

# Representation theory and the X-ray transform

**Projective differential geometry**

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# Summary of Lecture 1

## Curvature on $\mathbb{R}P_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

This Lecture!

$$\nabla_{[a}\nabla_{b]}\mu_c = g_{c[a}\mu_{b]}$$

## Curvature on $\mathbb{C}P_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

## Model embeddings

$\mu : \mathbb{R}P_n \hookrightarrow \mathbb{C}P_n$  totally geodesic

## 2-form lemma

$$\mu^*\psi_{ab} = 0 \forall \mu \iff \psi_{ab}^\perp = 0$$

## Curvature lemma

$$\mu^*\psi_{abcd} = 0 \forall \mu \iff \psi_{abcd}^\perp = 0$$

# Topics

- Tractor bundle and connection on  $\mathbb{RP}_n$
- Projective differential geometry
- Projective tractor bundles and connections
- Bernstein-Gelfand-Gelfand resolutions on  $\mathbb{RP}_n$
- Lie algebra cohomology
- Range of the Killing operator
- Higher Killing operators

# Tractors on real projective space

Consider the connection on  $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$  given by

$$\begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \xrightarrow{\nabla_a} \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \sigma \end{bmatrix}$$

Calculate

$$\begin{aligned} \nabla_a \nabla_b \begin{bmatrix} \sigma \\ \mu_c \end{bmatrix} &= \nabla_a \begin{bmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_c + g_{bc} \sigma \end{bmatrix} \\ &= \begin{bmatrix} \nabla_a (\nabla_b \sigma - \mu_b) - \nabla_b \mu_a - g_{ba} \sigma \\ \nabla_a (\nabla_b \mu_c + g_{bc} \sigma) + g_{ac} (\nabla_b \sigma - \mu_b) \end{bmatrix} \end{aligned}$$

$$\Rightarrow \nabla_{[a} \nabla_{b]} \begin{bmatrix} \sigma \\ \mu_c \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla_{[a} \nabla_{b]} \mu_c - g_{c[a} \mu_{b]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Flat!

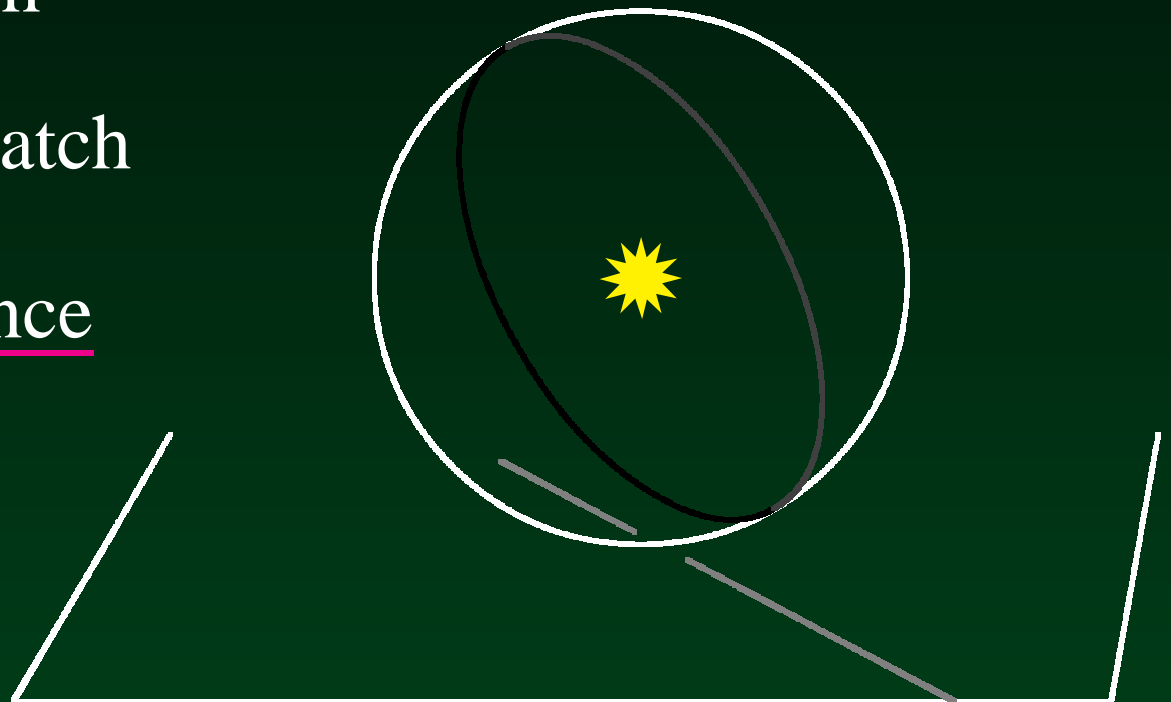
# Projective differential geometry

Suppose  $\nabla_a$  and  $\hat{\nabla}_a$  are torsion-free affine connections.

