Representation theory and the X-ray transform

Complex methods

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Aim of this lecture

To find kernel and range of X-ray transform on $\mathbb{RP}_n$. In particular, to prove

Theorem
Suppose $\omega_{ab\cdots c}$ is symmetric on $\mathbb{RP}_n$ for $n \geq 2$.

$$\oint_\gamma \omega_{ab\cdots c} = 0 \ \forall \text{ geodesics } \gamma \iff \omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)},$$

for some symmetric tensor $\phi_{b\cdots c}$.

Recall: bootstrap using $\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ allows us to prove the corresponding theorem for $\mathbb{CP}_n$.

We shall also find range of X-ray transform on $\mathbb{RP}_n$,

e.g. $\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\simeq} \ker \Box$ on $\mathrm{Gr}_2(\mathbb{R}^4)$.

Machine

Spectral Sequence $\boxed{E_1^{p,q} = \Gamma(\mathrm{Gr}_2(\mathbb{R}^{n+1}), \tau_*^q \widetilde{\text{Œ}}_\eta^p(k))}$

converging to E_∞^{p+q} where $E_\infty^r = 0$ except

- $0 \to \underbrace{\bigodot^k (\mathbb{R}^{n+1})^*}_{=0 \text{ if } k \leq -1} \to \Gamma(\mathbb{RP}_n, \mathcal{E}(k)) \to E_\infty^1 \to 0$
- if n is odd

$$0 \to E_\infty^{n-1} \to \Gamma(\mathbb{RP}_n, \mathcal{E}(k)) \to \overbrace{\bigodot^{-k-n-1} \mathbb{R}^{n+1}}^{=0 \text{ if } k \geq -n} \to 0$$

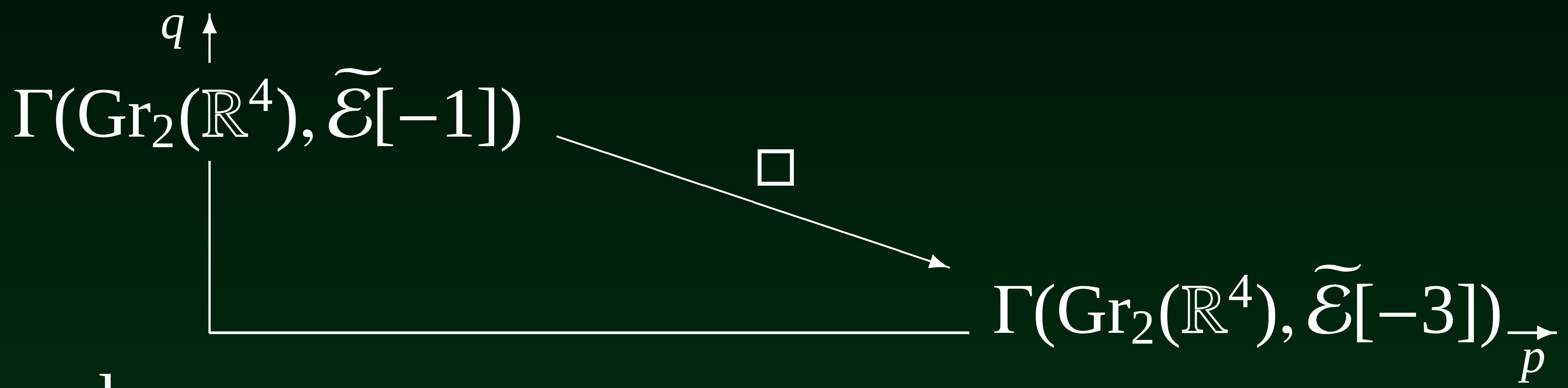
- if n is even and $k \leq -n - 1$

$$E_\infty^n \cong \bigodot^{-k-n-1} \mathbb{R}^{n+1}$$

Machine cont'd

For example ($k = -2$, $n = 3$) $\quad \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\cong} E_\infty^1$.

But $E_1^{p,q} = E_2^{p,q} =$



and so

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\cong}$$
$$\ker \square : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-1]) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3])$$

In particular, $\boxed{\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \to \ldots \text{ is injective}}$

Proof of $\oint_\gamma \omega_{ab\cdots c} = 0 \ \forall \gamma \iff \omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)}$

uses the following ingredients.

