

Representation theory and the X-ray transform

The X-ray transform on projective space

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Summary of Lectures 1 and 2

Curvature on \mathbb{RP}_n

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

Curvature on \mathbb{CP}_n

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Model embeddings $\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ totally geodesic

2-form lemma $\mu^*\psi_{ab} = 0 \forall \mu \iff \psi_{ab}^\perp = 0$

Curvature lemma $\mu^*\psi_{abcd} = 0 \forall \mu \iff \psi_{abcd}^\perp = 0$

On \mathbb{RP}_n for $n \geq 2$, $\omega_a = \nabla_a \phi \iff \nabla_{[a} \omega_{b]} = 0$

$\omega_{ab} = \nabla_{(a} \phi_{b)} \iff \pi(\nabla_{(a} \nabla_{c)} \omega_{bd} + g_{ac} \omega_{bd}) = 0$ &c.

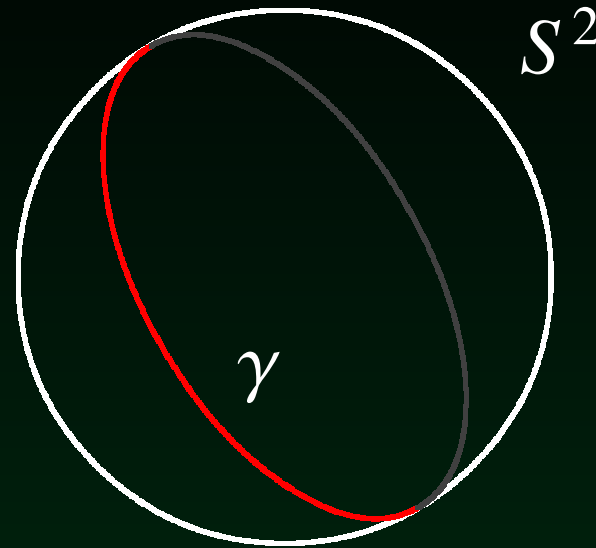
Topics

- Funk or Radon transforms on S^2 or \mathbb{R}^2
- X-ray transform on \mathbb{RP}_3
- X-ray transform on \mathbb{CP}_n
- X-ray transform on functions
- X-ray transform on 1-forms
- Symplectic geometry
- X-ray transform on symmetric tensors

Funk-Radon

- Funk (1913)

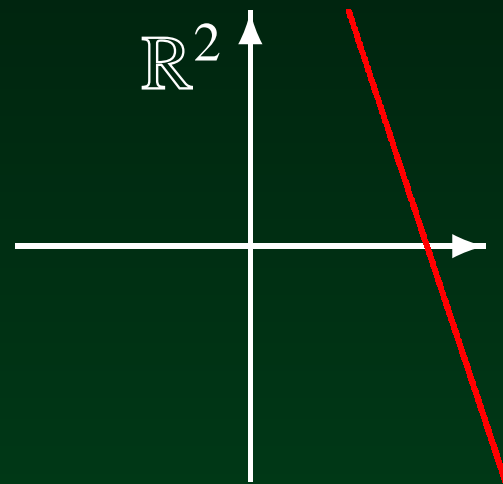
$$f \in \Gamma(S^2, \mathcal{E})$$



$$\phi(\gamma) = \oint_{\gamma} f$$

- Radon (1917)

$$f \in \Gamma_*(\mathbb{R}^2, \mathcal{E})$$



$$\phi(\gamma) = \int_{\gamma} f$$

Radon=Funk!

$$\mathcal{F} : \Gamma_{\text{even}}(S^2, \mathcal{E}) \xrightarrow{\cong} \Gamma_{\text{even}}(S^2, \mathcal{E})$$

$$\parallel \qquad \qquad \qquad \parallel$$

Better: $\Gamma(\mathbb{RP}_2, \mathcal{E}) \qquad \qquad \qquad \Gamma(\mathbb{RP}_2, \mathcal{E})$

