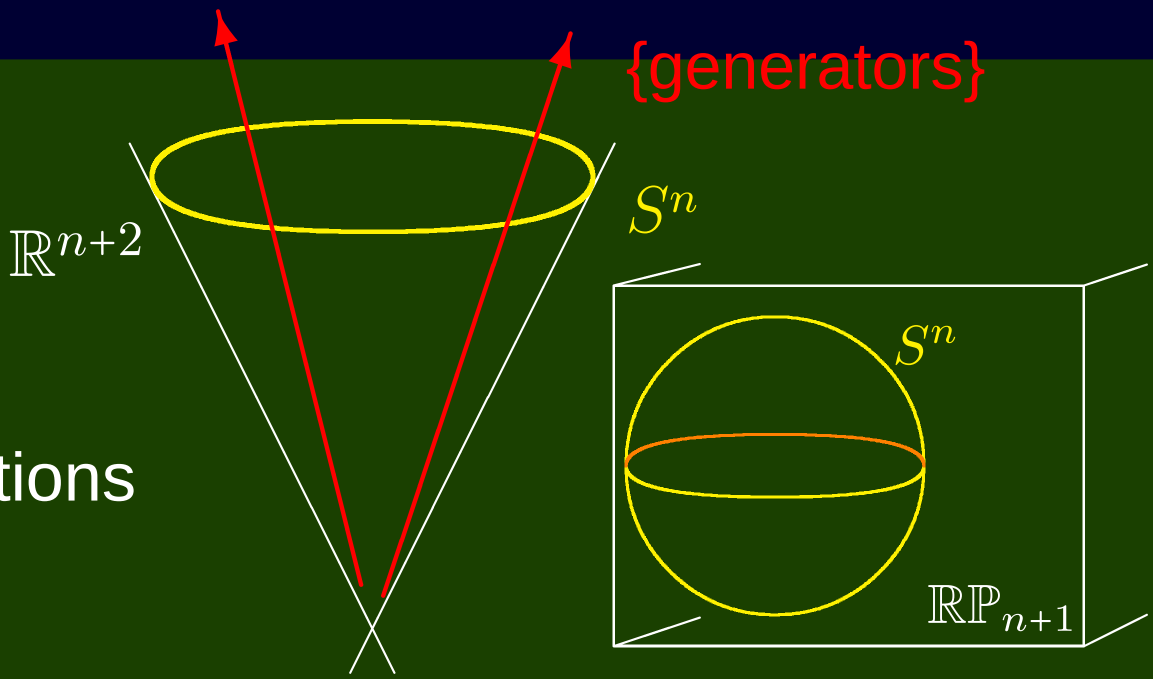
conformal



$$\mathbb{R}^n \ni x \mapsto \frac{1}{\|x\|^2+4}\begin{bmatrix} 4x \\ \|x\|^2 - 4 \end{bmatrix} \in S^n$$