- A generalisation of the spectral sequence for irreducible homogeneous vector bundles on $\mathbb{RP}_n$ rather than just line bundles.
- The BGG resolution on $\mathbb{RP}_n$.
- The BGG resolution on $\mathrm{Gr}_2(\mathbb{R}^{n+1})$.

NB

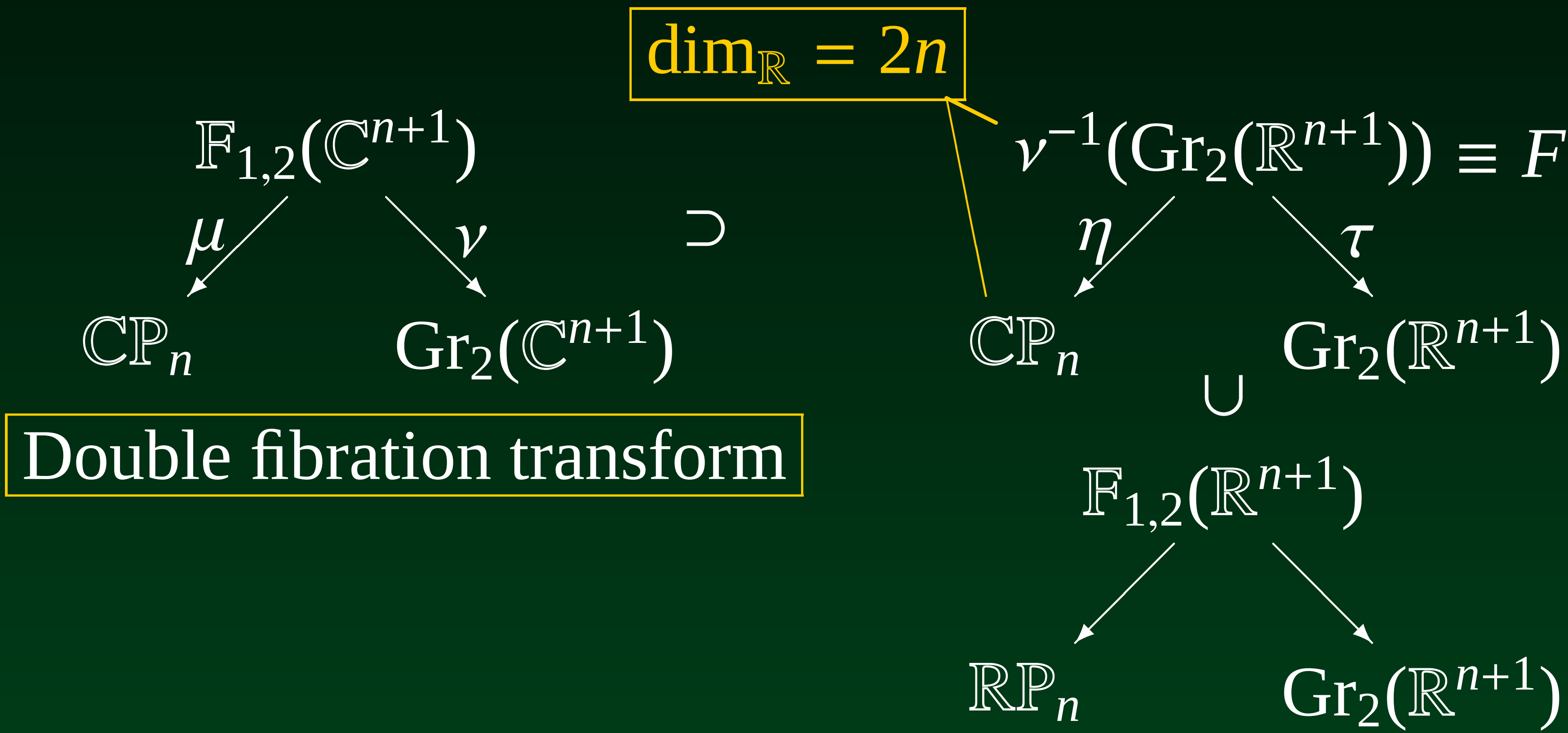
$$\mathrm{Gr}_2(\mathbb{R}^{n+1}) = \mathrm{SL}(n+1,\mathbb{R}) / \left\{ \begin{bmatrix} * & * & * \cdots * \\ * & * & * \cdots * \\ 0 & 0 & \\ \vdots & \vdots & \huge{*} \\ 0 & 0 & \end{bmatrix} \right\} = G/P,$$

where G is semisimple and P is parabolic.