Better still: $\mathcal{F} : \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) \xrightarrow{\cong} \Gamma(\mathbb{RP}_2^*, \widetilde{\mathcal{E}}(-1))$

Usual affine coordinates
projective equivalence!!

$$\mathbb{R}^2 \hookrightarrow \mathbb{RP}_2 \quad \rightsquigarrow$$

$$\begin{array}{ccc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} & \\ \downarrow & & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} & \end{array} \left. \vphantom{\begin{array}{ccc} \Gamma_*(\mathbb{R}^2, \mathcal{E}) & \xrightarrow{\mathcal{R}} & \\ \downarrow & & \\ \Gamma(\mathbb{RP}_2, \mathcal{E}(-2)) & \xrightarrow{\mathcal{F}} & \end{array}} \right\} \text{agree!}$$

John (1938)

The X-ray transform according to John

$$\Gamma_*(\mathbb{R}^3, \mathcal{E}) \ni f \mapsto \phi(\gamma) = \int_{\gamma} f$$

Better: $\Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \ni f \mapsto \phi(\gamma) = \oint_{\gamma} f$

NB invariance under $\mathrm{SL}(4, \mathbb{R})$ because

$$\Gamma(\mathbb{RP}_1, \mathcal{E}(-2)) \cong \Gamma(\mathbb{RP}_1, \Lambda^1) \xrightarrow{\int} \mathbb{R}$$

is invariant under $\mathrm{SL}(2, \mathbb{R})$.

X-ray transform on \mathbb{RP}_3

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \longrightarrow \Gamma(\text{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1])$$

Range?

Theorem (\approx John)

$$\phi = \mathcal{X}f \iff \square\phi = 0$$

where $\square : \widetilde{\mathcal{E}}[-1] \rightarrow \widetilde{\mathcal{E}}[-3]$

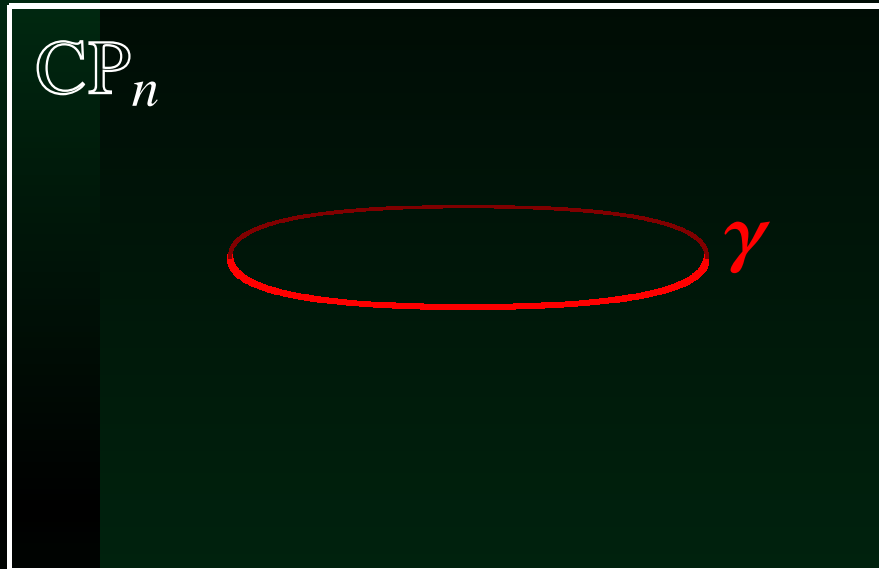
= ultrahyperbolic wave operator ($\text{SL}(4, \mathbb{R})$ -invariant).