trough

Conformal group



$\mathrm{SO}(n+1,1)$ acts on S^n
by conformal transformations

semisimple

parabolic

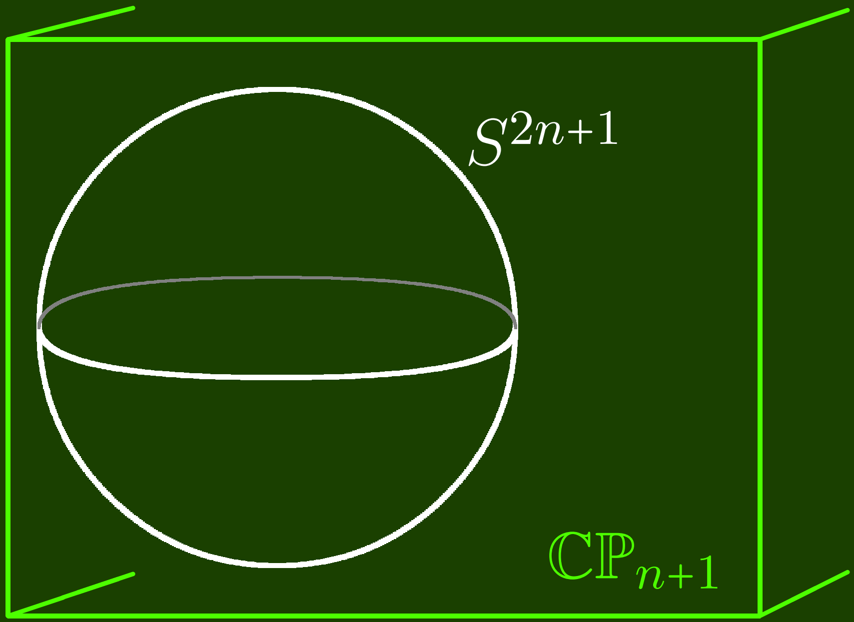
$$S^n = \mathrm{SO}(n+1,1)/P$$

flat model of conformal differential geometry

$$\mathbb{R}^n = (\mathrm{SO}(n) \ltimes \mathbb{R}^n)/\mathrm{SO}(n)$$

flat model of Riemannian differential geometry

The flat model: CR case



$$\{|z_0|^2 = |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2\}$$

semisimple

parabolic

$$S^{2n+1} = \mathrm{SU}(n+1,1)/P$$

flat model of CR geometry $\longleftarrow$ flat conformal geometry

Fefferman 1976

Parabolic geometry

- ‘Parabolic invariant theory . . . ,’ Fefferman 1979
- ‘. . . Parabolic geometries,’ Graham 1990
- ‘*ditto* and canonical Cartan connections,’ Čap & Schichl 2000
- ‘*ditto* I: background and general theory,’ Čap & Slovák 2009

Parabolic geometry $\leadsto G/P$ $\begin{cases} G \text{ semisimple} \\ P \text{ parabolic} \end{cases}$

Klein v. Cartan

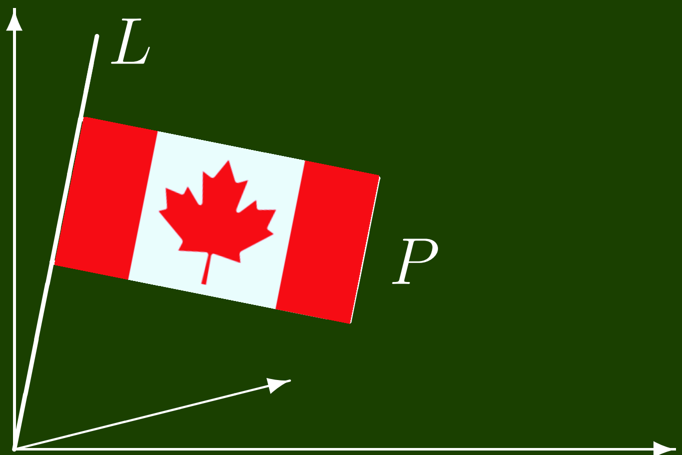
EG Conformal: $\mathrm{SO}(n+1,1)$ preserves $2x_0x_\infty + x_1{}^2 + \cdots + x_n{}^2$

$$\mathrm{SO}(n+1,1)/\left\{\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & * & \vdots & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}\right\}$$

‘block upper triangular’

