BATEMAN'S FORMULA

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In 1904, well before integral geometry as such was contemplated, Bateman [2, p. 457] wrote down the following formula:–

$$\phi(w,x,y,z)=\oint_\gamma f((w+ix)+(iy+z)\zeta,(iy-z)+(w-ix)\zeta,\zeta)d\zeta \tag{1}$$

for f a holomorphic function of three variables. He observed that differentiation under the integral sign implies that ϕ is harmonic and asserted that (1) gives the general harmonic function of four variables. By contemporary standards, the formulation is imprecise—where, for example, should f be defined? A precise formulation and proof of Bateman's assertion did not emerge for more than seventy years!

In 1938, John [4] wrote down the following formula:–

$$\phi(w,x,y,z)=\int_{-\infty}^{\infty} f(w+xs,y+zs,s)ds \tag{2}$$

for f a smooth function of three variables. He observed that differentiation under the integral sign implies that $\partial^2\phi/\partial w\partial z=\partial^2\phi/\partial x\partial y$. Not only that, but he gave a geometric interpretation of (2) and showed that this differential equation characterises the ϕ arising in this way. The geometric interpretation stems from the form of the integrand. Most straight lines in $\mathbb{R}^3$ can be written in the form $s\mapsto(w+xs,y+xz,s)$. One may view and extend (2) as $L\mapsto\int_L f$ for straight lines L in $\mathbb{R}^3$. This is one of the most basic transforms in real integral geometry. Nowadays, it is often called the 'X-ray transform' owing to its interpretation as describing the attenuation of X-rays encountering a patient with density $1/f$. Medical imaging calls for the inversion of transforms such as this.

The key to (1) is a similar geometric interpretation of the integrand. The Hopf fibration $\mathbb{RP}_3\to S^2$ has a higher dimensional version $\mathbb{CP}_3\xrightarrow{\tau}S^4$ sometimes known as the 'twistor fibration'. Any open $U\subseteq\mathbb{R}^4$ may be viewed as a subset of S^4 under stereographic projection and the 'Penrose transform' gives an isomorphism

$$H^1(\tau^{-1}(U),\mathcal{O}(-2))\xrightarrow{\cong}\{\text{harmonic }\phi\text{ on }U\}. \tag{3}$$

The holomorphic function f in Bateman's formula is interpreted as a Čech co-cycle representing a cohomology class on the left hand side of (3). See, for example, [6] for details of this isomorphism and many variations on this theme.

The evident formal analogy between (1) and (2) may be traced to geometry: we may view these formulae as arising from two different real forms of the complex correspondence between $\mathbb{CP}_3$ and $\mathrm{Gr}_2(\mathbb{C}^4)$. More precise links are, however, quite subtle. One possibility is discussed in [1], another in [3]. Nevertheless, the results are similar: in both cases the range of an integral geometric transform is charaterised by a differential equation. Determining the range of such transforms is a common goal in integral geometry.

There are similar results in higher dimensions but the story is most satisfactory in four dimensions. This is really because two stories coalesce, as reflected in the isomorphism of Lie groups $\mathrm{Spin}(6, \mathbb{C}) \cong \mathrm{SL}(4, \mathbb{C})$. In fact, this coincidence was one of the primary motivations for Penrose's 'twistor theory' [5, 6] as a physical theory: it is specific to the dimension of space-time.

References

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