Kernel? \mathcal{X} is injective on $\Gamma(\mathbb{RP}_n, \mathcal{E}(-2))$ for $n \geq 2$

As a Riemannian manifold under $\text{SO}(n+1)$

\parallel
 $\Gamma(\mathbb{RP}_n, \Lambda^0)$ Funk

X-ray transform on $\mathbb{C}\mathbb{P}_n$



$SU(n + 1)/S(U(1) \times U(n))$

Fubini-Study metric

$f =$ smooth function on $\mathbb{C}\mathbb{P}_n$

$\gamma =$ geodesic

$$f \xrightarrow{\mathcal{X}} \phi(\gamma) = \oint_{\gamma} f$$

Questions

- Kernel of \mathcal{X} ?
- What about $\omega \xrightarrow{\mathcal{X}} \phi(\gamma) = \oint_{\gamma} \omega$ for ω a 1-form?
- What about $\omega_{ab\dots c}$ a symmetric tensor?

X-ray transform on functions

Know $\oint_{\gamma} f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{RP}_n \Leftrightarrow f = 0.$

Suppose

Funk (1913)

$$\oint_{\gamma} f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_{\gamma} f = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{RP}_n \xrightarrow{\mu} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* f = 0 \quad \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence $f = 0$, i.e. \mathcal{X} is injective on functions on \mathbb{CP}_n

cf. Helgason, The Radon Transform, §2 Corollary 2.3

Approach (with Hubert Goldschmidt)



$\mathbb{R}P_n \hookrightarrow \mathbb{C}P_n$ induced
by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is
totally geodesic.

Translates by $SU(n+1)$ too!

↑ 'Model Embeddings' μ

- The X-ray transform on $\mathbb{R}P_n$ is well-understood.
- Pullback of tensors under μ is well-understood.
- Suitable global techniques on $\mathbb{C}P_n$ are available,
- compatible with similar techniques (BGG) on $\mathbb{R}P_n$.

X-ray transform on 1-forms

Know $\oint_{\gamma} \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{RP}_n \iff \omega = d\phi.$

Suppose

Michel (1978)

$$\oint_{\gamma} \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{CP}_n.$$

Then $\oint_{\gamma} \omega = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{RP}_n \xrightarrow{\mu} \mathbb{CP}_n$ for any model embedding. Hence,

$$\mu^* d\omega = 0 \quad \forall \text{ model embeddings } \mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n.$$

Hence, by 2-form lemma, $d\omega = \theta J$. Claim $\omega = d\phi$

1-forms cont'd

Theorem For $\omega \in \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1)$, $n \geq 2$, TFAE

(a) $\omega = d\phi$

(b) $d\omega = 0$

(c) $(d\omega)_\perp = 0$

(d) $d\omega = \theta J$

Proof (a) \Leftrightarrow (b) because $H^1(\mathbb{C}\mathbb{P}_n, \mathbb{R}) = 0$.

(c) \Leftrightarrow (d) by definition.

(b) \Rightarrow (c) is trivial.

(d) \Rightarrow (b) If $d\omega = \theta J$, then

$$0 = d^2\omega = d(\theta J) = d\theta \wedge J \Rightarrow d\theta = 0 \Rightarrow \theta = \text{constant.}$$

But if $\theta \neq 0$, then $d(\omega/\theta) = J$, a contradiction. \square

Symplectic geometry

Rumin-Seshadri complex

$$\begin{array}{ccccccccccc}
 \boxed{\Lambda^0} & \xrightarrow{d} & \boxed{\Lambda^1} & \xrightarrow{d_\perp} & \Lambda_{\perp}^2 & \xrightarrow{d_\perp} & \Lambda_{\perp}^3 & \xrightarrow{d_\perp} & \dots & \xrightarrow{d_\perp} & \Lambda_{\perp}^n \\
 & & & & & & & & & & \downarrow d_\perp^{(2)} \\
 \Lambda^0 & \xleftarrow{d_\perp} & \Lambda^1 & \xleftarrow{d_\perp} & \Lambda_{\perp}^2 & \xleftarrow{d_\perp} & \Lambda_{\perp}^3 & \xleftarrow{d_\perp} & \dots & \xleftarrow{d_\perp} & \Lambda_{\perp}^n
 \end{array}$$

\square local cohomology = \mathbb{R}

- $\Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda_{\perp}^2)$

is exact.