Def<sup>n</sup>  $\nabla_a$  and  $\hat{\nabla}_a$  are projectively equivalent iff they have the same geodesics (as unparameterised curves).

EG The round sphere is projectively flat by central projection

Affine coordinate patch  
 $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$  is a  
projective equivalence



# Projective tractors

- Can always choose  $\nabla_a$  s.t.  $R_{ab}$  is symmetric
- Then define a connection on  $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$  by

$$\begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \xrightarrow{\nabla_a} \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab} \sigma \end{bmatrix} \quad P_{ab} \equiv \frac{1}{n-1} R_{ab}$$

- It is projectively invariant (T.Y. Thomas)
- Equivalent to a Cartan connection (É. Cartan)
- It is flat iff  $\nabla_a$  is projectively flat
- Hence the flat tractor connection on  $\mathbb{RP}_n$
- NB:  $\mathbb{RP}_n$  is a homogeneous space  $SL(n+1, \mathbb{R})/P$

# Back to real projective space

Recall  $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$  with its flat connection.

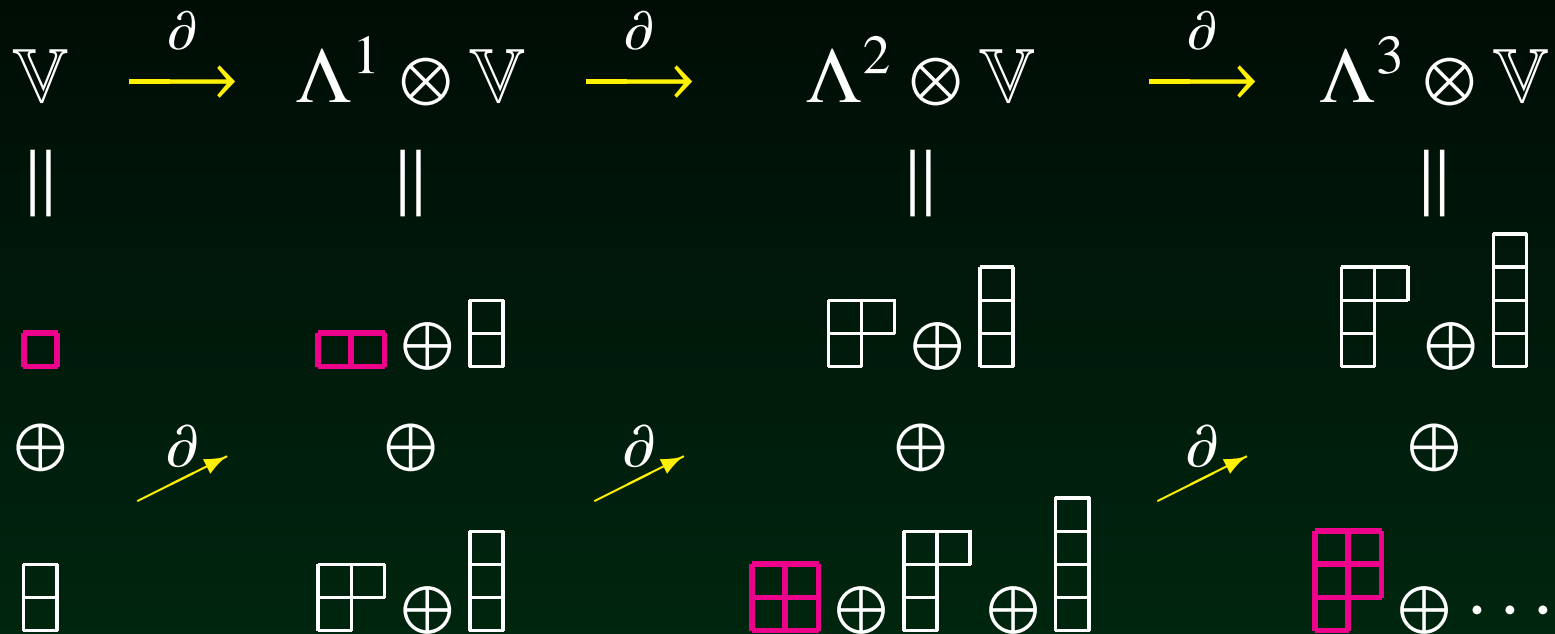
Thus,  $\mathbb{V} \equiv \Lambda^2 \mathbb{T} = \Lambda^1 \oplus \Lambda^2$  also has a flat connection