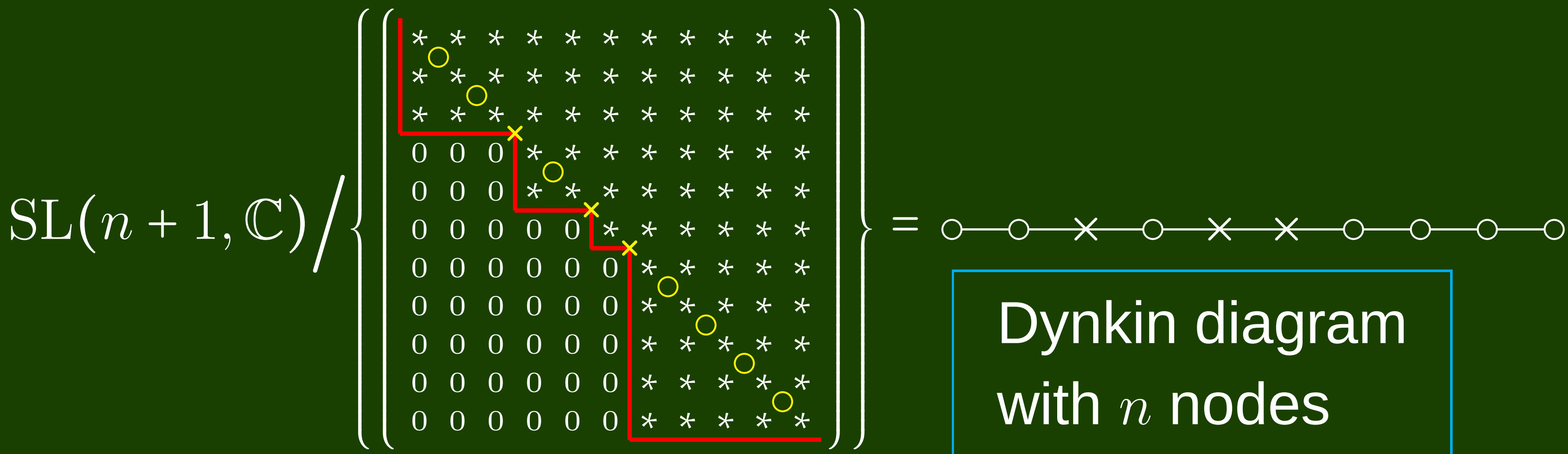
Flat models aka flag manifolds

- $\mathbb{RP}_2 = \{L \subset \mathbb{R}^3 | \dim_{\mathbb{R}} L = 1\}$
- $\mathbb{F}_{1,2}(\mathbb{R}^3) = \{L \subset P \subset \mathbb{R}^3 | \dim_{\mathbb{R}} L = 1, \dim_{\mathbb{R}} P = 2\}$



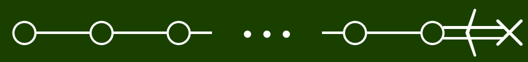
$$\mathbb{RP}_2 = \mathrm{SL}(3,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} \qquad \mathbb{F}_{1,2}(\mathbb{R}^3) = \mathrm{SL}(3,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

Complex flag manifolds



Dynkin diagram
with n nodes

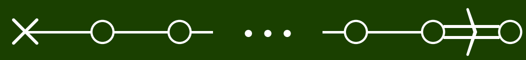
Lagrangian Grassmannians

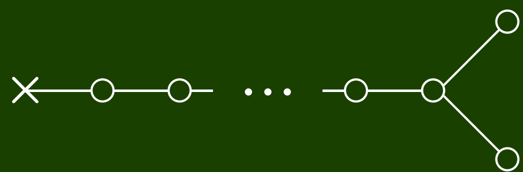


cHss

Quadrics

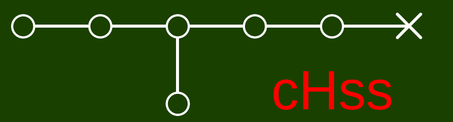
cHss



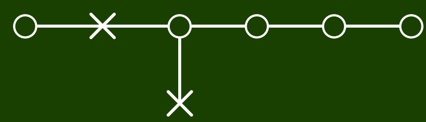


cHss

Exceptional examples



cHss



Real forms

$$\mathbb{F}_{1,2}(\mathbb{C}^3) = \mathrm{SL}(3,\mathbb{C}) / \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} = \times\!\!-\!\!\!-\!\!\times$$

$$\mathbb{F}_{1,2}(\mathbb{R}^3) = \mathrm{SL}(3,\mathbb{R}) / \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} = \times\!\!-\!\!\!-\!\!\times$$ (abuse notation)

$$S^3 = \mathrm{SU}(2,1)/P = \times\!\!-\!\!\!-\!\!\times$$ (severely abuse notation)

$\|$

flat model of 3-dim$^\ell$ CR geometry

flat model of . . .