Symplectic geometry cont'd

Suppose ∇ is a connection on \mathbb{V} such that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma = J_{ab}\Phi\Sigma \quad \Phi \in \text{End}\mathbb{V}.$$

Then we can couple the Rumin-Seshadri complex

$$\mathbb{V} \xrightarrow{\nabla} \Lambda^1 \otimes \mathbb{V} \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes \mathbb{V} \longrightarrow \dots$$

It's still a complex and[†] (if \mathbb{V} is a bundle on $\mathbb{C}\mathbb{P}_n$)

$$\Gamma(\mathbb{C}\mathbb{P}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^2_\perp \otimes \mathbb{V})$$

is exact.

[†] under further mild conditions

Symmetric 2-tensors

Know (Michel (1973))

$$\oint_{\gamma} \omega_{ab} = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{R}\mathbb{P}_n \iff \omega_{ab} = \nabla_{(a}\phi_{b)}.$$

Therefore (BGG from Lecture two),

$$\oint_{\gamma} \omega_{ab} = 0 \quad \forall \gamma \hookrightarrow \mathbb{R}\mathbb{P}_n \iff \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0.$$

Would like to show (Tsukamoto (1981))

$$\oint_{\gamma} \omega_{ab} = 0 \quad \forall \text{ geodesics } \gamma \hookrightarrow \mathbb{C}\mathbb{P}_n \iff \omega_{ab} = \nabla_{(a}\phi_{b)}.$$

Know (BGG & curvature lemma from Lecture one)

$$\oint_{\gamma} \omega_{ab} = 0 \quad \forall \gamma \hookrightarrow \mathbb{C}\mathbb{P}_n \implies (\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}))^{\perp} = 0.$$

Therefore, suffices to show, on $\mathbb{C}\mathbb{P}_n$

$$\omega_{ab} = \nabla_{(a}\phi_{b)} \iff (\pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}))^{\perp} = 0$$

Proof of

$$\omega_{ab} = \nabla_{(a}\phi_{b)} \iff \left(\pi\left(\nabla_{(a}\nabla_{b)}\omega_{cd} + g_{ab}\omega_{cd}\right)\right)^\perp = 0$$

Consider the connection on $\mathbb{V} \equiv \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^1$

$$\left[\begin{array}{c} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} + g_{ab} \sigma_c - g_{ac} \sigma_b + J_{ab} \rho_c - J_{ac} \rho_b - J_{bc} \rho_a + J_{bc} J_a^d \sigma_d \\ \nabla_a \rho_b + J_a^d \mu_{bd} \end{array} \right]$$

It satisfies[†] $(\nabla_a \nabla_b - \nabla_b \nabla_a) \Sigma = J_{ab} \Phi \Sigma$.

Now unravel the exactness of

$$\Gamma(\mathbb{C}P_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{C}P_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{C}P_n, \Lambda^2_\perp \otimes \mathbb{V})$$

by Heisenberg Lie algebra cohomology!

\Leftrightarrow BGG !!

[†] and further mild conditions

Further results and summary

Theorem

Suppose $\omega_{ab\dots c}$ is symmetric on $\mathbb{R}\mathbb{P}_n$ or $\mathbb{C}\mathbb{P}_n$ for $n \geq 2$.

$$\oint_{\gamma} \omega_{ab\dots c} = 0 \quad \forall \text{ geodesics } \gamma \iff \omega_{ab\dots c} = \nabla_{(a} \phi_{b\dots c)},$$

for some symmetric tensor $\phi_{b\dots c}$.

By combining

- pullback lemmata from Lecture 1
- BGG complexes from Lecture 2
- symplectic geometry from Lecture 3

we can bootstrap from $\mathbb{R}\mathbb{P}_n$ to $\mathbb{C}\mathbb{P}_n$. Therefore,

It suffices to prove the Theorem for $\mathbb{R}\mathbb{P}_n$ (Lecture 4).

THANK YOU

END OF PART THREE