$$\begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} \xrightarrow{\nabla_a} \begin{bmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} + g_{ab} \sigma_c - g_{ac} \sigma_b \end{bmatrix}.$$

$$\begin{array}{ccccccc} \mathbb{V} & \xrightarrow{\nabla} & \Lambda^1 \otimes \mathbb{V} & \xrightarrow{\nabla} & \Lambda^2 \otimes \mathbb{V} & \xrightarrow{\nabla} & \Lambda^3 \otimes \mathbb{V} \xrightarrow{\nabla} \dots \\ \parallel & & \parallel & & \parallel & & \parallel \\ \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 \dots \\ \oplus & \nearrow & \oplus & \boxed{\text{NB}} & \oplus & \nearrow & \oplus \\ \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2 \dots \end{array}$$

$$\Lambda^p \otimes \Lambda^2 \ni \mu_{a\dots bcd} \xrightarrow{\partial} \mu_{[a\dots bc]d} \in \Lambda^{p+1} \otimes \Lambda^1$$

# Decompose into irreducibles



Lie algebra cohomology! (Kostant 1961)

Suspend  
disbelief

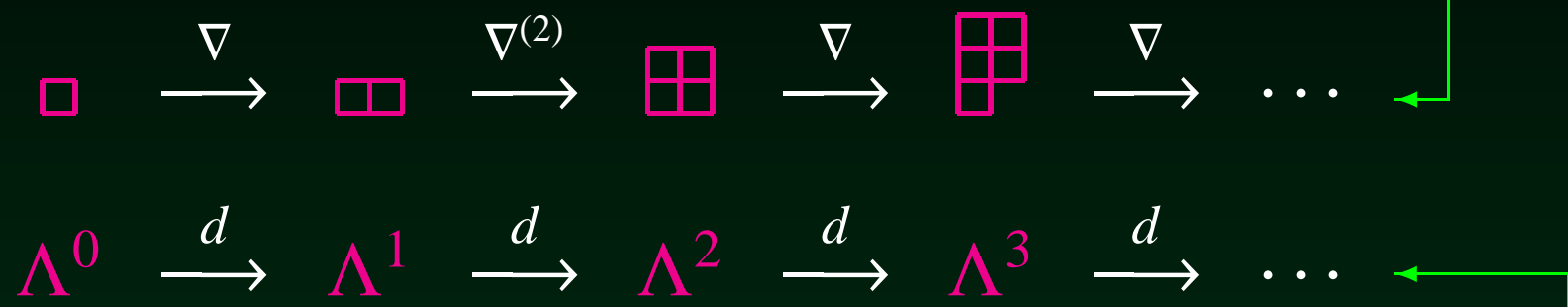
EG:  $\ker : \Lambda^2 \otimes \Lambda^2 \xrightarrow{d} \Lambda^3 \otimes \Lambda^1$   
 $= \{ \psi_{abcd} = \psi_{[ab][cd]} \text{ s.t. } \psi_{[abc]d} = 0 \}$   
 $= \{ \text{Riemann curvature tensors} \}!$



# BGG resolutions

Diagram chasing  $\rightsquigarrow$   
locally exact complexes

Riemannian deformation



Bernstein-Gelfand-Gelfand resolutions.

de Rham

$$\mathbb{R}P_n = \mathrm{SL}(n + 1, \mathbb{R}) / \left\{ \begin{bmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & * & & \\ 0 & & & \end{bmatrix} \right\} = G/P,$$

where  $G$  is semisimple and  $P$  is parabolic.