$\times\!\!\longleftarrow\!\!\times$ geometry

$$\begin{array}{ccc} & M = \mathbb{F}_{1,2}(\mathbb{R}^3) & \\ & \mu \swarrow \quad \searrow \nu & \\ \mathbb{RP}_2 = \mathbb{F}_1(\mathbb{R}^3) & & \mathbb{F}_2(\mathbb{R}^3) = \mathbb{RP}_2^* \end{array}$$

$\leadsto\leadsto$ $TM \supset H = T^{0,1} \oplus T^{1,0}$ ‘contact Lagrangian’ or ‘para-CR’

define $J : H \to H$ by $J|_{T^{0,1}} \equiv -\mathrm{Id}$ and $J|_{T^{1,0}} \equiv \mathrm{Id}$

NB $J^2 = \mathrm{Id}$ (cf. $J^2 = -\mathrm{Id}$ for CR geometry)

flat model $T^{0,1} = \mathrm{span}\{\partial/\partial x\}$ $T^{1,0} = \mathrm{span}\{\partial/\partial t + x\partial/\partial y\}$

symmetries

Lie 1888

Tresse 1896

Cartan 1924, 1932

- $\mathrm{SL}(3,\mathbb{R})$ or $\mathfrak{sl}(3,\mathbb{R})$ for the flat model
- dimension ≤ 8 in general
- none generically

Similarly for CR geometry

Invariant connections and ambience

$M = G/P$ homogeneous bundles $\mathcal{V} \leftrightarrow P$-modules $\mathbb{V}$

G-module $\mathbb{T}$ restrict to P and induce to G/P

tractor bundle $\mathcal{T}$ with G-invariant flat connection

conformal geometry
Tracey Thomas

vector	tensor	spinor
	tractor*	twistor

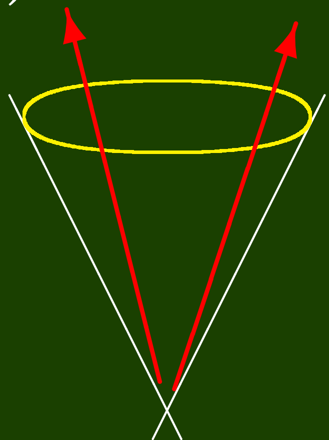
*A.P. Hodges 1991

curved analogues, e.g. in the conformal case

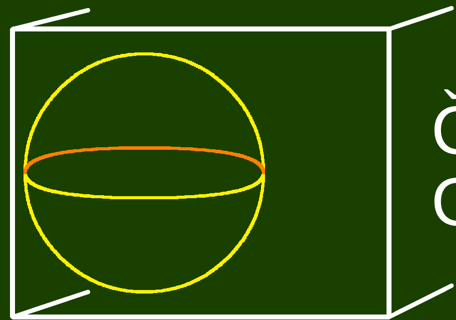
$\mathrm{SO}(n+1,1)$-module $\mathbb{R}^{n+2} \rightsquigarrow$ standard tractors