Complex analysis

comes into play in two ways

- constructing the spectral sequence,
- computing with the spectral sequence.



Real blow up

$$F = \left\{ (L, P) \text{ s.t. } \begin{array}{l} L \subset \mathbb{C}^{n+1} \text{ is a complex line} \\ P \subset \mathbb{R}^{n+1} \text{ is a real plane} \\ \Re(L) \subseteq P \text{ (generic equality)} \end{array} \right\}$$

$$\downarrow \eta \qquad\qquad\qquad \downarrow$$

$$\mathbb{CP}_n = \left\{ L \text{ s.t. } L \subset \mathbb{C}^{n+1} \text{ is a complex line} \right\}$$

$$\begin{array}{ccc} F & \supset & \mathbb{F}_{1,2}(\mathbb{R}^{n+1}) \\ \downarrow \eta & & \downarrow \\ \mathbb{CP}_n & \supset & \mathbb{RP}_n \end{array}$$

Real blow up of $\mathbb{CP}_n$ along $\mathbb{RP}_n$!

Involutive structure

- complex manifold Ω
  - $J : T\Omega \to T\Omega$ s.t. $J^2 = -\mathrm{Id}\ldots$

$$\begin{cases} T^{0,1} \subset \mathbb{C}T\Omega \text{ s.t. } [T^{0,1}, T^{0,1}] \subseteq T^{0,1} \\ 0 \to \Lambda^{1,0} \to \Lambda^{1} \to \Lambda^{0,1} \to 0 \text{ s.t. } \bar{\partial}^2 = 0 \end{cases}$$

- totally real submanifold $M \hookrightarrow \Omega$
  - $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \Omega$ and $TM \cap JTM = 0$

$\equiv$ involutive structure $\overset{\checkmark}{\longrightarrow}$

$$\begin{array}{ccc} \widetilde{\Omega} & \supset & \Sigma \\ \downarrow \eta & & \downarrow \\ \Omega & \supset & M \end{array}$$

real blow-up

Involutive cohomology $H^r_{\bar{\partial}}(\widetilde{\Omega})$ (cf. Dolbeault, $\bar{\partial}_b, \ldots$)

Direct images

$$\begin{array}{ccc} & F & \\ \eta \swarrow & & \searrow \tau \\ \mathbb{RP}_n \subset \mathbb{CP}_n & & \mathrm{Gr}_2(\mathbb{R}^{n+1}) \end{array}$$

- pullback by η
- pushdown by τ
- fibres of $\tau \cong \mathbb{CP}_1$

- $0 \to \Gamma(\mathbb{CP}_n, \mathcal{O}(V)) \to \Gamma(\mathbb{RP}_n, \mathcal{E}(V)) \to H^1_{\underline{\partial}}(F, \widetilde{V}) \to H^1(\mathbb{CP}_n, \mathcal{O}(V)) \to 0$
- Spectral sequence

$$E_1^{p,q} = \Gamma(\mathrm{Gr}_2(\mathbb{R}^{n+1}), \tau_*^q \text{Œ}_\eta^p(\widetilde{V})) \Longrightarrow H^{p+q}_{\underline{\partial}}(F, \widetilde{V})$$

  - Œ = partially holomorphic functions on F
  - $\tau_*^q \text{Œ}_\eta^p(\widetilde{V})$ computed by Bott-Borel-Weil

Further Reading

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- M.G. Eastwood and C.R. Graham, The involutive structure on the blow-up of $\mathbb{R}^n$ in $\mathbb{C}^n$, Commun. Anal. Geom. 7 (1999) 613–626.
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THANK YOU

THE END