# Lie algebra cohomology

$\mathfrak{u} = \text{Lie algebra}$

$\mathcal{V} = \mathfrak{u}\text{-module}$



$$0 \rightarrow \mathcal{V} \xrightarrow{\partial} \text{Hom}(\mathfrak{u}, \mathcal{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{u}, \mathcal{V}) \xrightarrow{\partial} \dots$$

$$\partial v(X) = Xv \quad \partial \phi(X \wedge Y) = \phi([X, Y]) - X\phi(Y) + Y\phi(X) \quad \dots$$

$$\rightsquigarrow H^p(\mathfrak{u}, \mathcal{V})$$

$$\mathfrak{sl}(n+1, \mathbb{R}) \ni \left[ \begin{array}{c|ccc} * & * & \dots & * \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array} \right]$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\text{Let } \mathfrak{u} = \mathfrak{g}_{-1} \quad (\Rightarrow \mathfrak{u}^* = \mathfrak{g}_1)$$

$$\text{Let } \mathcal{V} = \Lambda^2 \mathbb{R}^{n+1}|_{\mathfrak{g}_{-1}}$$

$$0 \rightarrow \mathcal{V} \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathcal{V} \xrightarrow{\partial} \Lambda^2 \mathfrak{g}_1 \otimes \mathcal{V} \xrightarrow{\partial}$$

# Geometric import

Kostant's Bott-Borel-Weil Theorem  $\implies$

$H^p(\mathfrak{g}_{-1}, \mathcal{V}) = \square, \square\square, \square\square\square, \dots$  as  $SL(n, \mathbb{R})$ -modules

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathcal{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathcal{V} & \xrightarrow{\partial} & \Lambda^2 \mathfrak{g}_1 \otimes \mathcal{V} & \xrightarrow{\partial} & \Lambda^3 \mathfrak{g}_1 \otimes \mathcal{V} \\
 & & \downarrow \zeta & & \downarrow \zeta & & \downarrow \zeta & & \downarrow \zeta \\
 0 & \rightarrow & \mathbb{V} & \xrightarrow{\partial} & \Lambda^1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^2 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^3 \otimes \mathbb{V} \\
 & & \parallel & & \parallel & & \parallel & & \parallel \\
 & & \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 \\
 & & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus \\
 & & \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2
 \end{array}$$

Previously suspected tensor identities are justified !!

# Back to BGG

de Rham

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2$$

$$\phi \mapsto \nabla_a \phi$$

$$\omega_a \mapsto \nabla_{[a} \omega_{b]}$$

On  $\mathbb{RP}_n$  for  $n \geq 2$ ,

$$\omega_a = \nabla_a \phi \Leftrightarrow \nabla_{[a} \omega_{b]} = 0$$

Riemannian deformation  $\phi_a \mapsto \nabla_{(a} \phi_{b)} = \underline{\text{Killing}}$

$$\square \xrightarrow{\nabla} \square \xrightarrow{\nabla^{(2)}} \square \oplus \square$$

$$\phi_a \mapsto \nabla_{(a} \phi_{b)}$$

$$\omega_{ab} \mapsto \pi(\nabla_{(a} \nabla_{c)} \omega_{bd} + g_{ac} \omega_{bd})$$