$\equiv$ Cartan connection

ambience



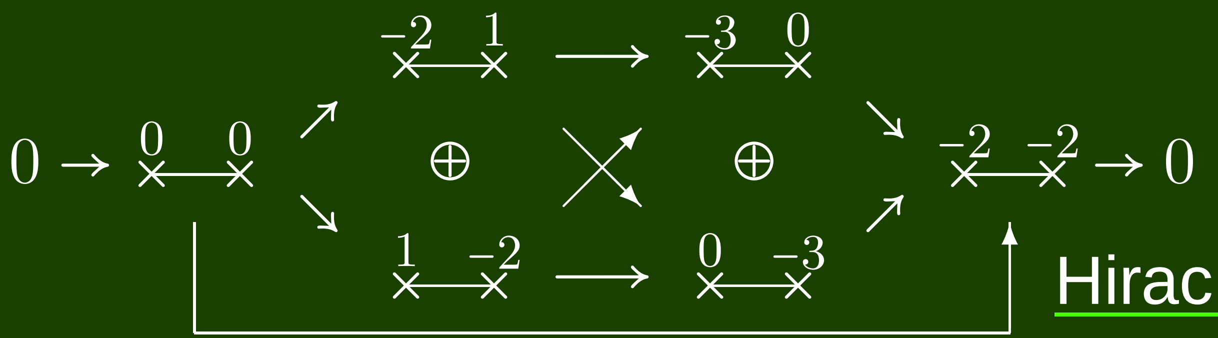
Fefferman-Graham



Čap-Gover

Invariant differential operators

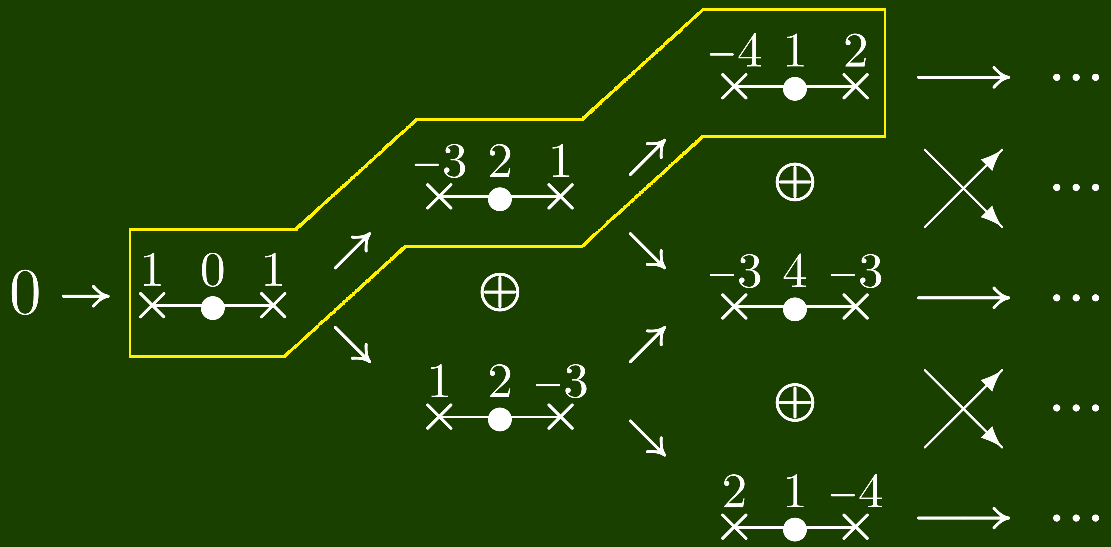
3-dim$^\ell$ CR case BGG complex



Rumin complex

Hirachi operator

5-dim$^\ell$ CR case curved adjoint BGG sequence



Akahori-Garfield-Lee CR deformation complex

(E+Seshadri)

Where next? – beyond parabolic?

- prolongation of overdetermined systems on
  - (Riemannian) manifolds (Branson-Čap-E-Gover)
  - contact manifolds (E-Gover)

    BGG machinery / Lie algebra cohomology (Kostant)
  - filtered manifolds (Neusser)
- differential complexes (Bryant-E-Gover-Neusser, . . . , Calin-Chang, . . .)

EG Rumin-Seshadri complex (Tseng-Yau) for M^{2n} symplectic

$$\begin{array}{c} 0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2_\perp \to \cdots \to \Lambda^n_\perp \\ \hspace{20em}\downarrow \\ 0 \leftarrow \Lambda^0 \leftarrow \Lambda^1 \leftarrow \Lambda^2_\perp \leftarrow \cdots \leftarrow \Lambda^n_\perp \end{array}$$

second order!

elliptic

+projective

- conformally Fedosov manifolds (E-Slovák)

THE END

THANK YOU