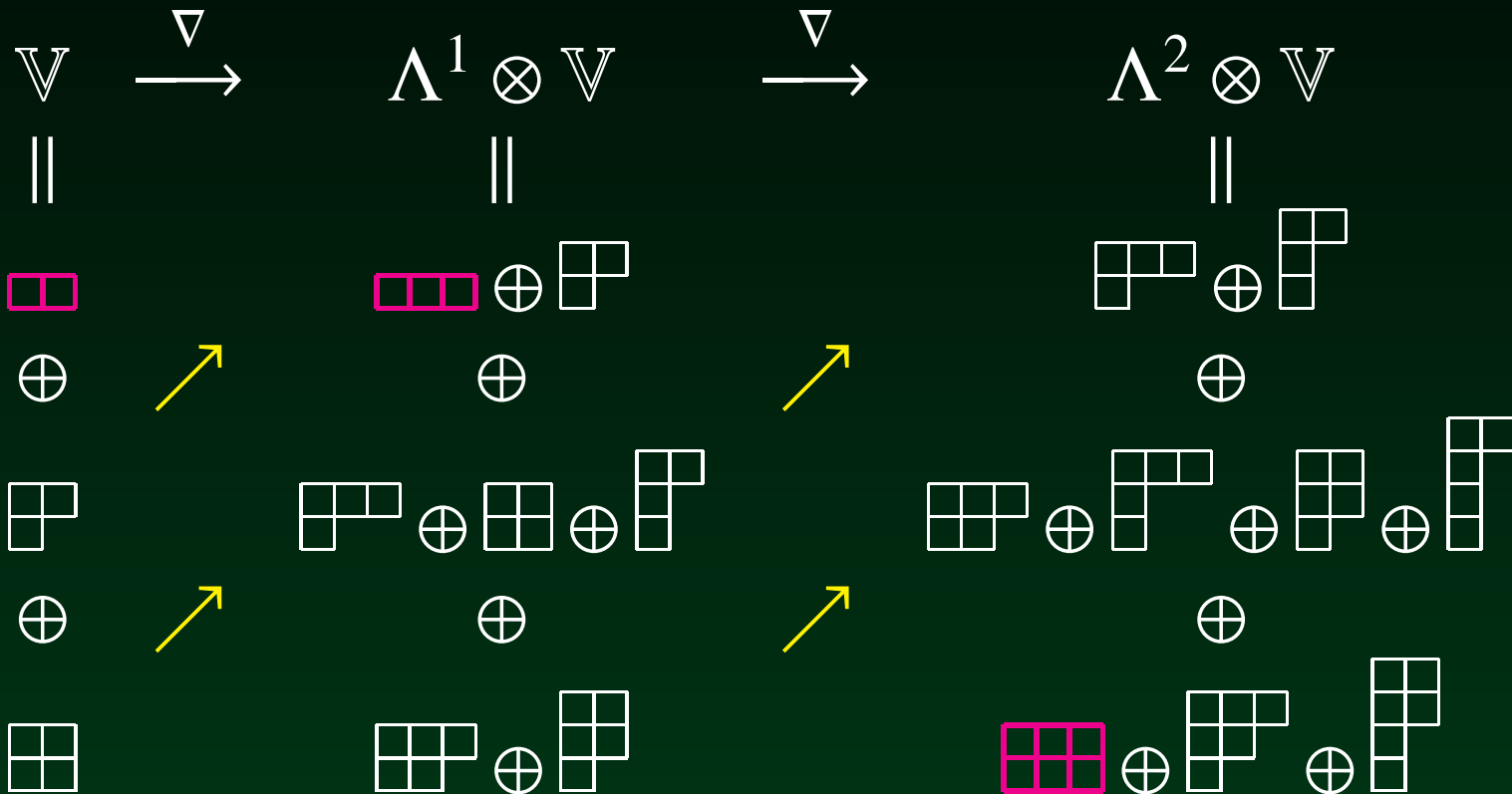
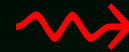
$$\omega_{ab} = \nabla_{(a} \phi_{b)} \Leftrightarrow \pi(\nabla_{(a} \nabla_{c)} \omega_{bd} + g_{ac} \omega_{bd}) = 0,$$

where  $\pi(\Phi_{acbd} = \Phi_{[ab][cd]}) \quad \square \otimes \square \longrightarrow \square \oplus \square$

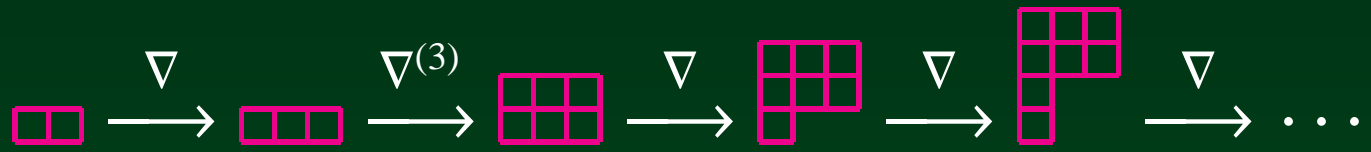
# Higher Killing operators

prolong using Lie algebra cohomology

$$\phi_{bc} = \phi_{(bc)} \mapsto \nabla_{(a}\phi_{bc)}$$



BGG



# Summary

## Range of Killing operators on $\mathbb{RP}_n$ for $n \geq 2$

$$\omega_a = \nabla_a \phi \Leftrightarrow \pi(\nabla_a \omega_b) = \nabla_{[a} \omega_{b]} = 0$$

$$\omega_{ab} = \nabla_{(a} \phi_{b)} \Leftrightarrow \pi(\nabla_{(a} \nabla_{c)} \omega_{bd} + g_{ac} \omega_{bd}) = 0$$

$$\omega_{abc} = \nabla_{(a} \phi_{bc)} \Leftrightarrow \pi(\nabla_{(a} \nabla_c \nabla_{e)} \omega_{bdf} + 4g_{(ac} \nabla_{e)} \omega_{bdf}) = 0$$

## Combinatorics

1						
1	1					
1	4					
1	10	9				
1	20	64				
1	35	259	225			
1	56	784	2304			
1	84	1974	12916	11025		
⋮	⋮	⋮	⋮	⋮	⋮	⋮

THANK YOU

END Of PART TWO